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Crossed bimodules over rings and Shukla cohomology

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Abstract. In this paper we present some applications of Ann-category theory to classification of crossed bimodules over rings, classification of ring extensions of the type of a crossed bimodule.

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1. Introduction

Crossed modules over groups were introduced by J. H. C. Whitehead [15]. A crossed module over a group G with kernel a G-module M represents an element in the cohomology $H^3(G, M)$ [8]. The results on group extensions of the type of a crossed module were also represented by the cohomology of groups [6].

Later, H-J. Baues [2] introduced crossed modules over **k**-algebras. Crossed modules over **k**-algebras which are **k**-split with the same kernel M and cokernel B were classified by Hochschild cohomology $H^3_{Hoch}(B, M)$ [3].

In [4] the field **k** is replaced by a commutative ring $\mathbb{K}$, and crossed modules over $\mathbb{K}$-algebras were called *crossed bimodules*. In particular, if $\mathbb{K} = \mathbb{Z}$ one obtains crossed bimodules over rings.

Crossed modules over groups can be defined over rings in a different way under the name of *E-systems*. The notion of an E-system is weaker than that of a crossed bimodule over rings.

Crossed modules over groups are often studied in the form of $\mathcal{G}$-groupoids [5], or strict 2-groups [1]. From this point, we represent E-systems in the form of strict Ann-categories (also called strict 2-rings). Hence, one can use the results on Ann-category theory to study crossed bimodules over rings.

The plan of this paper is, briefly, as follows. Section 2 is dedicated to review definitions and some basic facts concerning Ann-categories. In Section 3, we introduce the concept of an E-system and prove that there is an isomorphism between the category of regular E-systems and that of crossed bimodules over rings. The relation among these concepts and crossed **C**-modules in the sense of T. Porter[10] is also discussed. The next section is devoted to showing a categorical equivalence of the

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category of E-systems and a subcategory of the category of strict Ann-categories, which is an extending of the result of R. Brown and C. Spencer [5].

The group extensions of the type of a crossed module were dealt with by R. Brown and O. Mucuk [6]. The similar results for ∂-extensions by an algebra R were done by H-J. Baues and T. Pirashvili [4] in a particular case. In Section 5 we solve this problem for ring extensions of the type of an E-system by Shukla cohomology groups. Our classification result contains the result in [4] when R is a ring.

2. Ann-categories

We state a minimum of necessary concepts and facts of Ann-categories and Ann-functors (see [11]).

A *Gr-category* (or a *categorical group*) is a monoidal category in which all objects are invertible and the background category is a groupoid. A *Picard* category (or a *symmetric* categorical group) is a Gr-category equipped with a symmetry constraint which is compatible with associativity constraint.

Definition 1. *An Ann-category consists of*

(i) *a category* $\mathcal{A}$ *together with two bifunctors* $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$*;*

(ii) *a fixed object* $0 \in \mathrm{Ob}(\mathcal{A})$ *together with natural isomorphisms* $\mathbf{a}_+, \mathbf{c}, \mathbf{g}, \mathbf{d}$ *such that* $(\mathcal{A}, \oplus, \mathbf{a}_+, \mathbf{c}, (0, \mathbf{g}, \mathbf{d}))$ *is a Picard category;*

(iii) *a fixed object* $1 \in \mathrm{Ob}(\mathcal{A})$ *together with natural isomorphisms* $\mathbf{a}, \mathbf{l}, \mathbf{r}$ *such that* $(\mathcal{A}, \otimes, \mathbf{a}, (1, \mathbf{l}, \mathbf{r}))$ *is a monoidal category;*

(iv) *natural isomorphisms* $\mathfrak{L}, \mathfrak{R}$ *given by*

$$\mathfrak{L}_{A,X,Y} : A \otimes (X \oplus Y) \longrightarrow (A \otimes X) \oplus (A \otimes Y),$$
$$\mathfrak{R}_{X,Y,A} : (X \oplus Y) \otimes A \longrightarrow (X \otimes A) \oplus (Y \otimes A)$$

such that the following conditions hold:

(Ann - 1) for $A \in \mathrm{Ob}(\mathcal{A})$*, the pairs* $(L^A, \breve{L}^A), (R^A, \breve{R}^A)$ *defined by*

$$\begin{array}{llll} L^A & = A \otimes - & R^A & = - \otimes A \\ \breve{L}^A_{X,Y} & = \mathfrak{L}_{A,X,Y} & \breve{R}^A_{X,Y} & = \mathfrak{R}_{X,Y,A} \end{array}$$

are $\oplus$*-functors which are compatible with* $\mathbf{a}_+$ *and* $\mathbf{c}$*;*
(Ann - 2) for all $A, B, X, Y \in \mathrm{Ob}(\mathcal{A})$*, the following diagrams commute*

$$\begin{array}{ccccc} (AB)(X \oplus Y) & \xleftarrow{\mathbf{a}_{A,B,X\oplus Y}} & A(B(X \oplus Y)) & \xrightarrow{id_A \otimes \breve{L}^B} & A(BX \oplus BY) \\ \downarrow{\scriptstyle \breve{L}^{AB}} & & & & \downarrow{\scriptstyle \breve{L}^{A}} \\ (AB)X \oplus (AB)Y & & \xleftarrow{\mathbf{a}_{A,B,X} \oplus \mathbf{a}_{A,B,Y}} & & A(BX) \oplus A(BY) \end{array}$$

$$\begin{array}{ccccc}
(X\oplus Y)(BA) & \xrightarrow{\mathbf{a}_{X\oplus Y,B,A}} & ((X\oplus Y)B)A & \xrightarrow{\breve{R}^B\otimes id_A} & (XB\oplus YB)A \\
\downarrow{\scriptstyle \breve{R}^{BA}} & & & & \downarrow{\scriptstyle \breve{R}^{A}} \\
X(BA)\oplus Y(BA) & & \xrightarrow{\mathbf{a}_{X,B,A}\oplus \mathbf{a}_{Y,B,A}} & & (XB)A\oplus (YB)A
\end{array}$$

$$\begin{array}{ccccc}
(A(X\oplus Y)B & \xleftarrow{\mathbf{a}_{A,X\oplus Y,B}} & A((X\oplus Y)B) & \xrightarrow{id_A\otimes \breve{R}^B} & A(XB\oplus YB) \\
\downarrow{\scriptstyle \breve{L}^{A}\otimes id_B} & & & & \downarrow{\scriptstyle \breve{L}^{A}} \\
(AX\oplus AY)B & \xrightarrow{\breve{R}^B} & (AX)B\oplus (AY)B & \xleftarrow{\mathbf{a}\oplus\mathbf{a}} & A(XB)\oplus A(YB)
\end{array}$$

$$\begin{array}{ccccc}
(A\oplus B)X\oplus (A\oplus B)Y & \xleftarrow{\breve{L}^{A\oplus B}} & (A\oplus B)(X\oplus Y) & \xrightarrow{\breve{R}^{X\oplus Y}} & A(X\oplus Y)\oplus B(X\oplus Y) \\
\downarrow{\scriptstyle \breve{R}^{X}\oplus\breve{R}^{Y}} & & & & \downarrow{\scriptstyle \breve{L}^{A}\oplus\breve{L}^{B}} \\
(AX\oplus BX)\oplus (AY\oplus BY) & & \xrightarrow{\mathbf{v}} & & (AX\oplus AY)\oplus (BX\oplus BY)
\end{array}$$

where $\mathbf{v}=\mathbf{v}_{U,V,Z,T}:(U\oplus V)\oplus(Z\oplus T)\longrightarrow(U\oplus Z)\oplus(V\oplus T)$ *is a unique morphism constructed from* $\oplus,\mathbf{a}_+,\mathbf{c},id$ *of the symmetric monoidal category* $(\mathcal{A},\oplus)$*;*
(Ann - 3) for the unit $1\in\mathrm{Ob}(\mathcal{A})$ *of the operation* $\otimes$*, the following diagrams commute*

$$\begin{array}{ccc}
1(X\oplus Y) & \xrightarrow{\breve{L}^1} & 1X\oplus 1Y \\
& {\scriptstyle \mathbf{l}_{X\oplus Y}}\searrow \quad \swarrow{\scriptstyle \mathbf{l}_X\oplus\mathbf{l}_Y} & \\
& X\oplus Y &
\end{array}
\qquad
\begin{array}{ccc}
(X\oplus Y)1 & \xrightarrow{\breve{R}^1} & X1\oplus Y1 \\
& {\scriptstyle \mathbf{r}_{X\oplus Y}}\searrow \quad \swarrow{\scriptstyle \mathbf{r}_X\oplus\mathbf{r}_Y} & \\
& X\oplus Y. &
\end{array}$$

An Ann-category $\mathcal{A}$ *is regular if its symmetry constraint satisfies the condition* $\mathbf{c}_{X,X}=id$*, and strict if all of its constraints are identities.*

Example 1. *Let* $\mathcal{A}=(\mathcal{A},\oplus)$ *be a Picard category whose unity and associativity constraints are identities. Denote by End*($\mathcal{A}$) *a category whose objects are symmetric monoidal functors from* $\mathcal{A}$ *to* $\mathcal{A}$ *and whose morphisms are* $\oplus$*-morphisms. Then, End*($\mathcal{A}$) *is a Picard category together with the operation* $\oplus$ *on monoidal functors and on morphisms. In this* $\oplus$*-category, the unity and associativity constraints are identities, the commutativity constraint is given by*

$$(\mathbf{c}^*_{F,G})_X=\mathbf{c}_{FX,GX},\quad X\in\mathrm{Ob}(\mathcal{A}),\ F,G\in\mathrm{End}(\mathcal{A}).$$

The operation $\otimes$ *on End*($\mathcal{A}$) *is naturally defined being the composition of functors. Then, End*($\mathcal{A}$) *together with two operations* $\oplus,\otimes$ *is an Ann-category in which the left distributivity constraint is given by*

$$(\mathfrak{L}^*_{F,G,H})_X=\breve{F}_{GX,HX},\quad X\in\mathrm{Ob}(\mathcal{A}),$$

and other constraints are identities (for details, see [12]).

Example 2. *Let R be a ring with an unit and M be an R-bimodule. The pair $\mathcal{I} = (R, M)$ is a category whose objects are elements of R and whose morphisms are automorphisms $(r, a) : r \to r$, $r \in R, a \in M$. The composition of morphisms is given by the addition in M. Two operations $\oplus$ and $\otimes$ on $\mathcal{I}$ are defined by*

$$\begin{aligned} x \oplus y &= x + y,\ (x, a) + (y, b) = (x + y, a + b), \\ x \otimes y &= xy,\ (x, a) \otimes (y, b) = (xy, xb + ay). \end{aligned}$$

The constraints of $\mathcal{I}$ are identities, except for left distributivity and commutativity constraints which are given by

$$\begin{aligned} \mathcal{L}_{x,y,z} &= (\bullet, \lambda(x, y, z)) : x(y + z) \to xy + xz, \\ \mathbf{c}_{x,y} &= (\bullet, \eta(x, y)) : x + y \to y + x, \end{aligned}$$

where $\lambda : R^3 \to M, \eta : R^2 \to M$ are functions satisfying the appropriate coherence conditions.

Here are standard consequences of the axioms of an Ann-category.

Lemma 1. *For every Ann-category $\mathcal{A}$ there exist uniquely isomorphisms*

$$\hat{L}^A : A \otimes 0 \to 0, \qquad \hat{R}^A : 0 \otimes A \to 0,$$

where $A \in \mathrm{Ob}(\mathcal{A})$, such that $\oplus$-functors $(L^A, \breve{L}, \hat{L}^A)$ and $(R^A, \breve{R}, \hat{R}^A)$ are compatible with unit constraints $(0, \mathbf{g}, \mathbf{d})$.

It is easy to see that if $(\mathcal{A}, \oplus)$ and $(\mathcal{A}', \oplus)$ are Gr-categories, then every $\oplus$-functor $(F, \breve{F}) : \mathcal{A} \to \mathcal{A}'$, which is compatible with associativity constraints, is a monoidal functor. Thus, we state the following definition.

Definition 2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories. An Ann-functor $(F, \breve{F}, \widetilde{F}, F_*) : \mathcal{A} \to \mathcal{A}'$ consists of a functor $F : \mathcal{A} \to \mathcal{A}'$, natural isomorphisms*

$$\breve{F}_{X,Y} : F(X \oplus Y) \to F(X) \oplus F(Y),\ \widetilde{F}_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y),$$

and an isomorphism $F_* : F(1) \to 1'$ such that $(F, \breve{F})$ is a symmetric monoidal functor for the operation $\oplus$, $(F, \widetilde{F}, F_*)$ is a monoidal functor for the operation $\otimes$, and the following diagrams commute

$$\begin{array}{ccccc} F(X(Y \oplus Z)) & \xrightarrow{\widetilde{F}} & FX.F(Y \oplus Z) & \xrightarrow{id \otimes \widetilde{F}} & FX(FY \oplus FZ) \\ \Big\downarrow{\scriptstyle F(\mathfrak{L})} & & & & \Big\downarrow{\scriptstyle \mathfrak{L}'} \\ F(XY \oplus XZ) & \xrightarrow{\breve{F}} & F(XY) \oplus F(XZ) & \xrightarrow{\widetilde{F} \oplus \widetilde{F}} & FX.FY \oplus FX.FZ \end{array}$$

$$\begin{array}{ccccc}
F((X\oplus Y)Z) & \xrightarrow{\widetilde{F}} & F(X\oplus Y).FZ & \xrightarrow{\breve{F}\otimes id} & (FX\oplus FY)FZ \\
\downarrow{\scriptstyle F(\mathfrak{R})} & & & & \downarrow{\scriptstyle \mathfrak{R}'} \\
F(XZ\oplus YZ) & \xrightarrow{\breve{F}} & F(XZ)\oplus F(YZ) & \xrightarrow{\widetilde{F}\oplus\widetilde{F}} & FX.FZ\oplus FY.FZ.
\end{array}$$

These diagrams are called the compatibility of the functor F with the distributivity constraints.

An *Ann-morphism* (or a *homotopy*)

$$\theta : (F, \breve{F}, \widetilde{F}, F_*) \to (F', \breve{F}', \widetilde{F}', F'_*)$$

between Ann-functors is an $\oplus$-morphism, as well as an $\otimes$-morphism.

If there exist an Ann-functor $(F', \breve{F}', \widetilde{F}', F'_*) : \mathcal{A}' \to \mathcal{A}$ and Ann-morphisms $F'F \stackrel{\sim}{\to} id_{\mathcal{A}}$, $FF' \stackrel{\sim}{\to} id_{\mathcal{A}'}$, we say that $(F, \breve{F}, \widetilde{F}, F_*)$ is an *Ann-equivalence*, and $\mathcal{A}$, $\mathcal{A}'$ are *Ann-equivalent*.

For an Ann-category $\mathcal{A}$, the set $R = \pi_0\mathcal{A}$ of isomorphism classes of the objects in $\mathcal{A}$ is a ring with two operations $+, \times$ induced by the functors $\oplus, \otimes$ on $\mathcal{A}$, and the set $M = \pi_1\mathcal{A} = \mathrm{Aut}(0)$ is a group with the composition denoted by $+$. Moreover, M is a R-bimodule with the actions

$$sa = \lambda_X(a), \quad as = \rho_X(a),$$

for $X \in s, s \in \pi_0\mathcal{A}, a \in \pi_1\mathcal{A}$, and λ_X, ρ_X satisfy

$$\begin{aligned}
\lambda_X(a) \circ \hat{L}^X &= \hat{L}^X \circ (id \otimes \mathbf{a}) : X.0 \to 0, \\
\rho_X(a) \circ \hat{R}^X &= \hat{R}^X \circ (\mathbf{a} \otimes id) : 0.X \to 0.
\end{aligned}$$

We recall briefly some main facts of the construction of the reduced Ann-category $S_{\mathcal{A}}$ of $\mathcal{A}$ via the structure transport (for details, see [11]). The objects of $S_{\mathcal{A}}$ are the elements of the ring $\pi_0\mathcal{A}$. A morphism is an automorphism $(s, a) : s \to s,\ s \in \pi_0\mathcal{A}, a \in \pi_1\mathcal{A}$. The composition of morphisms is given by

$$(s, a) \circ (s, b) = (s, a + b).$$

For each $s \in \pi_0\mathcal{A}$, choose an object $X_s \in \mathrm{Ob}(\mathcal{A})$ such that $X_0 = 0, X_1 = 1$, and choose an isomorphism $i_X : X \to X_s$ such that $i_{X_s} = id_{X_s}$. We obtain two functors

$$\begin{cases} G : \mathcal{A} \to S_{\mathcal{A}} \\ G(X) = [X] = s \\ G(X \stackrel{f}{\to} Y) = (s, \gamma_{X_s}^{-1}(i_Y f i_X^{-1})), \end{cases} \qquad \begin{cases} H : S_{\mathcal{A}} \to \mathcal{A} \\ H(s) = X_s \\ H(s, u) = \gamma_{X_s}(u), \end{cases}$$

for $X, Y \in s,\ f : X \to Y$, and γ_X

$$\gamma_X(\mathbf{a}) = \mathbf{g}_X \circ (\mathbf{a} \oplus id) \circ \mathbf{g}_X^{-1}. \tag{1}$$

Two operations on $S_{\mathcal{A}}$ are given by

$$\begin{aligned} s \oplus t &= G(H(s) \oplus H(t)) = s + t, \\ s \otimes t &= G(H(s) \otimes H(t)) = st, \\ (s,a) \oplus (t,b) &= G(H(s,a) \oplus H(t,b)) = (s+t, a+b), \\ (s,a) \otimes (t,b) &= G(H(s,a) \otimes H(t,b)) = (st, sb+at), \end{aligned}$$

for $s, t \in \pi_0\mathcal{A}$, $a, b \in \pi_1\mathcal{A}$. Clearly, they do not depend on the choice of the representative (X_s, i_X).

The constraints in $S_{\mathcal{A}}$ are defined by sticks. A *stick* of $\mathcal{A}$ is a representative (X_s, i_X) such that

$$\begin{aligned} &i_{0\oplus X_t} = \mathbf{g}_{X_t}, \qquad i_{X_s\oplus 0} = \mathbf{d}_{X_s}, \\ &i_{1\otimes X_t} = \mathbf{l}_{X_t}, \qquad i_{X_s\otimes 1} = \mathbf{r}_{X_s}, i_{0\otimes X_t} = \widehat{R}^{X_t}, \quad i_{X_s\otimes 0} = \widehat{L}^{X_s}. \end{aligned}$$

The unit constraints in $S_{\mathcal{A}}$ are $(0, id, id)$ and $(1, id, id)$. The family of the rest ones, $h = (\xi, \eta, \alpha, \lambda, \rho)$, is defined by the compatibility of the constraints $\mathbf{a}_+, \mathbf{c}, \mathbf{a}, \mathfrak{L}, \mathfrak{R}$ of $\mathcal{A}$ with the functor H and isomorphisms

$$\breve{H} = i^{-1}_{X_s\oplus X_t}, \widetilde{H} = i^{-1}_{X_s\otimes X_t}. \tag{2}$$

Then $(H, \breve{H}, \widetilde{H}) : S_{\mathcal{A}} \to \mathcal{A}$ is an Ann-equivalence. Besides, the functor $G : \mathcal{A} \to S_{\mathcal{A}}$ together with isomorphisms

$$\breve{G}_{X,Y} = G(i_X \oplus i_Y), \quad \widetilde{G}_{X,Y} = G(i_X \otimes i_Y) \tag{3}$$

is also an Ann-equivalence. We refer to $S_{\mathcal{A}}$ as an Ann-category of *type* (R, M), and $(H, \breve{H}, \widetilde{H})$, $(G, \breve{G}, \widetilde{G})$ are *canonical* Ann-equivalences. The family of constraints $h = (\xi, \eta, \alpha, \lambda, \rho)$ of $S_{\mathcal{A}}$ is called a *structure* of the Ann-category of type (R, M).

Mac Lane [7] and Shukla [14] cohomomology groups at low dimensions are used to classify Ann-categories and regular Ann-categories, respectively. A structure h of the Ann-category $S_{\mathcal{A}}$ is an element in the group of Mac Lane 3-cocycles $Z^3_{MacL}(R, M)$. In the case when $\mathcal{A}$ is regular, $h \in Z^3_{Shu}(R, M)$.

Proposition 1 ([11, Proposition 11])**.** *Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories.*

(*i*) *Every Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ induces an Ann-functor $S_F : S_{\mathcal{A}} \to S_{\mathcal{A}'}$ of type (p, q), where*

$$\begin{aligned} p &= F_0 : \pi_0\mathcal{A} \to \pi_0\mathcal{A}', \ [X] \mapsto [FX], \\ q &= F_1 : \pi_1\mathcal{A} \to \pi_1\mathcal{A}', \ u \mapsto \gamma^{-1}_{F0}(Fu), \end{aligned}$$

for γ is a map given by the relation (1).

(*ii*) *F is an equivalence if and only if F_0, F_1 are isomorphisms.*

(iii) The Ann-functor S_F satisfies

$$S_F = G' \circ F \circ H,$$

where H, G' are canonical Ann-equivalences.

Let $\mathcal{S} = (R, M, h), \mathcal{S}' = (R', M', h')$ be Ann-categories. Since $\breve{F}_{x,y} = (\bullet, \tau(x, y))$, $\widetilde{F}_{x,y} = (\bullet, \nu(x, y))$, then $g_F = (\tau, \nu)$ is a pair of maps *associated* with $(\breve{F}, \widetilde{F})$, we thus can regard an Ann-functor $F : \mathcal{S} \to \mathcal{S}'$ as a triple (p, q, g_F). It follows from the compatibility of F with the constraints that

$$q_*h - p^*h' = \partial(g_F),$$

where q_*, p^* are canonical homomorphisms,

$$Z^3_{MacL}(R, M) \xrightarrow{q_*} Z^3_{MacL}(R, M') \xleftarrow{p^*} Z^3_{MacL}(R', M').$$

Further, two Ann-functors $(F, g_F), (F', g_{F'})$ are homotopic if and only if $F = F'$, that is, they are the same type of (p, q), and there exists a function $t : R \to M'$ such that $g_{F'} = g_F + \partial t$.

We denote by

$$\mathrm{Hom}^{Ann}_{(p,q)}[\mathcal{S}, \mathcal{S}']$$

the set of homotopy classes of Ann-functors of type (p, q) from $\mathcal{S}$ to $\mathcal{S}'$.

Let $F : \mathcal{S} \to \mathcal{S}'$ be an Ann-functor of type (p, q), then the function

$$k = q_*h - p^*h' \in Z^3_{MacL}(R, M') \tag{4}$$

is called an *obstruction* of F.

Theorem 1 ([13, Theorem 4.4, 4.5])**.** *A functor $F : \mathcal{S} \to \mathcal{S}'$ of type (p, q) is an Ann-functor if and only if its obstruction $\overline{k}$ vanishes in $H^3_{MacL}(R, M')$. Then, there exists a bijection*

$$\mathrm{Hom}^{Ann}_{(p,q)}[\mathcal{S}, \mathcal{S}'] \leftrightarrow H^2_{MacL}(R, M')(= H^2_{Shu}(R, M')).$$

3. Crossed bimodules over rings and regular E-systems

The results on crossed bimodules can be found in [2, 3, 4, 9]. We shall show a characteristic of crossed bimodules when the base ring $\mathbb{K}$ is the ring of integers $\mathbb{Z}$. Based on this characteristic, we can establish the relation between crossed bimodules over rings and Ann-category theory in the next section.

Definition 3 (see [9])**.** *A* crossed bimodule *is a triple (B, D, d), where D is an associative $\mathbb{K}$-algebra, B is a D-bimodule and $d : B \to D$ is a homomorphism of D-bimodules such that*

$$d(b)b' = bd(b'), \ b, b' \in B. \tag{5}$$

A *morphism* $(k_1, k_0) : (B, D, d) \to (B', D', d')$ of crossed bimodules is a pair $k_1 : B \to B'$, $k_0 : D \to D'$, where k_1 is a group homomorphism, k_0 is a $\mathbb{K}$-algebra homomorphism such that

$$k_0 d = d' k_1 \tag{6}$$

and

$$k_1(xb) = k_0(x)k_1(b),\ k_1(bx) = k_1(b)k_0(x), \tag{7}$$

for all $x \in D, b \in B$.

The condition (7) shows that k_1 is a homomorphism of D-bimodules, where B' is a D-bimodule with the action $xb' = k_0(x)b',\ b'x = b'k_0(x)$.

Below, the base ring $\mathbb{K}$ is the ring of integers $\mathbb{Z}$, and a crossed bimodule (B, D, d) is called a *crossed bimodule over rings*. Thus, D is a ring with unit.

In order to introduce the concept of an E-system, we now recall some terminologies due to Mac Lane [7]. The set of all bimultiplications of a ring A is a ring denoted by M_A. For each element c of A, a bimultiplication μ_c is defined by

$$\mu_c a = ca, a\mu_c = ac, a \in A$$

we call μ_c an *inner bimultiplication.* Then $C_A = \{c \in A | \mu_c = 0\}$ is called the *bicenter* of A.

The bimultiplications σ and τ are *permutable* if for every $a \in A$,

$$\sigma(a\tau) = (\sigma a)\tau,\quad \tau(a\sigma) = (\tau a)\sigma. \tag{8}$$

We now introduce the main concept of the present paper which can be seen as a version of the concept of a crossed module over rings.

Definition 4. *An E-system is a quadruple (B, D, d, θ), where $d : B \to D,\ \theta : D \to M_B$ are the ring homomorphisms such that the following diagram commutes*

$$\begin{array}{ccc} B & \xrightarrow{\ d\ } & D \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\theta} & \\ & M_B & \end{array} \tag{9}$$

and the following relations hold for all $x \in D, b \in B$,

$$d(\theta_x b) = x.d(b),\quad d(b\theta_x) = d(b).x. \tag{10}$$

An E-system (B, D, d, θ) is *regular* if θ is a 1-homomorphism (a homomorphism carries the identity to the identity), and the elements of $\theta(D)$ are permutable.

A *morphism* $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ of E-systems consists of ring homomorphisms $f_1 : B \to B',\ f_0 : D \to D'$ such that

$$f_0 d = d' f_1 \tag{11}$$

and f_1 is an *operator homomorphism*, that is,

$$f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b),\ f_1(b\theta_x) = f_1(b)\theta'_{f_0(x)}. \tag{12}$$

In this paper, an E-system (B, D, d, θ) is sometimes denoted by $B \xrightarrow{d} D$, or $B \to D$.

Example 3. *If B is a two-sided ideal in D, then (B, D, d, θ) is a regular E-system, where d is an inclusion, $\theta : D \to M_B$ is given by the bimultiplication type, that is,*

$$\theta_x b = xb,\ b\theta_x = bx,\ \ x \in D, b \in B.$$

Example 4. *Let D be a ring, B be a D-bimodule, $\mathbf{0} : B \to D$ is the zero homomorphism of D-bimodules. B can be considered as a ring with zero multiplication defined by $b.b' = \mathbf{0}(b)b' = b\mathbf{0}(b') = 0$, for all $b, b' \in B$. Then, $(B, D, \mathbf{0}, \theta)$ is a regular E-system, where θ is given by the action of D-bimodules.*

Example 5. *Let B be a ring, M_B be the ring of bimultiplications of B, and $\mu : B \to M_B$ be the homomorphism which carries an element b in B to an inner bimultiplication of B. Then (B, M_B, μ, id) is an E-system. In general, this E-system is not regular.*

Standard consequences of the axioms of an E-system are as below.

Proposition 2. *Let (B, D, d, θ) be an E-system.*

(i) $\mathrm{Ker}d \subset C_B$.

(ii) $\mathrm{Im}d$ *is an ideal in* D.

(iii) *The homomorphism θ induces a homomorphism $\varphi : D \to M_{\mathrm{Ker}d}$ given by*

$$\varphi_x = \theta_x|_{\mathrm{Ker}d}.$$

(iv) $\mathrm{Ker}d$ *is a* $\mathrm{Coker}d$*-bimodule with the actions*

$$sa = \varphi_x(a), \quad as = (a)\varphi_x, \quad a \in \mathrm{Ker}d,\ x \in s \in \mathrm{Coker}d.$$

To state the relation between regular E-systems and crossed bimodules over rings, one recalls the following definition.

Definition 5. *A functor $\Phi : \mathbf{C} \to \mathbf{C}'$ is an isomorphism of categories if it is bijective on objects and on morphism sets.*

Theorem 2. *The categories of regular E-systems and of crossed bimodules over rings are isomorphic.*

Proof. Let ${}_sB = (B, D, d, \theta)$ be a regular E-system. The abelian additive group B is a D-bimodule with the actions

$$xb = \theta_x b, \quad bx = b\theta_x, \tag{13}$$

for $x \in D, b \in B$. It is then easy to check that the axioms of a crossed bimodule hold. For example, the relation (5) follows from the relation (9),

$$d(b)b' = \theta_{d(b)}(b') \overset{(9)}{=} \mu_b(b') = bb' = b\mu_{b'} \overset{(9)}{=} b\theta_{d(b')} = bd(b'),$$

since $\mu_b, \mu_{b'}$ are inner bimultiplications of the ring B. Besides, the regularity of the E-system (B, D, d, θ) is necessary and sufficient for the two-sided module B to be a D-bimodule.

Conversely, if ${}_cB = (B, D, d)$ is a crossed bimodule then B has a ring structure with the multiplication

$$b * b' := d(b)b' = bd(b'),\ b, b' \in B. \tag{14}$$

Clearly, $d : B \to D$ is a ring homomorphism since for all $b, b' \in B$,

$$d(b * b') = d(d(b)b') = d(b)d(b').$$

The map $\theta : D \to M_B$ is defined by the D-bimodule actions (13). Then, θ is a homomorphism with image in M_B, the elements of $\theta(D)$ are permutable since B is a D-bimodule. The homomorphism θ satisfies the condition (10) since d is a homomorphism of bimodules. Thus, the correspondence ${}_sB \mapsto {}_cB$ is bijective on objects.

Now, if $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ is a morphism of E-systems, it is then clear that (f_1, f_0) satisfies the relation (6).

Further, for all $x \in D, b \in B$, one has

$$f_1(xb) \overset{(13)}{=} f_1(\theta_x b) \overset{(12)}{=} \theta'_{f_0(x)} f_1(b) \overset{(13)}{=} f_0(x)f_1(b) = xf_1(b).$$

Similarly, one obtains $f_1(bx) = f_1(b)x$. This means that the pair (f_1, f_0) is a morphism of crossed bimodules.

Conversely, let $(k_1, k_0) : (B, D, d) \to (B', D', d')$ be a morphism of crossed bimodules. We show that k_1 is a ring homomorphism. According to the determination of the multiplication on the ring B, we have

$$k_1(b * b') \overset{(14)}{=} k_1(d(b)b') \overset{(7)}{=} k_0(d(b))k_1(b') \overset{(6)}{=} d'(k_1(b))k_1(b') \overset{(14)}{=} k_1(b) * k_1(b'),$$

for all $b, b' \in B$. Besides, the pair (k_1, k_0) also satisfies (12). □

By the above proposition, the notion of an E-system can be seen as a weaker version of the notion of a crossed bimodule over rings.

We now discuss the relationship among the above concepts and the concept of a crossed module of D-structures in the category $\mathbf{C}$ of Ω-groups (see [10]). For convenience, such a crossed module is called a crossed $\mathbf{C}$-module. T. Porter proved that there is an equivalence between the category of crossed $\mathbf{C}$-modules and that of internal categories in $\mathbf{C}$. A crossed $\mathbf{C}$-module can be described as follows.

Proposition 3 ([10, Proposition 2]). *Given a D-structure on B, $d : B \to D$ is a crossed $\mathbf{C}$-module if and only if the following conditions are satisfied for all $b, b_1, b_2 \in B, x \in D, * \in \Omega'_2 \subset \Omega$*

(i) $d((-x) \cdot b) = -x + d(b) + x$;

(ii) $(-d(b_1)) \cdot b_2 = -b_1 + b_2 + b_1$;

(*iii*) $d(b_1) * b_2 = b_1 * b_2 = b_1 * d(b_2)$;

(*iv*) $\begin{cases} d(xb) = x * d(b) \\ d(bx) = d(b) * x. \end{cases}$

Here $*$ is a binary operation which is not the group operation $+$, the actions $x \cdot b, x * b$ are given by

$$x \cdot b = s(x) + b - s(x),$$
$$x * b = s(x) * b,$$

where s is the morphism in the split exact sequence

$$0 \to B \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} D \to 0.$$

To establish the link between these crossed **C**-modules and crossed modules over rings, we take **C** to be a category whose objects are rings. The morphisms of **C** are ring homomorphisms which are not necessarily 1-homomorphisms.

Proposition 4. *Every crossed* **C***-module is a crossed bimodule over rings.*

Proof. Let $d : B \to D$ be a crossed **C**-module. Then d is a ring homomorphism, and D acts on B by

$$xb = s(x)b, \ bx = bs(x), \ x \in D, b \in B. \tag{15}$$

The map $\theta : D \to M_B$ is given by

$$\theta_x(b) = xb, (b)\theta_x = bx.$$

Since s is a ring homomorphism, so is θ. The relation (9) follows from the condition (iii). Indeed, for $b, b' \in B$

$$(\theta d)(b)(b') = \theta_{db}(b') = (db)b' = bb' = \mu_b(b').$$

It follows from (iv) that $d(\theta_x(b)) = d(xb) = xd(b)b$. This means the relation (10) holds, and therefore (B, D, d, θ) is an E-system. □

One can see that a crossed **C**-module $d : B \to D$ satisfies most of the conditions of a crossed bimodule over rings. We first see that B is a D-bimodule with the action (15) By (iv), the ring homomorphism $d : B \to D$ is a D-bimodule. The relation (5) follows directly from the condition (iii). Note that the ring D is not necessarily unitary and if it has a unit, the ring B is not assumed to be a unitary D-bimodule. These investigations show that the concept of a crossed **C**-module can be seen as a weaken version of the concept of a crossed bimodule over rings.

Remark 1. *Since* **C** *can be any of categories of* Ω*-groups, use of crossed* **C***-modules has resulted in various contexts. However, in each particular case there is a certain restriction. For example, by Proposition 3 [10] Kerd is singular; while for crossed* *modules over groups, (or crossed modules over rings) Kerd is a subgroup in the center (or the bicenter) of B.*

Since rings with unit are not Ω-groups, one cannot seek a relation among the category of crossed **C***-modules, cohomology of algebras and cohomology of rings.*

4. Strict Ann-categories and E-systems

Crossed modules over groups are often studied in the form of strict 2-groups (see [1, 5, 6]). In this section, we prove that E-systems and strict Ann-categories are equivalent.

For every E-system (B, D, d, θ) we can construct a strict Ann-category $\mathcal{A} = \mathcal{A}_{B\to D}$, called the Ann-category *associated to* the E-system (B, D, d, θ), as follows. One sets

$$\mathrm{Ob}(\mathcal{A}) = D,$$

and for two objects x, y of $\mathcal{A}$,

$$\mathrm{Hom}(x, y) = \{b \in B/y = d(b) + x\}.$$

The composition of morphisms is given by

$$(x \xrightarrow{b} y \xrightarrow{c} z) = (x \xrightarrow{b+c} z).$$

Two operations $\oplus, \otimes$ on objects are given by the operations $+, \times$ on the ring D. For the morphisms, we set

$$(x \xrightarrow{b} y) \oplus (x' \xrightarrow{b'} y') = (x + x' \xrightarrow{b+b'} y + y'),$$
$$(x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (xx' \xrightarrow{bb'+b\theta_{x'}+\theta_x b'} yy').$$

Based on the definition of an E-system, it is easy to verify that $\mathcal{A}$ is an Ann-category with the strict constraints.

Conversely, for every strict Ann-category $(\mathcal{A}, \oplus, \otimes)$ one can define an E-system $C_{\mathcal{A}} = (B, D, d, \theta)$. Indeed, let

$$D = \mathrm{Ob}(\mathcal{A}), \quad B = \{0 \xrightarrow{b} x |\ x \in D\}.$$

Then, D is a ring with two operations

$$x + y = x \oplus y, \quad xy = x \otimes y,$$

and B is a ring with two operations

$$b + c = b \oplus c, \quad bc = b \otimes c.$$

The homomorphisms $d: B \to D$ and $\theta: D \to M_B$ are defined by

$$d(0 \xrightarrow{b} x) = x,$$
$$\theta_y(0 \xrightarrow{b} x) = (0 \xrightarrow{id_y \otimes b} yx),$$
$$(0 \xrightarrow{b} x)\theta_y = (0 \xrightarrow{b \otimes id_y} yx).$$

The quadruple (B, D, d, θ) defined as above is an E-system.

In the following lemmas, let $\mathcal{A}_{B \to D}$ and $\mathcal{A}_{B' \to D'}$ be Ann-categories associated to E-systems (B, D, d, θ) and (B', D', d', θ'), respectively.

Lemma 2. *Let $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ be a morphism of E-systems.*

(*i*) *There is a functor $F : \mathcal{A}_{B \to D} \to \mathcal{A}_{B' \to D'}$ defined by*

$$F(x) = f_0(x),\ F(b) = f_1(b),\ x \in \mathrm{Ob}(\mathcal{A}_{B \to D}), b \in \mathrm{Mor}(\mathcal{A}_{B \to D}).$$

(*ii*) *The functor F together with isomorphisms*

$$\breve{F}_{x,y} : F(x+y) \to Fx + Fy, \quad \widetilde{F}_{x,y} : F(xy) \to FxFy$$

are Ann-functor if $\breve{F}_{x,y}$ and $\widetilde{F}_{x,y}$ are constants in $\mathrm{Ker}d'$ *and for all $x, y \in D$ the following conditions hold:*

$$\theta'_{Fx}(\widetilde{F}) = (\widetilde{F})\theta'_{Fy} = \widetilde{F}, \tag{16}$$
$$\theta'_{Fx}(\breve{F}) = (\breve{F})\theta'_{Fy} = \breve{F} + \widetilde{F}. \tag{17}$$

Then, we say that F is an Ann-functor of *form* (f_1, f_0).

Proof. i) Every element $b \in B$ can be considered as a morphism $(0 \xrightarrow{b} db)$ in $\mathcal{A}_{B \to D}$. Then,

$$(F0 \xrightarrow{F(b)} F(db))$$

is a morphism in $\mathcal{A}_{B' \to D'}$. By the construction of the Ann-category associated to an E-system, F is a functor.

ii) We define the natural isomorphisms

$$\breve{F}_{x,y} : F(x+y) \to F(x) + F(y),\ \widetilde{F}_{x,y} : F(xy) \to F(x)F(y)$$

such that $F = (F, \breve{F}, \widetilde{F})$ becomes an Ann-functor. First we see that

$$F(x) + F(x') = F(x + x'),$$

so $d'(\breve{F}_{x,x'}) = 0$. Analogously, $d'(\widetilde{F}_{x,x'}) = 0$, thus

$$\breve{F}_{x,x'}, \widetilde{F}_{x,x'} \in \mathrm{Ker}d' \subset C_{B'}. \tag{18}$$

Now, for two morphisms $(x \xrightarrow{b} y)$ and $(x' \xrightarrow{b'} y')$ in $\mathcal{A}_{B\to D}$, we have:

$$\begin{aligned}
\bullet\ F(b\oplus b') &= F(x+x' \xrightarrow{b+b'} y+y')\\
&= \big(f_0(x+x') \xrightarrow{f_1(b+b')} f_0(y+y')\big),\\
F(b)\oplus F(b') &= \big(f_0(x) \xrightarrow{f_1(b)} f_0(y)\big)\oplus\big(f_0(x') \xrightarrow{f_1(b')} f_0(y')\big)\\
&= \big(f_0(x)+f_0(x') \xrightarrow{f_1(b)+f_1(b')} f_0(y)+f_0(y')\big).
\end{aligned}$$

Since f_1 is a ring homomorphism, one obtains

$$F(b\oplus b') = F(b)\oplus F(b'). \tag{19}$$

By (18) and (19), the commutative diagram

$$\begin{array}{ccc}
F(x+x') & \xrightarrow{\breve{F}_{x,x'}} & F(x)+F(x')\\
{\scriptstyle F(b\oplus b')}\downarrow & & \downarrow{\scriptstyle F(b)\oplus F(b')}\\
F(y+y') & \xrightarrow[\breve{F}_{y,y'}]{} & F(y)+F(y')
\end{array} \tag{20}$$

follows from $\breve{F}_{x,x'} = \breve{F}_{y,y'}$.

$$\begin{aligned}
\bullet\ F(b\otimes b') &= F(xx' \xrightarrow{bb'+b\theta_{x'}+\theta_x b'} yy') = \big(f_0(xx') \xrightarrow{f_1(bb'+b\theta_{x'}+\theta_x b')} f_0(yy')\big),\\
F(b)\otimes F(b') &= \big(f_0(x) \xrightarrow{f_1(b)} f_0(y)\big)\otimes\big(f_0(x') \xrightarrow{f_1(b')} f_0(y')\big)\\
&= \big(f_0(x)f_0(x') \xrightarrow{f_1(b)f_1(b')+f_1(b)\theta'_{f_0(x')}+\theta'_{f_0(x)}f_1(b')} f_0(y)f_0(y')\big).
\end{aligned}$$

By (12), $f_1(\theta_x b') = \theta'_{f_0(x)} f_1(b')$ and $f_1(b\theta_{x'}) = f_1(b)\theta'_{f_0(x')}$, hence

$$F(b\otimes b') = F(b)\otimes F(b'). \tag{21}$$

By (18) and (21), the commutative diagram

$$\begin{array}{ccc}
F(xx') & \xrightarrow{\widetilde{F}_{x,x'}} & F(x)F(x')\\
{\scriptstyle F(b\otimes b')}\downarrow & & \downarrow{\scriptstyle F(b)\otimes F(b')}\\
F(yy') & \xrightarrow[\widetilde{F}_{y,y'}]{} & F(y)F(y')
\end{array} \tag{22}$$

follows from $\widetilde{F}_{x,x'} = \widetilde{F}_{y,y'}$. The equalities (16) and (17) come from the compatibility of $(F,\widetilde{F})$ with the associativity constraint and the distributivity ones, respectively. $\square$

An Ann-functor F is *single* if $F(0) = 0', F(1) = 1'$ and $\breve{F}, \widetilde{F}$ are constants. Then we state the converse of Lemma 2.

Lemma 3. *Let* $(F, \breve{F}, \widetilde{F}) : \mathcal{A}_{B\to D} \to \mathcal{A}_{B'\to D'}$ *be a single Ann-functor. Then, there is a morphism of E-systems* $(f_1, f_0) : (B \to D) \to (B' \to D')$*, where*

$$f_1(b) = F(b), \quad f_0(x) = F(x),$$

for $b \in B, x \in D$.

Proof. Since $F(0) = 0', F(1) = 1'$ and $\breve{F}, \widetilde{F}$ are constants, it is easy to see that $\breve{F}, \widetilde{F}$ are in Kerd'. By the determination of a morphism in $\mathcal{A}_{B'\to D'}$,

$$F(x + y) = F(x) + F(y), \;\; F(xy) = F(x)F(y),$$

so f_0 is a ring homomorphism.

Since $\breve{F}$ is a constant in Kerd', the commutative diagram (20) implies

$$F(b \oplus b') = F(b) \oplus F(b').$$

This means that $f_1(b + b') = f_1(b) + f_1(b')$.

Since $\widetilde{F}$ is a constant in Kerd', the commutative diagram (22) implies

$$F(b \otimes b') = F(b) \otimes F(b').$$

By the definition of $\otimes$,

$$f_1(bb') + f_1(b\theta_{x'}) + f_1(\theta_x b') = f_1(b)f_1(b') + f_1(b)\theta'_{f_0(x')} + \theta'_{f_0(x)}f_1(b'). \tag{23}$$

In this relation, taking $b = 0$ and then $b' = 0$ yield

$$f_1(\theta_x b') = \theta'_{f_0(x)}f_1(b'), \; f_1(b\theta_{x'}) = f_1(b)\theta'_{f_0(x')}.$$

Thus, (12) holds. Then, the equation (23) turns into $f_1(bb') = f_1(b)f_1(b')$, that is, f_1 is a ring homomorphism. The rule (11) also holds. Indeed, for all morphisms $(x \xrightarrow{b} y)$ in $\mathcal{A}_{B\to D}$, $y = d(b) + x$. It follows that

$$f_0(y) = f_0(d(b) + x) = f_0(d(b)) + f_0(x).$$

Besides, $(f_0(x) \xrightarrow{f_1(b)} f_0(y))$ is a morphism in $\mathcal{A}_{B'\to D'}$, so

$$f_0(y) = d'(f_1(b)) + f_0(x).$$

Thus, $f_0(d(b)) = d'(f_1(b))$ for all $b \in B$. □

Lemma 4. *Two Ann-functors* $(F, \breve{F}, \widetilde{F}), (G, \breve{G}, \widetilde{G}) : \mathcal{A}_{B\to D} \to \mathcal{A}_{B'\to D'}$ *of the same form are homotopic.*

Proof. Suppose that F and G are two Ann-functors of form (f_1, f_0). By Lemma 2, $\breve{F}, \breve{G}$ are constants. We prove that $\alpha = \breve{G} - \breve{F}$ is a homotopy between F and G.

It is easy to check the naturality of α and the compatibility of α with the addition. Besides, α is compatible with the multiplication. In other words, the following diagram commutes

$$\begin{array}{ccc} F(xy) & \xrightarrow{\widetilde{F}} & F(x)F(y) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\otimes\alpha} \\ G(xy) & \xrightarrow[\widetilde{G}]{} & G(x)G(y) \end{array} . \tag{24}$$

Indeed, by Lemma 2,

$$\begin{aligned} \widetilde{G} - \widetilde{F} =& (\theta'_{Fx}(\breve{G}) - \breve{G}) - (\theta'_{Fx}(\breve{F}) - \breve{F}) \\ =& \theta'_{Fx}(\alpha) - \alpha. \end{aligned}$$

Since $\alpha \in \mathrm{Ker} d' \subset C_{B'}$, so

$$\begin{aligned} \alpha \otimes \alpha =& \alpha.\alpha + (\alpha)\theta'_{Gy} + \theta'_{Gx}(\alpha) \\ =& (\alpha)\theta'_{Gy} + \theta'_{Gx}(\alpha). \end{aligned}$$

For $y = 0$, or $x = 0$ we have

$$\alpha \otimes \alpha = (\alpha)\theta'_{Gy} = \theta'_{Gx}(\alpha).$$

Thus,

$$\widetilde{G} - \widetilde{F} = \alpha \otimes \alpha - \alpha,$$

that is, (24) holds. □

Two Ann-functors $(F, \breve{F}, \widetilde{F})$ and $(G, \breve{G}, \widetilde{G})$ are *strong homotopic* if they are homotopic and $F = G$. By Lemma (4), one obtains the following fact.

Corollary 1. *Two Ann-functors $F, G : \mathcal{A}_{B\to D} \to \mathcal{A}_{B'\to D'}$ are strong homotopic if and only if they are of the same form.*

We write **Annstr** for the category of strict Ann-categories and their single Ann-functors. We can define the *strong homotopy category* Ho**Annstr** to be the quotient category with the same objects, but morphisms are strong homotopy classes of single Ann-functors. We write $\mathrm{Hom}_{\mathbf{Annstr}}[\mathcal{A}, \mathcal{A}']$ for the homsets of the homotopy category, that is,

$$\mathrm{Hom}_{\mathbf{Annstr}}[\mathcal{A}, \mathcal{A}'] = \frac{\mathrm{Hom}_{\mathbf{Annstr}}(\mathcal{A}, \mathcal{A}')}{\text{strong homotopies}}.$$

Denote by **ESyst** the category of E-systems, we obtain the following result which is an extending of Theorem 1 [5]

Theorem 3 (Classification Theorem)**.** *There exists an equivalence*

$$\begin{array}{rcl} \Phi : \mathbf{ESyst} & \to & Ho\mathbf{Annstr} \\ (B \to D) & \mapsto & \mathcal{A}_{B\to D} \\ (f_1, f_0) & \mapsto & [F], \end{array}$$

where $F(x) = f_0(x), F(b) = f_1(b)$, *for* $x \in \mathrm{Ob}\mathcal{A}, b \in \mathrm{Mor}\mathcal{A}$.

Proof. By Corollary 1, the correspondence Φ on homsets,

$$\mathrm{Hom}_{\mathbf{ESyst}}(B \to D, B' \to D') \to \mathrm{Hom}_{\mathbf{Annstr}}[\mathcal{A}_{B\to D}, \mathcal{A}_{B'\to D'}],$$

is a map. Since a homotopy between Ann-functors is strong, Φ is an injection. By Lemma 9, every single Ann-functor $F : \mathcal{A}_{B\to D} \to \mathcal{A}_{B'\to D'}$ determines a morphism of E-systems (f_1, f_0), and clearly $\Phi(f_1, f_0) = [F]$, thus Φ is surjective on homsets.

Let $C_{\mathcal{A}}$ be an E-system associated to a strict Ann-category $\mathcal{A}$. By the construction of an Ann-category associated to an E-system, $\Phi(C_{\mathcal{A}}) = \mathcal{A}$ (rather than an isomorphism). Hence, Φ is an equivalence of categories. □

5. Ring extensions of the type of an E-system

In this section we consider the ring extensions of the type of an E-system, which are analogous to the group extensions of the type of a crossed module [6].

Definition 6. *Let* (B, D, d, θ) *be an E-system. A ring* extension *of* B *by* Q *of* type $B \to D$ *is a diagram of ring homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 0, \\ & & \| & & \downarrow{\scriptstyle \varepsilon} & & & & \\ & & B & \xrightarrow{d} & D & & & & \end{array}$$

where the top row is exact, the quadruple (B, E, j, θ') *is an E-system where* θ' *is given by the bimultiplication type, and the pair* (id, ε) *is a morphism of E-systems.*

Two extensions of B by Q of type $B \xrightarrow{d} D$ are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 0, & \qquad & E & \xrightarrow{\varepsilon} & D \\ & & \| & & \downarrow{\scriptstyle \eta} & & \| & & & & & & \\ 0 & \longrightarrow & B & \xrightarrow{j'} & E' & \xrightarrow{p'} & Q & \longrightarrow & 0, & \qquad & E' & \xrightarrow{\varepsilon'} & D \end{array} \tag{25}$$

and $\varepsilon'\eta = \varepsilon$. Obviously, η is an isomorphism.

In the diagram

$$\begin{array}{cccccccccc} \mathcal{E}: & 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 0, \\ & & & \| & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \psi} & & \\ & & & B & \xrightarrow{d} & D & \xrightarrow{q} & \mathrm{Coker}d & & \end{array} \tag{26}$$

where q is a canonical projection, since the top row is exact and $q \circ \varepsilon \circ j = q \circ d = 0$, there is a ring homomorphism $\psi : Q \to \mathrm{Coker}d$ such that the right-hand side square commutes. Moreover, ψ depends only on the equivalence class of the extension $\mathcal{E}$. Our purpose is to study the set

$$\mathrm{Ext}_{B\to D}(Q,B,\psi)$$

of equivalence classes of extensions of B by Q of type $B\to D$ inducing ψ. The results use the obstruction theory of Ann-functors

Let $\mathcal{A}=\mathcal{A}_{B\to D}$ be the Ann-category associated to an E-system $B\to D$. Clearly, $\pi_0\mathcal{A}=\mathrm{Coker}d$, $\pi_1\mathcal{A}=\mathrm{Ker}d$ and therefore the reduced Ann-category $S_{\mathcal{A}}$ is of form

$$S_{\mathcal{A}}=(\mathrm{Coker}d,\mathrm{Ker}d,k),$$

where $\overline{k}\in H^3_{Shu}(\mathrm{Coker}d,\mathrm{Ker}d)$ since $\mathcal{A}$ and $S_{\mathcal{A}}$ are regular Ann-categories. The homomorphism $\psi:Q\to\mathrm{Coker}d$ induces an *obstruction*,

$$\psi^*k\in Z^3_{Shu}(Q,\mathrm{Ker}d), \tag{27}$$

which plays a fundamental role to state Theorem 4. This is the main result of this section, an extending of [6, Theorem 5.2]. Besides, a particular case of a regular E-system when $Q=\mathrm{Coker}d$ and $\psi=id_{\mathrm{Coker}d}$ is a *∂-extension* [4], so our result contains [4, Theorem 4.4.2].

Theorem 4. *Let (B,D,d,θ) be a regular E-system, $\psi:Q\to\mathrm{Coker}d$ be a ring homomorphism. Then, the vanishing of $\overline{\psi^*k}$ in $H^3_{Shu}(Q,\mathrm{Ker}d)$ is necessary and sufficient for there to exist a ring extension of B by Q of type $B\to D$ inducing ψ. Further, if $\overline{\psi^*k}$ vanishes then there is a bijection*

$$\mathrm{Ext}_{B\to D}(Q,B,\psi)\leftrightarrow H^2_{Shu}(Q,\mathrm{Ker}d).$$

The first assertion is based on the following lemmas.

Lemma 5. *For every Ann-functor $(F,\breve{F},\widetilde{F}):\mathrm{Dis}Q\to\mathcal{A}$ there exists an extension $\mathcal{E}_F$ of B by Q of type $B\to D$ inducing $\psi:Q\to\mathrm{Coker}d$.*

Such extension $\mathcal{E}_F$ is called an *associated* extension to Ann-functor F.

Proof. By Proposition 1, $(F,\breve{F},\widetilde{F})$ induces an Ann-functor $K:\mathrm{Dis}Q\to S_{\mathcal{A}}$ of type $(\psi,0)$. Let $(H,\breve{H},\widetilde{H}):S_{\mathcal{A}}\to\mathcal{A}$ be a canonical Ann-functor defined by the stick (x_s,i_x). By (2), we have

$$H(s)=x_s,\ H(s,b)=b,\ \breve{H}_{s,r}=-i_{x_s+x_r},\ \widetilde{H}_{s,r}=-i_{x_s\cdot x_r}.$$

Also by Proposition 1, $(F,\breve{F},\widetilde{F})$ is homotopic to the composition

$$\mathrm{Dis}Q\xrightarrow{K}S_{\mathcal{A}}\xrightarrow{H}\mathcal{A}.$$

So one can choose $(F,\breve{F},\widetilde{F})$ being this composition. By the determination of $\breve{H}K$

and $\widetilde{HK}$,

$$\breve{F}_{u,v} = f(u,v) = f'(u,v) - i_{x_s+x_r}, \tag{28}$$
$$\widetilde{F}_{u,v} = g(u,v) = g'(u,v) - i_{x_s \cdot x_r} \in B, \tag{29}$$

where $u, v \in Q, s = \psi(u), r = \psi(v), f'(u,v) = \breve{K}_{u,v}, g'(u,v) = \widetilde{K}_{u,v}$. By the compatibility of $(F, \breve{F}, \widetilde{F})$ with the strict constraints of Dis$Q$ and $\mathcal{A}$, the functions f and g are the "normal" ones satisfying

$$f(u, v+t) + f(v,t) - f(u,v) - f(u+v,t) = 0, \tag{30}$$
$$f(u,v) = f(v,u), \tag{31}$$
$$\theta_{Fu} g(v,t) - g(uv,t) + g(u,vt) - g(u,v)\theta_{Ft} = 0, \tag{32}$$
$$g(u, v+t) - g(u,v) - g(u,t) + \theta_{Fu} f(v,t) - f(uv,ut) = 0, \tag{33}$$
$$g(u+v,t) - g(u,t) - g(v,t) + f(u,v)\theta_{Ft} - f(ut,vt) = 0. \tag{34}$$

The function $\varphi : Q \to M_B$ defined by

$$\varphi(u) = \theta_{Fu} = \theta_{x_s} \ \ (s = \psi(u))$$

satisfies the relations

$$\varphi(u) + \varphi(v) = \mu_{f(u,v)} + \varphi(u+v), \tag{35}$$
$$\varphi(u)\varphi(v) = \mu_{g(u,v)} + \varphi(uv). \tag{36}$$

We only prove the relation (35), the proof of (36) follows from (29) in the same way. Since $f'(u,v) = \breve{K}_{u,v} \in \text{Ker}d$, then by Proposition 2, $f'(u,v) \in C_B$. By (28), one has $\mu_{f(u,v)} = \mu(-i_{x_s+x_r})$. Thus,

$$\begin{aligned} \varphi(u) + \varphi(v) &= \theta_{x_s} + \theta_{x_r} = \theta_{x_s+x_r} \\ &= \theta[d(-i_{x_s+x_r}) + x_{s+r}] = \theta[d(-i_{x_s+x_r})] + \theta_{x_{s+r}} \\ &= \mu(-i_{x_s+x_r}) + \varphi(u+v) \overset{(28)}{=} \mu_{f(u,v)} + \varphi(u+v). \end{aligned}$$

Since the family of functions (φ, f, g) satisfies the relations (30) - (36), we have a crossed product $E_0 = [B, \varphi, f, g, Q]$, that means $E_0 = B \times Q$, and two operations are

$$\begin{aligned} (b,u) + (b',u') &= (b + b' + f(u,u'), u + u'), \\ (b,u).(b',u') &= (b.b' + b\varphi(u') + \varphi(u)b' + g(u,u'), uu'). \end{aligned}$$

The set E_0 satisfies the axioms of a ring, in which note that the associativity for the multiplication in E_0 holds if and only if the E-system $B \to D$ is regular. Indeed, one can calculate the triple products as follows:

$$\begin{aligned}[(b,u)(b',u')](b'',u'') &= ((bb')b'' + b\varphi(u')\varphi(u'') + [\varphi(u)b']\varphi(u'') \\ &\quad + g(u,u')\varphi(u'') + \varphi(uu')b'' + g(uu',u''), (uu')u''), \\ (b,u)[(b',u')(b'',u'')] &= (b(b'b'') + b\varphi(u'u'') + \varphi(u)[b'\varphi(u'')] \\ &\quad + \varphi(u)\varphi(u')b'' + \varphi(u)g(u,u') + g(u,u'u''), u(u'u'')),\end{aligned}$$

By (32), (36), associative law for the multiplication in B, Q, and commutative law for the addition in B, especially by the relation (8), $[\varphi(u)b']\varphi(u'') = \varphi(u)[b'\varphi(u'')]$, we get the associative law for product in E_0. Then, there is an exact sequence of ring homomorphisms

$$\mathcal{E}_F: \quad 0 \to B \stackrel{j_0}{\to} E_0 \stackrel{p_0}{\to} Q \to 0,$$

where $j_0(b) = (b, 0)$; $p_0(b, u) = u$, $b \in B, u \in Q$. Since $j_0(B)$ is a two-sided ideal in E_0, $B \xrightarrow{j_0} E_0$ is an E-system, where $\theta_0 : E_0 \to M_B$ is given by the bimultiplication type.

We define a ring homomorphism $\varepsilon : E_0 \to D$ by

$$\varepsilon(b,u) = db + x_{\psi(u)}, \ (b,u) \in E_0,$$

where $x_{\psi(u)}$ is a representative of u in D. We show that the pair (id_B, ε) satisfies the rules (11), (12). Clearly, $\varepsilon \circ j_0 = d$. Besides, for all $(b,u) \in E_0, c \in B$,

$$\begin{aligned}\theta_0(b,u)(c) &= j_0^{-1}[(b,u)(c,0)] = bc + \varphi(u)c, \\ \theta_{\varepsilon(b,u)}(c) &= \theta_{db + x_{\psi(u)}}c = bc + \varphi(u)c.\end{aligned}$$

Thus, $\theta_0(b,u)(c) = \theta_{\varepsilon(b,u)}(c)$. Analogously, $c\theta_0(b,u) = c\theta_{\varepsilon(b,u)}$. So (id_B, ε) is a morphism of E-systems, that is, one has an extension (26), where E is replaced by E_0.

For all $u \in Q$ we have $q\varepsilon(0,u) = q(x_{\psi(u)}) = \psi(u)$, then the extension $\mathcal{E}_F$ induces $\psi : Q \to \text{Coker } d$. □

The proof of Theorem 4

Proof. Let us recall that $\mathcal{A}$ is the Ann-category associated to the regular E-system $B \stackrel{d}{\to} D$. Then, its reduced Ann-category is $S_{\mathcal{A}} = (\text{Coker}d, \text{Ker}d, k)$, where $k \in Z^3_{Shu}(\text{Coker}d, \text{Ker}d)$. The pair

$$(\psi, 0) : (Q, 0, 0) \to (\text{Coker}d, \text{Ker}d, k)$$

has $-\psi^*k$ as an obstruction. By the assumption, $\overline{\psi^*k} = 0$, hence by Proposition 1 the pair $(\psi, 0)$ determines an Ann-functor $(\Psi, \breve{\Psi}, \widetilde{\Psi}) : \text{Dis}Q \to S_{\mathcal{A}}$. Then the composition of $(\Psi, \breve{\Psi}, \widetilde{\Psi})$ and $(H, \breve{H}, \widetilde{H}) : S_{\mathcal{A}} \to \mathcal{A}$ is an Ann-functor $(F, \breve{F}, \widetilde{F}) : \text{Dis}Q \to \mathcal{A}$, and by Lemma 5 we obtain an associated extension $\mathcal{E}_F$.

Conversely, suppose that there is an extension as in the diagram (26). Let $\mathcal{A}'$ be the Ann-category associated to the E-system $B \to E$. By Proposition 1, there is an Ann-functor $F : \mathcal{A}' \to \mathcal{A}$. Since the reduced Ann-category of $\mathcal{A}'$ is Dis$Q$, so by Proposition 1, F induces an Ann-functor of type $(\psi, 0)$ from Dis$Q$ to $(\mathrm{Coker}d, \mathrm{Ker}d, k)$. Now, by Proposition 1, the obstruction of the pair $(\psi, 0)$ must vanish in $H^3_{Shu}(Q, \mathrm{Ker}d)$, that is, $\overline{\psi^* k} = 0$. □

The final assertion of Theorem 4 follows from the next theorem.

Theorem 5 (Schreier theory for ring extensions of the type of an E-system)**.** *There is a bijection*

$$\Omega : \mathrm{Hom}^{Ann}_{(\psi,0)}[\mathrm{Dis}Q, \mathcal{A}] \to \mathrm{Ext}_{B \to D}(Q, B, \psi).$$

Proof. *Step 1: The Ann-functors $(F, \breve{F}, \widetilde{F})$, $(F', \breve{F}', \widetilde{F}')$ are homotopic if and only if their corresponding associated extensions $\mathcal{E}_F, \mathcal{E}_{F'}$ are equivalent.*

Let two Ann-functors $F, F' : \mathrm{Dis}Q \to \mathcal{A}$ be homotopic by a homotopy $\alpha : F \to F'$. Then, by the definition of an Ann-morphism, the following diagrams commute

$$\begin{array}{ccc} F(u+v) & \xrightarrow{\breve{F}_{u,v}} & F(u)+F(v) \\ {\scriptstyle \alpha_{u+v}}\downarrow & & \downarrow{\scriptstyle \alpha_u+\alpha_v} \\ F'(u+v) & \xrightarrow[\breve{F}'_{u,v}]{} & F'(u)+F'(v), \end{array} \qquad \begin{array}{ccc} F(uv) & \xrightarrow{\widetilde{F}_{u,v}} & F(u)F(v) \\ {\scriptstyle \alpha_{uv}}\downarrow & & \downarrow{\scriptstyle \alpha_u\otimes\alpha_v} \\ F'(uv) & \xrightarrow[\widetilde{F}'_{u,v}]{} & F'(u)F'(v). \end{array}$$

By the definition of the operation $\otimes$ on $\mathcal{A}$,

$$\alpha_u \otimes \alpha_v = \alpha_u \alpha_v + \alpha_u \theta_{Fv} + \theta_{Fu} \alpha_v.$$

Then, since $f(u,v) = \breve{F}_{u,v}, f'(u,v) = \breve{F}'_{u,v}, g(u,v) = \widetilde{F}_{u,v}, g'(u,v) = \widetilde{F}'_{u,v}$, we have

$$f'(u,v) - f(u,v) = \alpha_u - \alpha_{u+v} + \alpha_v, \tag{37}$$
$$g'(u,v) - g(u,v) = \alpha_u \alpha_v + \alpha_u \theta_{Fv} + \theta_{Fu} \alpha_v - \alpha_{uv}. \tag{38}$$

Now, we set

$$\begin{aligned} \alpha^* : E_F &\to E_{F'} \\ (b,u) &\mapsto (b - \alpha_u, u). \end{aligned}$$

Note that $\theta_{F'u} = \mu_{\alpha_u} + \theta_{Fu}$, and by the relations (37), (38), the correspondence α^* is an isomorphism. Besides, the diagram (25) commutes in which E and E' are replaced by E_F and $E_{F'}$, respectively.

Finally, $\varepsilon' \alpha^* = \varepsilon$. Indeed, since $\alpha : F \to F'$ is a homotopy, then $Fu = x_{\psi(u)} = F'u$. Thus $x_{\psi(u)} = d(\alpha_u) + x_{\psi(u)}$, or $d(\alpha_u) = 0$. Hence,

$$\begin{aligned} \varepsilon' \alpha^*(b,u) &= \varepsilon'(b - \alpha_u, u) = d(b - \alpha_u) + x_{\psi(u)} \\ &= d(b) - d(\alpha_u) + x_{\psi(u)} = d(b) + x_{\psi(u)} = \varepsilon(b,u). \end{aligned}$$

That means two extensions $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ are equivalent.

Conversely, if $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ are equivalent, there exists a ring isomorphism $(b, u) \mapsto (b - \alpha_u, u)$. Then, we have a homotopy $\alpha : F \to F'$ by retracing our steps.

Step 2: Ω is a surjection.

Let $\mathcal{E}$ be an extension E of B by Q of type (B, D, d, θ) inducing $\psi : Q \to \operatorname{Coker} d$ (see the commutative diagram (26)). We prove that $\mathcal{E}$ is equivalent to an extension $\mathcal{E}_F$ which is associated to an Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathrm{Dis}Q \to \mathcal{A}$.

Let $\mathcal{A}' = \mathcal{A}_{B \to E}$ be the Ann-category associated to the E-system (B, E, j, θ'). By Lemma 2, the pair (id_B, ε) in the diagram (26) determines a single Ann-functor $(K, \breve{K}, \widetilde{K}) : \mathcal{A}' \to \mathcal{A}$.

Since $\pi_0\mathcal{A}' = Q, \pi_1\mathcal{A}' = 0$, the reduced Ann-category $S_{\mathcal{A}'}$ is nothing else but the Ann-category $\mathrm{Dis}Q$. Choose a stick $(e_u, i_e), e \in E, u \in Q$, of $\mathcal{A}'$ (that is, $\{e_u\}$ is a representative of Q in E). By (2), the canonical Ann-functor $(H', \breve{H}', \widetilde{H}') : \mathrm{Dis}Q \to \mathcal{A}'$ is given by

$$H'(u) = e_u,\ \breve{H}'_{u,v} = -i_{e_u + e_v} = g'(u, v),\ \widetilde{H}'_{u,v} = -i_{e_u . e_v} = h'(u, v).$$

The composition $F = K \circ H'$ is an Ann-functor $\mathrm{Dis}Q \to \mathcal{A}$, where

$$F(u) = \varepsilon(e_u),\ \breve{F}_{u,v} = \breve{H}'_{u,v} = g'(u, v),\ \widetilde{F}_{u,v} = \widetilde{H}'_{u,v} = h'(u, v).$$

According to the proof of Theorem 4, we construct an extension $\mathcal{E}_F$ of the crossed product $E_0 = [B, \varphi, g', h', Q]$ which is associated to $(F, \breve{F}, \widetilde{F})$.

We now prove that $\mathcal{E}$ and $\mathcal{E}_F$ are equivalent, that is, there is a commutative diagram

$$\begin{array}{lccccccccc}
\mathcal{E}_F : & 0 \longrightarrow & B & \xrightarrow{j_0} & E_0 & \xrightarrow{p_0} & Q & \longrightarrow 0 & \qquad & E_0 \xrightarrow{\varepsilon_0} D \\
 & & \| & & \downarrow{\scriptstyle \eta} & & \| & & & \\
\mathcal{E} : & 0 \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow 0 & \qquad & E \xrightarrow{\varepsilon} D
\end{array}$$

and $\varepsilon\eta = \varepsilon_0$.

Indeed, since every element of E can be written uniquely as $b + e_u, b \in B$, we can define a map

$$\eta : E_0 \to E,\ (b, u) \mapsto b + e_u.$$

We next verify that η is a ring isomorphism. The representatives e_u have the following properties

$$\varphi(u)c = \theta'_{e_u} c,\ c\varphi(u) = c\theta'_{e_u},\ c \in B, \tag{39}$$

$$e_u + e_v = -i_{e_u + e_v} + e_{u+v} = g'(u, v) + e_{u+v}, \tag{40}$$

$$e_u . e_v = -i_{e_u . e_v} + e_{u.v} = h'(u, v) + e_{uv}. \tag{41}$$

(The relation (39) holds since the pair (id_B, ε) is a morphism of E-systems. The relations (40), (41) hold thanks to the definition of a morphism in $\mathcal{A}'$.) Now, we have

$$\begin{aligned}
\eta[(b,u)+(c,v)] &= \eta(b+c+g'(u,v),u+v) = b+c+g'(u,v)+e_{u+v}\\
&\overset{(40)}{=} b+c+e_u+e_v = (b+e_u)+(c+e_v) = \eta(b,u)+\eta(c,v).\\
\eta[(b,u)(c,v)] &= \eta(bc+b\varphi(v)+\varphi(u)c+h'(u,v),uv)\\
&= bc+b\varphi(v)+\varphi(u)c+h'(u,v)+e_{uv}\\
&\overset{(39),(41)}{=} bc+b\theta'_{e_v}+\theta'_{e_u}c+e_ue_v\\
&= bc+b.e_v+e_u.c+e_u.e_v\\
&= (b+e_u).(c+e_v) = \eta(b,u).\eta(c,v).
\end{aligned}$$

Finally, choose the representative e_u such that $\varepsilon(e_u)=x_{\psi(u)}$ (since it follows from (26) that

$$q(\varepsilon(e_u)) = \psi p(e_u) = \psi(u).$$

Thus,

$$\varepsilon\eta(b,u) = \varepsilon(b+e_u) = \varepsilon(b)+\varepsilon(e_u) = d(b)+x_{\psi(u)} = \varepsilon_0(b,u),$$

that is, $\mathcal{E}$ and $\mathcal{E}_F$ are equivalent. □

Now, the bijection mentioned in Theorem 4 is obtained as follows. Note that there is a natural bijection

$$\text{Hom}[\text{Dis}Q,\mathcal{A}] \leftrightarrow \text{Hom}[\text{Dis}Q,S_{\mathcal{A}}].$$

Then, since $\pi_0(\text{Dis}Q)=Q$ and $\pi_1(S_{\mathcal{A}})=\text{Ker}d$, Theorem 5 and Theorem 1 imply

$$\text{Ext}_{B\to D}(Q,B,\psi) \leftrightarrow H^2_{Shu}(Q,\text{Ker}d).$$

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