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Measuring DMU-efficiency by modified cross-efficiency approach

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Abstract. A fundamental weakness of the Data Envelopment Analysis (DEA) is its weak discrimination in cases when a small number of decision making units are compared. Therefore, in such cases the basic DEA model (optimistic and pessimistic) is used in combination with other methods or additional constraints are added to the model. In this paper, the cross-efficiency method was combined with a self-ranking procedure which uses the Potential Method (PM). The results are similar to those of Wang and others [15] based on the Geometric Mean (GM). Another interesting result is that a preorder "domination by inputs" and ordinal ranking derived from it produce results very close to those obtained by GM and PM.
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1. Introduction. DEA

Data Envelopment Analysis (DEA) is a well known tool for evaluating the performance of manufacturing and service activities. It is widely used for evaluations of banks, hospitals and production plans with multiple inputs and outputs. The main idea is to calculate the efficiency score of such Decision Making Unit (DMU) as:

$$\text{eff} = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}.$$

In situations when the weights are determined or known, the most efficient DMU is easily determined. Otherwise, the problem is much more sophisticated. The model proposed by Cooper at all. [8] calculates the weights and the relative efficiency score of a test DMU_o, $o \in \{1, \ldots, n\}$ by solving the maximization problem given bellow, assuming that there are n DMUs, m inputs and s outputs:

$$\max_{u \geq 0, v \geq 0} \theta_o := \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}, \tag{1}$$

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$$\text{s.t. } \theta_j := \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad \forall j \in \{1,\dots,n\},$$
$$u_r, v_i \geq 0, \quad \forall r \in \{1,\dots,s\}, i \in \{1,\dots,m\},$$

where

$$y_{ro} = \text{an amount of } r\text{-th output for } o\text{-th DMU}_o$$
$$x_{io} = \text{an amount of } i\text{-th input for } o\text{-th DMU}_o.$$

The fractional maximization (1) may be converted to an LP problem:

$$\begin{aligned} \max \theta_o = & \sum_{r=1}^{s} u_r y_{ro}, \\ \text{s.t. } & \sum_{i=1}^{m} v_i x_{io} = 1 \quad , \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad \forall j \\ & u_r, v_i \geq 0 \quad \forall r, i, \end{aligned} \tag{2}$$

which is iterated n times to identify the relative efficiency of DMU$_o$ its input and output weights that maximize its efficiency score. DMU is *efficient* (*optimistic efficient*) if it obtains the score equal to 1, otherwise it is *inefficient*. In further text we denote a vector of optimal output weights for DMU$_o$ by u^o and the vector of the optimal input weights for DMU$_o$ by v^o.

Pessimistic efficiency may be obtained by the following pessimistic DEA model:

$$\begin{aligned} \min_{u\geq 0, v\geq 0} \psi_o := & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}, \\ \text{s.t. } \psi_j := & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \geq 1, \quad \forall j \in \{1,\dots,n\}, \\ & u_r, v_i \geq 0, \quad \forall r \in \{1,\dots,s\}, i \in \{1,\dots,m\}. \end{aligned} \tag{3}$$

The equivalent pessimistic LP model is:

$$\begin{aligned} \min \psi_o = & \sum_{r=1}^{s} u_r y_{ro}, \\ \text{s.t. } & \sum_{i=1}^{m} v_i x_{io} = 1 \quad , \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0 \quad \forall j \\ & u_r, v_i \geq 0 \quad \forall r, i, \end{aligned} \tag{4}$$

Beside its discriminatory power DEA allows the identification of targets for improvements (benchmarking). Both have certain limitations. Benchmarks od inefficient DMU may be different in the operating structure or even nonexistent in reality. Doyle and Green [9] tried to overcome this problem using performance-based clustering methods to identify more appropriate benchmarks. This problem is not discussed here.

At a glance, traditional DEA models offer efficiency score as a value function for ranking inefficient DMUs. This may be wrong because some inefficient DMU may be better overall performers than certain efficient ones. This is mainly because of the unrestricted flexibility of the weights (Wong and Beasley [16]).

1.1. Cross-efficiency

The aim of this article is to improve the discriminatory power of DEA using the cross-efficiency approach introduced by Sexton et al. [14]. The idea of this approach is to evaluate the performance of one DMU with respect to the optimal input and optimal output weights of another DMU. The cross-efficiency matrix (CEM) may be built to capture all cross-performances and used to calculate 'overall' preferences among DMUs. Despite the non-uniqueness of optimal weights, CEM is well defined. The matrix element c_{ij} of CEM in i-ith row and j-th column of CEM represents the efficiency of DMU_j when evaluated with the optimal weights of DMU_i, i.e.

$$c_{ij} = \text{eff}_j(u^i, v^i) \tag{5}$$

We expect that 'good' DMU has several high values in its column.

Some authors (Boussofiane et. al. [3]) computed the means of each column of the cross-efficiency matrix C to differentiate between good and poor performers. Jing and Zhao [10] use those weights to calculate the weighted mean of the rows of CEM to obtain new (improved) relative weights. Iterating their argument, we may proceed to obtain 'robust' weights of the aggregation operator

$$\Phi_C(u) := \frac{uC}{\|uC\|_1},$$

where $\|\cdot\|_1$ represents 1-norm. It is obvious from Perron's theorem that the iterative procedure

$$u_{n+1} = \Phi_C(u_n), n \in \mathbb{N},$$

converges to the unique positive left eigenvector of C which is independent of $u_0 \in \Sigma$. This eigenvector represents the required weights. We shall use this idea to calculate the 'robust' weights as a fixed point of a self-aggregation operator.

1.2. Self-ranking

Our approach resembles the Multicriteria Decision Making Analysis (MCDA). CEM may be considered as a decision table with columns which correspond to the set of states, and rows which corresponds to the set of actions. Evidently, both sets are equal (the set od all DMUs) and CEM offers all information about the feedback

connectivity between DMUs. The point is that the probabilities of the states are not known, and whatever multicriteria aggregation procedure Φ we use, we should be aware that real weights should be *robust* in the following sense

$$\Phi(w) = w \text{ and } w > 0, \tag{6}$$

where $\Phi : \Sigma \to \Sigma$ is a self-aggregation procedure defined on the standard simplex $\Sigma := \{\xi \in \mathbf{R}^n \mid \sum_{j=1}^{n} \xi_j = 1, \xi_j \geq 0 \; \forall j\}$. In other words, the robust priority vector is a fixed point of the *self-aggregation* operator. The non-negativity of the fixed point means that all states should be 'counted'.

In principle, priorities may be calculated for all DMUs or only for efficient DMUs. In the first case we talk about *global ranking*, otherwise we talk about *reduced ranking* (of the efficient DMUs only). Both of them use CEM. Instead of a decision table we consider a network with feedback. The nodes of the network are DMUs, and each node influences the other nodes with information given in CEM. We may restrict the zone of influence of each node to all nodes, except the node in consideration (figure 1). The reason for that is the high self-estimation of each node in CEM

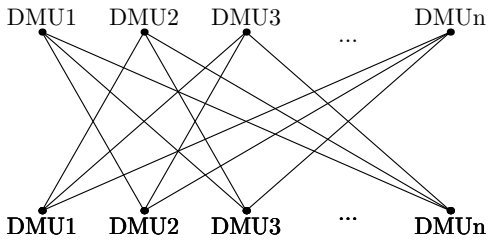

Figure 1: *The influence zone of each node in the network with feedback*

that we want to avoid. The same effect is achieved if we delete the main diagonal in CEM. Classical aggregation procedures, like the weighted arithmetic (geometric) mean or the eigenvalue method (Saaty [13]) may not be applied now without further improvements because of the missing data. A potential method (PM), developed by the first author, is a natural extension of AHP for such networks. PM is intuitively clear and more flexible than the eigenvalue method.

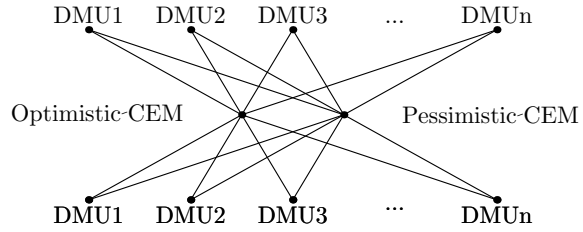

Figure 2: *The zones of influence in the self-dual optimistic-pessimistic hierarchy*

Moreover, we may combine *optimistic* CEM and *pessimistic* CEM introduced by Parkan & Wang [12] and Paradi et all. [11]. In that case the zone of influence of each node is given by the self-dual hierarchy in figure 2. The relative weights of

optimistic and *pessimistic* CEM in that hierarchy may be given arbitrary, but we used equal weights.

1.3. Direct ranking

Yet another ranking procedure may be applied, the *direct PM ranking*, which use a modified input-output modification of PM and overrides CEM. The details may be found in Čaklović [5]. Roughly speaking, input data may be considered as a logarithm of some other values. This allows us to interpret the difference of the input and output flow at each node (DMU) as a ratio of the weighted geometric mean of inputs and the weighted geometric mean of outputs for that node. The weights should be done a priori. The advantage of this method is that negative values are allowed as input, as well as positive values.

In order to decide whether data should be preferred for consideration as nominal values or as the logarithm of the nominal values, it is necessary to estimate the productivity change for each approach separately. To avoid the discussion about measuring the productivity change we refer the reader to Angelidis & Lyroudi [2]. In practice, there is no significant difference in direct ranking if we use nominal values or their logarithm.

2. The model. Potential method and self-duality.

2.1. Single-criterion case

In the Potential Method, a decision maker's preferences are captured in a directed weighted graph $(V, \mathcal{A})$, where the set of nodes V is the set of alternatives, and the elements of the arcs set $\mathcal{A}$ are pairs of compared nodes. An arc $\alpha = (a, b)$ is directed towards the more preferred node (here a). The (non-negative) weight of an arc α represents the strength of the preference on some scale, we named it the *preference flow* (in notation $\mathcal{F}_\alpha$). This means that $\mathcal{F}$ is a non-negative function on the set of arcs. Formally, a decision process is a procedure (function) which associates (to $\mathcal{F}$) a function X on the set of nodes V. In the literature, X has different names: a *value function*, *utility*, *ranking*... we named it *potential* because of a certain similarity of the model with the electric circuit.

In further text we identify the vector space of nodes by $\mathbb{R}^n$, where n is the number of nodes and the vector space generated by arcs by $\mathbb{R}^m$, where m is the number of arcs. The incidence matrix A is $m \times n$ matrix defined as:

$$a_{\alpha,v} = \begin{cases} -1, \text{ if } \alpha \text{ leaves node } v \\ 1, \text{ if } \alpha \text{ enters node } v \\ 0, \text{ otherwise.} \end{cases}$$

We shall write a_{ij} where i is the index of i-th arc and j is the index of j-th node.

Definition 1. *A preference flow is consistent if the algebraic sum of the flow components along each cycle equals zero.*

Consistency may be named 'weighted transitivity'. If the flow is a constant function, then the consistency is equivalent to the transitivity of a binary relation generated by the flow. The following statements are equivalent (left to the reader):

1. $\mathcal{F}$ is consistent.
2. $\mathcal{F}$ is a linear combination of columns of the incidence matrix.
3. There exists $X \in \mathbb{R}^n$ such that $AX = \mathcal{F}$.
4. The scalar product $y^\tau \mathcal{F} = 0$ for each cycle y, i.e. $\mathcal{F}$ is orthogonal to the cycle space.

We examine the consistency of a given flow $\mathcal{F}$ by solving the equation

$$AX = \mathcal{F}. \tag{7}$$

Readers familiar with the Analytic Hierarchy Process (AHP) may know that a positive reciprocal preference matrix H is consistent if

$$h_{ij}h_{jk} = h_{ik}, \quad i,j,k = 1,\ldots,n. \tag{8}$$

Considering a logarithm of this relation one can recognize condition (7).

An inconsistent flow $\mathcal{F}$ we approximate by the consistent one and find the potential of the consistent approximation of $\mathcal{F}$. This leads to the normal equation

$$A^\tau AX = A^\tau \mathcal{F}. \tag{9}$$

Definition 2. *A solution of the equation* (7) *or* (9)*, if the previous one does not exists, is called a potential of* $\mathcal{F}$*.*

For a consistent flow it is easy to find a potential X using the spanning tree of the preference graph. Evidently, X is not unique because the constant column is an element of the kernel $N(A)$. The uniqueness[‡] may be obtained adding a constraint

$$\sum_{i=1}^{n} X_i = 0.$$

The weights w_X, associated with the potential X are given by

$$w_X = \frac{a^{X\xi}}{\|a^{X\xi}\|_1}, a > 0. \tag{10}$$

An argument for such a formula has its origin in the relation of a consistent positive reciprocal matrix A and its logarithm $\mathcal{F}$ which is the potential difference. To obtain the weights from the potential X, we simply return to the "exponential" domain.

[‡]For connected graphs only. Otherwise, the potential may be calculated for each connected component separately.

2.2. Aggregation of preference flows

Suppose there are several preference graphs $(V, \mathcal{A}_i)$ and the corresponding preference flows $\mathcal{F}^{(i)}$ over the same set of vertices. Use w_i to denote the weight of i-th graph.

The overall preference flow (consensus) is a kind of a convex combination of individual flows $\mathcal{F}^{(i)}$. Before the formal summation we should define the "overall arc(s)". After that, the "overall preference intensity" of the arcs may be calculated. The formal definition is the following: For a given pair $\alpha = (u, v)$ of alternatives we calculate

$$F_\alpha := \sum_{\substack{i=1 \\ \pm\alpha\in\mathcal{A}_i}}^{k} w_i \mathcal{F}_\alpha^{(i)}, \tag{11}$$

where the item $w_i\mathcal{F}_\alpha^{(i)}$ is taken into account if and only if $\alpha \in \mathcal{A}_i$ or $-\alpha \in \mathcal{A}_i$. If this sum is non-negative, then we include α in the set $\mathcal{A}$ of arcs of the consensus graph, and we put $\mathcal{F}(\alpha) := F_\alpha$. If it is negative, we define $-\alpha = (v, u)$ as an arc in $\mathcal{A}$ and $\mathcal{F}(-\alpha) := -F_\alpha$. Obviously, $\mathcal{F} \geq 0$. If F_α is not defined then u and v are not adjacent in the consensus graph.

2.3. Self-assessment

An elementary example of self-assessment is a group of decision makers ranking themselves. Each member of the group may include himself in his preference graph or not, depending upon the rules, and he may use its own criteria in ranking of others.

To simplify, we denote a group of decision makers by $G = \{1, 2, \ldots, n\}$ and the preference flow of the decision maker i by $\mathcal{F}^{(i)}$. Moreover, the standard simplex in $\mathbb{R}^n$ is denoted by $\Sigma = \{\xi \mid \sum_i \xi_i = 1, \xi \geq 0\}$. If $\xi \in \Sigma$ is an a priori given group ranking and $\mathcal{F}_G$ is the consensus flow, then

$$\sum_{i\in G} \xi_i \mathcal{F}^{(i)} = \mathcal{F}^G. \tag{12}$$

Because of the linearity, the same relation should take place for potentials, i.e.

$$\sum_{g\in G} \xi_i X^{(i)} = X^G. \tag{13}$$

If X denotes the matrix with columns $X^{(i)}$, $i = 1, \ldots, n$ then, the left-hand side of the above equation may be written as a product $X\xi$ between matrix X and column ξ. A function

$$\Phi_X : \xi \mapsto \frac{a^{X\xi}}{\|a^{X\xi}\|_1} \tag{14}$$

defined on the standard simplex Σ to itself is well-defined and the group ranking derived from the consensus flow should be the fixed point of Φ_X, i.e

$$\xi = \Phi_X(\xi). \tag{15}$$

The existence of the fixed point is a consequence of the Brouwer's fixed point theorem, but the uniqueness needs some restrictions on Φ_X.

Theorem 1. *Suppose that*

$$2 \ln a \|X\|_\infty < 1. \tag{16}$$

Then, Φ_X is a contraction and for each $\xi \in \Sigma$ the sequence $(\Phi_X)^n(\xi)$ converges to the unique fixed point $\xi_0 \in \Sigma$ of Φ_X.

The proof may be found in [4]. The condition (16) may be satisfied by changing parameter a. Moreover, for complete graphs, it may be shown that the norm $\|X\|_\infty$ is smaller as n rises. In that case, normal equation (9) may be solved explicitly

$$X_i = \frac{1}{n} \sum_{i=1}^{n} (A^\tau \mathcal{F})_i.$$

The right-hand side of the equation above is the flow difference for i-th node, i.e. the difference between the input and output flow for i-th node, divided by the number of nodes in the graph.

An example of self-assessment in context different than bank ranking, may be found in Čaklović [6].

2.4. Aggregation of data in CEF self-assessment

As an application of Theorem 1, we calculate the importance (score) of each DMU from the cross-efficiency table (5). Define a family of preference flows $\mathcal{F}^{(i)}, i = 1, \ldots, n$ by

$$\mathcal{F}^{(i)}(j,k) = c_{ij} - c_{ik}. \tag{17}$$

There are two options: (1) The first is to use logarithmic data, i.e. to substitute $c_{ij} \leftarrow \log c_{ij}, \forall i, j$ which implies

$$\begin{aligned} \mathcal{F}^G(j,k) &= \sum_i \xi_i \log \frac{c_{ij}}{c_{ik}} \\ &= \log \prod_i \left(\frac{c_{ij}}{c_{ik}} \right)^{\xi_i}, \end{aligned}$$

where $\xi \in \Sigma$ denotes an unknown weight vector. In other words, the component of the consensus flow is the logarithm of the quotient of geometric means of the corresponding columns. By substituting $\mathcal{F}^G$ into equations (12) and using (13), (14) we obtain equation (15) which may be solved iteratively.

(2) The second option is to use original values c_{ij} in formula (17). In the following numerical example we used original values.

3. A numerical example

The data in Table 4 was obtained from the *China Industrial Economy Statistical Yearbook* (2001) [1]. The *Original value of Fixed Assets*, *Current Assets* and the *Gross Industrial Output* value are shown in 100 million RMB (Chinese monetary unit) and calculated at current prices. The *Number of Staff and Workers at year end* is expressed in units of 10 thousand persons.

Wang and others [15] used the data to measure the efficiency of DMUs from two different points of view: optimistic and pessimistic, and calculated the overall efficiency as the geometric mean. This method may fail if some of the DMUs are simultaneously at the optimistic and pessimistic frontier.

Table 2 shows DEA optimistic and DEA pessimistic scores[§], the rank position of each DMU in geometric mean scores (GM column) and the rank position in self-ranking obtained from the cross-efficiency table (optimistic, pessimistic and aggregated). CEF ranking (in this example) is independent of whether we restrict the zone of influence for each DMU, see the Figure 1.

The aggregation of CEF optimistic and CEF pessimistic values is done according to hierarchy 2. This hierarchy is a self-dual hierarchy and the geometric mean is calculated in each step of the procedure described in theorem 1. This is equivalent to the procedure of self-ranking with the geometric mean of two CEF matrices. The result of such aggregation is given in the last column of Table 2.

3.1. Symmetric distance matrix

The symmetric distance of two rank lists is defined as a cardinal number of the symmetric difference (XOR) of two relations induced by those lists. A strong domination counts 2 in the distance, and the tie counts 1.

Dist.	DEA_{opt}	DEA_{pes}	GM	CEF_{opt}	CEF_{pes}	CEF_{agg}
DEA_{opt}	0	127	61	63	131	43
DEA_{pess}	127	0	66	86	124	96
GM	61	66	0	36	110	30
CEF_{opt}	63	86	36	0	82	26
CEF_{pes}	131	124	110	82	0	108
CEF_{agg}	43	96	30	26	108	0

Table 1: *Symmetric distance matrix for various rankings. The distance does not depend on whether we restrict or the zone of influence for each DMU, see the Figure 1.*

The symmetric distance between GM and CEF_{agg} scores is 30 which means that those two rankings are quite close, regarding that we are dealing with a great number of DMUs. Table 1 shows pairwise distances between various rankings: DEA optimistic, DEA pessimistic, GM, CEF (optimistic), CEF (pessimistic) and CEF (aggregated).

[§]Using CCR model.

		Values		Rank position			
No	DMU	DEA Opt	DEA Pess	GM	CEF Opt	CEF Pess	CEF Agg
1	Beijing	0.84	1.99	12	8	3	8
2	Tianjing	0.89	2.27	7	7	5	7
3	Hebei	0.84	2.26	8	10	13	9
4	Liaoning	0.7	2.1	14	14	11	14
5	Shanghai	1.	2.21	6	4	1	5
6	Jiangsu	0.98	2.96	2	3	6	2
7	Zhejiang	1.	3.05	1	1	4	1
8	Fujian	0.94	2.91	5	6	8	6
9	Shandong	1.	2.83	3	5	9	4
10	Guangdong	1.	2.82	4	2	2	3
11	Guangxi	0.72	1.95	15	18	17	16
12	Hainan	0.74	2.02	13	12	10	13
13	Shanxi	0.49	1.18	30	30	30	30
14	Neimenggu	0.61	1.56	25	26	27	24
15	Jilin	0.66	2.03	18	17	16	18
16	Heilongjiang	0.77	2.24	11	13	15	12
17	Anhui	0.67	1.82	20	20	20	20
18	Jiangxi	0.63	1.52	24	24	28	21
19	Henan	0.77	1.8	16	16	19	15
20	Hubei	0.78	2.36	9	11	12	11
21	Hunan	0.72	1.74	19	19	21	19
22	Chongqing	0.56	1.73	23	21	23	22
23	Sichuang	0.56	1.75	22	23	24	26
24	Guizhou	0.47	1.46	29	29	29	29
25	Yunnan	0.67	2.04	17	15	14	17
26	Tibet	0.32	1.	31	31	31	31
27	Shaanxi	0.54	1.68	27	27	25	27
28	Gansu	0.58	1.64	26	25	26	25
29	Qinghai	0.52	1.37	28	28	18	28
30	Ningxia	0.57	1.75	21	22	22	23
31	Sinkiang	0.82	2.23	10	9	7	10

Table 2: *Optimistic and pessimistic DEA scores and the geometric mean (GM) compared to Cross-Efficiency (optimistic and pessimistic view)*

The distance may be understood also as the Kendall tau rank correlation coefficient (or some other type of correlation) — the distance matrix is 'proportional' to

the given one.

3.2. Ordinal ranking

An interesting question is what may be said about DMUs ranking taking regarding only *domination by inputs*? First, we rescale the data in Table 4 in such a way that the output value for each DMU becomes 1. Then, for two DMUs a and b we say that a *dominates* b, in notation $a \succcurlyeq b$, if $x(a) \geq x(b)$ by components, where $x(a)$ and $x(b)$ are input vectors of a and b. Domination $\succcurlyeq$ on the set of all DMUs is a partial order (transitive, reflexive and antisymmetric relation).

This order (in our example) is represented as a directed graph in which an ordered pair (b, a) is an arc iff $a \succcurlyeq b$ (figure 3). To simplify, transitive arcs, i.e. the arcs that have a parallel directed path are not shown. The nodes Zhejiang, Shanghai and Shandong are not dominated by some other nodes. They are simultaneously efficient in DEA optimistic approach.

One possible ranking of the nodes in such a graph may be done calculating the difference Φ between ingoing and outgoing arcs for each node. A function $\Phi : V \to \mathbb{R}$ we call the *flow difference*. In table 3 we calculated a symmetric distance from the ordering Ord_Φ generetad by Φ and other orderings: $\mathrm{DEA}_{opt}, \mathrm{DEA}_{pes}$, GM, $\mathrm{CEF}_{opt}, \mathrm{CEF}_{pes}$ and CEF_{agg}. The closest one is CEF_{opt} followed by CEF_{agg} and GM.

Dist.	Ord_Φ
DEA_{opt}	91
DEA_{pes}	98
GM	56
CEF_{opt}	50
CEF_{pes}	108
CEF_{agg}	54

Table 3: *Symmetric distance from ordinal ranking generated by Φ to other types of rankings*

Ordinal ranking of this type seems to be informative and not far from other rankings because the relation of weak domination has 296 elements (pairs), quite enough information to perform some ranking list. We are not discussing other ordinal aggregation procedures, Cook & others [7] gave a review of them and suggested some interesting algorithms.

Theorem 2. *If $(S, \succcurlyeq)$ is a partial order on a finite set S, then a flow difference Φ satisfies: $a \succcurlyeq b \implies \Phi(a) \geq \Phi(b)$.*

Moreover, partial $\succcurlyeq$ may be extended up to a complete (total) order $\succcurlyeq'$ such that

$$a \succcurlyeq' b \iff \Phi(a) \geq \Phi(b).$$

Proof. Because of the reflexivity of $\succcurlyeq$ the following functions $U, V : S \to \mathbb{N}$ are well

defined:

$$V(a) := \#\{x \in S \mid a \succcurlyeq x\}$$
$$U(a) := \#\{x \in S \mid x \succcurlyeq a\}.$$

Evidently,

$$\Phi(a) = V(a) - U(a),$$
$$a \succcurlyeq b \implies \Phi(a) \geq \Phi(b)$$

and

$$\Phi(a) > \Phi(b) \implies \neg(b \succcurlyeq a)$$

which implies that $\succcurlyeq$ may be extended up to a complete (total) order $\succcurlyeq'$ by

$$a \succcurlyeq' b \iff \Phi(a) \geq \Phi(b).$$

The claim is now straightforward;

$$a \succcurlyeq b \implies a \succcurlyeq' b \implies \Phi(a) \geq \Phi(b).$$

□

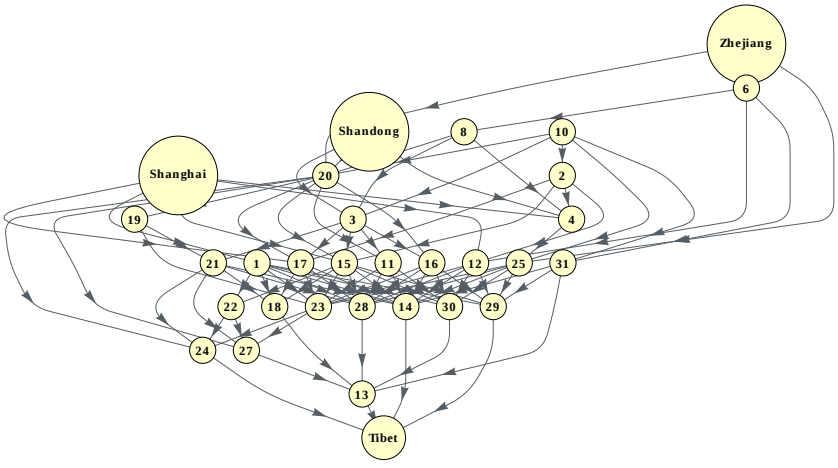

Figure 3: *Partial ordering domination by input represented as a digraph. Transitive arcs are removed for simplicity. Zhejiang, Shanghai and Shandong respectively are efficient in DEA optimistic approach.*

DMU	Inputs			Output
	Original value of ***fixed assets***	***Current assets***	Number of ***staff*** and workers at year end	Gross industrial ***output value***
Beijing	2402.79	2005.63	113.13	2565.38
Tianjing	2488.60	1787.41	120.19	2606.38
Hebei	3532.84	2000.19	269.75	3426.05
Liaoning	5372.79	3155.9	295.18	4249.46
Shanghai	5373.06	4370.38	204.94	6204.52
Jiangsu	6181.57	5499.34	518.19	10452.87
Zhejiang	3753.68	3377.81	323.22	6603.65
Fujian	2032.18	1401.27	155.55	2616.12
Shandong	6297.37	4076.79	522.37	8311.53
Guangdong	8005.77	6891.49	572.79	12480.93
Guangxi	1296.41	684.22	91.25	1003.24
Hainan	264.43	156.70	12.00	202.87
Shanxi	2170.68	1221.05	183.56	1216.86
Neimenggu	1385.74	603.7	85.34	748.97
Jilin	1833.04	1291.93	134.85	1679.91
Heilongjiang	3233.51	1567.07	195.17	2460.88
Anhui	1880.95	1212.57	162.61	1661.44
Jiangxi	1154.45	730.33	108.85	932.21
Henan	3447.01	2216.51	345.20	3494.96
Hubei	2989.31	1941.97	230.36	3064.43
Hunan	1947.23	1107.19	166.71	1627.94
Chongqing	1151.58	865.97	90.79	962.32
Sichuang	2917.04	1845.51	208.00	2076.96
Guizhou	913.15	676.07	68.34	631.64
Yunnan	1409.92	812.81	77.07	1063.36
Tibet	59.58	25.62	2.92	16.43
Shaanxi	1730.35	1084.49	124.98	1184.58
Gansu	1165.68	713.65	91.25	840.58
Qinghai	505.81	223.06	15.87	196.08
Ningxia	362.21	212.73	22.41	239.11
Sinkiang	1387.56	578.79	46.52	852.01

Table 4: *Data for 31 provinces, municipalities and autonomous regions of China in the year 2000*

4. Conclusion

In this paper we integrated the self-ranking procedure of the Potential Method and the classical DEA CCR-model for better differentiation of DMUs. A key connection between those two methods is a cross-efficiency matrix defined by (5). A cross-efficiency matrix uses the input and output weights of each DMU to calculate the performances of other units, which is a preparation step before making a consensus over the group of all DMUs. The consensus is done using an iterative process which calculates group weights as a fixed point of a self-aggregation procedure.

The same procedure may be done using optimistic and pessimistic cross-efficiency matrices as criteria to obtain a self-dual hierarchy 2 and calculate the fixed point of this hierarchy. The results seem to be close to those by Wang and others [15] obtained as the geometric mean of pessimistic and optimistic DEA-performances. The geometric mean method may fail if some DMUs are simultaneously at the optimistic and pessimistic frontier.

The Potential Method allows excluding 'self-perception' of each DMU and using only off-diagonal entries in cross-efficiency matrix. The results do not differ much from the results obtained using full cross-efficiency matrices.

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