MATHEMATICAL COMMUNICATIONS
Math. Commun. **17**(2012), 599–611

Norm estimates for resolvents of non-selfadjoint operators having Hilbert-Schmidt inverse ones

Michael Gil'[1,*]

[1] *Department of Mathematics, Ben Gurion University of the Negev, P. O. Box 653, Beer-Sheva 84 105, Israel*

Received November 14, 2011; accepted May 16, 2012

Abstract. The paper is devoted to an invertible linear operator whose inverse is a Hilbert - Schmidt operator and imaginary Hermitian component is bounded. Numerous regular differential and integro-differential operators satisfy these conditions. A sharp norm estimate for the resolvent of the considered operator is established. It gives us estimates for the semigroup and so-called Hirsch operator functions. The operator logarithm and fractional powers are examples of Hirsch functions. In addition, we investigate spectrum perturbation and suggest the multiplicative representation for the resolvent of the considered operator.

AMS subject classifications: 47A10, 47A30, 47A56, 47A60, 47B10

Key words: linear operator, resolvent, multiplicative representation, spectrum perturbations, fractional powers, operator logarithm, semigroup

1. Introduction and statement of the main result

Norm estimates for the resolvent and functions of normal and selfadjoint linear operators are well known [13]. But if an operator is non-selfadjoint, such estimates are known in particular cases only. One of the first norm estimates for the function of a non-Hermitian matrix was established by I.M. Gel'fand and G.E. Shilov [11] in connection with their investigations of partial differential equations, but that estimate is not sharp; it is not attained for any matrix. The problem of obtaining a sharp estimate for the norm of a matrix-valued function has been repeatedly discussed in the literature, cf. [7]. In the late 1970s, the author has obtained sharp estimates for the resolvent and matrix-valued functions regular on the convex hull of the spectrum. They are attained in the case of normal matrices. Later, these estimates were extended to the Schatten-von Neumann operators, operators with the Schatten - von Neumann Hermitian components, and operators represented as a sum of a unitary operator and a Hilbert-Schmidt one. For the details see [12], and references therein.

Let H be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and the unit operator I. For a linear unbounded operator A in H $Dom(A)$ is the domain, A^* is the adjoint of A; $\sigma(A)$ denotes the spectrum of A and A^{-1} is the

*Corresponding author. *Email address:* `gilmi@bezeqint.net` (M. Gil')

inverse to A, $R_\lambda(A) = (A - I\lambda)^{-1}$ ($\lambda \notin \sigma(A)$) is the resolvent; $A_R := (A + A^*)/2$ and $A_I := (A - A^*)/2i$. Recall that K is a Hilbert-Schmidt operator if

$$N_2(K) := [Trace(KK^*)]^{1/2} < \infty.$$

Besides, $N_2(K)$ is called the Hilbert-Schmidt norm. The ideal of Hilbert-Schmidt operators is denoted by HS_2.

Everywhere below it is assumed that A is invertible operator, $Dom(A) = Dom(A^*)$,

$$A^{-1} \in HS_2 \text{ and } \|A_I\| < \infty. \tag{1}$$

The aim of this paper is to derive a sharp norm estimate for the resolvent of the considered operator. It gives us estimates for the semigroup and so called Hirsch operator functions. The operator logarithm and fractional powers are examples of Hirsch functions. In addition, we investigate spectrum perturbations and suggest the multiplicative representation for the resolvent of the considered operator. Numerous regular differential and integro-differential operators satisfy conditions (1), cf. [17] (see also Section 7 below).

Note that instead of the invertibility and condition $A^{-1} \in HS_2$, in our reasonings below, we can require the condition $(A - aI)^{-1} \in HS_2$ for a regular a.

Put

$$\tau(A) := 2\|A_I\|N_2(A^{-1}), \quad \rho(A, \lambda) := \inf_{s \in \sigma(A)} |\lambda - s|$$

and

$$\psi(A, \lambda) := \inf_{s \in \sigma(A)} \left|1 - \frac{\lambda}{s}\right| \quad (\lambda \in \mathbb{C}).$$

Now we are in a position to formulate our main result.

Theorem 1. *Under condition* (1)*, the inequality*

$$\|(A - \lambda I)^{-1}\| \leq \frac{1}{\rho(A, \lambda)} \Phi\left(\frac{\tau(A)}{\psi(A, \lambda)}\right) \quad (\lambda \notin \sigma(A)),$$

is valid, where

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{x^k}{\sqrt{k!}} \quad (x > 0). \tag{2}$$

The proof of this theorem is presented in the next section. If $A_I = 0$, then $\tau(A) = 0$ and we obtain the equality $\|(A - \lambda I)^{-1}\| = \rho^{-1}(A, \lambda)$.

Below we show that instead of $\Phi(x)$, in the previous theorem one can take

$$\hat{\Phi}(x) = e^{\frac{1}{2}(1+x^2)} \quad (x > 0). \tag{3}$$

About the recent results on the resolvent see the interesting papers [4, 5] and references given therein.

2. Proof of Theorem 1

Lemma 1. *Under the hypothesis of Theorem 1, operator A^{-1} has a complete system of the roots vectors.*

Proof. For any real c with $-ic \notin \sigma(A)$ we have

$$(A + icI)^{-1} = (I + i(A_R + icI)^{-1}A_I)^{-1}(A_R + icI)^{-1}.$$

But $(A + icI)^{-1} = A^{-1}(I + icA^{-1})^{-1} \in HS_2$. So $(A_R + icI)^{-1} \in HS_2$, and by the Keldysh theorem, cf. [13, Theorem V. 8.1] operator $(A + icI)^{-1}$ has a complete system of the roots vectors. Since $(A+icI)^{-1}$ and A^{-1} commute, A^{-1} has a complete system of the roots vectors. As claimed. □

From the previous lemma it follows that there is the orthogonal normal (Schur) basis $\{e_k\}$, in which A^{-1} is represented by a triangular matrix (see [13, Lemma I.4.1]). Denote

$$P_k = \sum_{j=1}^{k} (., e_j)e_j \, .$$

Then

$$A^{-1}P_k = P_kA^{-1}P_k \ (k = 1, 2, \dots). \tag{4}$$

Besides,

$$\Delta P_k A^{-1} \Delta P_k = \lambda_k^{-1} \Delta P_k \ \ (\Delta P_k = P_k - P_{k-1}, \ k = 1, 2, ...; P_0 = 0). \tag{5}$$

Here $\lambda_k = \lambda_k(A)$ are the eigenvalues of A taken with their (finite) multiplicities. Put

$$D = \sum_{k=1}^{\infty} \lambda_k \Delta P_k \ \ (\Delta P_k = P_k - P_{k-1}, \ k = 1, 2, ...; P_0 = 0) \text{ and } V = A - D.$$

Lemma 2. *Under the hypothesis of Theorem 1, one has $N_2(VD^{-1}) \le \tau(A)$.*

Proof. We have

$$AP_kf = P_kAP_kf \ (k = 1, 2, ...; \ f \in Dom(A)). \tag{6}$$

Indeed, $A^{-1}P_k$ is an invertible $k \times k$ matrix, and therefore, $A^{-1}P_kH$ is dense in P_kH. Since $\Delta P_jP_k = 0$ for $j > k$, we have $0 = \Delta P_jAA^{-1}P_k = \Delta P_jAP_kA^{-1}P_k$. Hence $\Delta P_jAf = 0$ for any $f \in P_kH$. This implies (6).

Due to (5) we can write

$$\Delta P_k = \Delta P_k \Delta P_k = \Delta P_k A A^{-1} \Delta P_k = \lambda_k^{-1} \Delta P_k A \Delta P_k.$$

Thus,

$$\Delta P_k A \Delta P_k = \Delta P_k D \Delta P_k = \lambda_k \Delta P_k.$$

But $\Delta P_k A P_{k-1} = \Delta P_k D P_{k-1} = 0$. So $\Delta P_k A P_k = \Delta P_k D P_k$ and thus $\Delta P_k V P_k = 0$. Hence,

$$VP_k = P_{k-1} V P_k. \tag{7}$$

Take into account that $\|VD^{-1}e_k\| = |\lambda_k|^{-1}\|Ve_k\|$, $Ve_k = P_{k-1}Ve_k$ and $P_{k-1}V^*e_k = 0$. Then

$$\|Ve_k\| = 2\|P_{k-1}V_I e_k\| \le 2\|V_I e_k\| \quad (V_I = (V - V^*)/2i).$$

Furthermore, $V_I e_k = (A_I - D_I)e_k = A_I e_k - Im\ \lambda_k e_k$ and

$$2i(V_I e_k, e_k) = (Ve_k, e_k) - (V^* e_k, e_k) = 0.$$

So

$$\|A_I e_k\|^2 = \|V_I e_k + Im\ \lambda_k e_k\|^2 = \|V_I e_k\|^2 + (Im\ \lambda_k)^2.$$

Hence,

$$\|Ve_k\|^2 \le 4\|V_I e_k\|^2 = 4(\|A_I e_k\|^2 - (Im\ \lambda_k)^2) \le 4\|A_I\|^2.$$

But by the Weyl inequalities, cf. [13],

$$\sum_{k=1}^{\infty} |\lambda_k|^{-2} \le N_2^2(A^{-1}).$$

Consequently,

$$N_2^2(VD^{-1}) = \sum_{k=1}^{\infty} \|VD^{-1}e_k\|^2 = \sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^2}\|Ve_k\|^2 \le 4\|A_I\|^2 N_2^2(A^{-1}) = \tau^2(A).$$

As claimed. □

Lemma 3. *Under the hypothesis of Theorem 1, the inequality*

$$\|(A - z)^{-1}\| \le \frac{1}{\rho(A, \lambda)} \Phi(N_2(VD^{-1})/\psi(A, \lambda)) \quad (\lambda \notin \sigma(A))$$

is valid.

Proof. With $Iz = z, z \notin \sigma(A)$, we have

$$(A - z)^{-1} = (D + V - z)^{-1} = (D - z)^{-1}(I + VD^{-1}(I - D^{-1}z)^{-1})^{-1}. \tag{8}$$

Put $J := VD^{-1}(I - D^{-1}z)^{-1}$. Due to the previous lemma VD^{-1} is a Hilbert-Schmidt operator and due to (7) $JP_k = P_{k-1}JP_k$. So J is the limit in the operator norm of the operators $J_k = JP_k = P_{k-1}JP_k$ as $k \to \infty$. Clearly J_k^2 has the property $J^2P_k = P_{k-2}J_k^2P_k$, $J_k^3P_k = P_{k-3}J_k^2P_k$, and thus we get $J_k^k = 0$. But the limit of nilpotent operators in the operator norm is a Volterra (compact quasinilpotent) operator, cf. [13, Theorem I.4.2]. So the operator $VD^{-1}(I - D^{-1}z)^{-1}$ $(z \notin \sigma(A))$ is quasinilpotent. By [12, Theorem 6.4.1] we have

$$\|(I + VD^{-1}(I - D^{-1}z)^{-1})^{-1}\| \le \Phi(N_2(VD^{-1}(I - D^{-1}z)^{-1})).$$

Since D is normal, we get

$$\|(I - D^{-1}z)^{-1}\| = \frac{1}{\rho(1, A^{-1}z)}.$$

But $\rho(1, A^{-1}z) = \psi(A, z)$. Thus

$$N_2(VD^{-1}(I - D^{-1}z)^{-1}) \le N_2(VD^{-1})\|(I - D^{-1}z)^{-1}\| = \frac{N_2(VD^{-1})}{\psi(A, z)}.$$

Now (8) proves the lemma. □

The assertion of Theorem 1 follows from Lemmas 2 and 3.

Due to Theorem 6.4.2 [12],

$$\|(I + VD^{-1}(I - D^{-1}z)^{-1})^{-1}\| \le \hat{\Phi}(N_2(VD^{-1}(I - D^{-1}z)^{-1})).$$

So Lemma 2 and (8) allow us to replace Φ by $\hat{\Phi}$.

3. Spectral variations

In this section we investigate the spectral variation of the considered operators under bounded and unbounded perturbations. Immediately from Theorem 1 we get

Corollary 1. *Under the hypothesis of Theorem 1, let C be a linear operator with the domain $Dom(A)$ and operator $C - A$ be bounded. If, in addition,*

$$\|C - A\|\Phi\left(\frac{\tau(A)}{\psi(A, \lambda)}\right) < \rho(A, \lambda) \ \ (\lambda \notin \sigma(A)), \tag{9}$$

then λ is a regular point for C, and

$$\|R_\lambda(C)\| \le \frac{\Phi\left(\tau(A)/\psi(A, \lambda)\right)}{\rho(A, \lambda) - \|C - A\|\Phi\left(\tau(A)/\psi(A, \lambda)\right)}.$$

Consider now unbounded perturbations.

Let A_1 and A_2 be boundedly invertible operators in a Banach space X. Then the quantity

$$rsv_{A_1}(A_2) := \sup_{s \in \sigma(A_2)} \inf_{\mu \in \sigma(A_1)} \left|\frac{1}{s} - \frac{1}{\mu}\right|$$

is said to be *the relative spectral variation of A_2 with respect to A_1*. It is assumed that

$$\nu := \|(A_1 - A_2)A_1^{-1}\| < 1. \tag{10}$$

Lemma 4. *Let A_1 be invertible, and condition* (10) *hold. Then A_2 is invertible,*

$$\|A_2^{-1}\| \le \frac{\|A_1^{-1}\|}{1 - \nu} \ \text{and} \ \|A_1^{-1} - A_2^{-1}\| \le \chi := \frac{\|A_1^{-1}\|\nu}{1 - \nu}.$$

Proof. We have $A_2 = A_1 + (A_2 - A_1) = (I + (A_2 - A_1)A_1^{-1})A_1$. Now from (10), the invertibility of A_2 follows. But $A_1^{-1} - A_2^{-1} = -A_2^{-1}(A_1 - A_2)A_1^{-1}$. Hence we get the required result. $\square$

Lemma 5. *Let A_1 be invertible, condition* (10) *hold and there be a continuous strongly increasing function $\phi : [0,\infty) \to [0,\infty)$, such that $\phi(\infty) = \infty$, $\phi(0) = 0$, and $\|(A_1^{-1} - \lambda I)^{-1}\| \le \phi(1/\rho(A_1^{-1},\lambda))$, where $\rho(A_1^{-1},\lambda) = \inf_{s\in\sigma(A_1^{-1})}|s-\lambda|$. Then $rsv_{A_1}(A_2) \le z_0(\chi)$, where $z_0(\chi)$ is the unique positive root of the equation $\chi\phi(1/x) = 1$.*

Proof. Due to [12, Lemma 8.4.2], we have

$$\sup_{s_1\in\sigma(A_2^{-1})}\ \inf_{\mu_1\in\sigma(A_1^{-1})} |s_1 - \mu_1| \le z_0(\chi).$$

But $\sigma(A_1) = \{\frac{1}{s}, s \in A_1^{-1}\}$. This proves the lemma. $\square$

Now let us return to the operator A in H, satisfying (1). By Theorem 6.4.2 from [12] and [12, Lemma 6.5.2],

$$\|(A^{-1} - \lambda I)^{-1}\| \le \frac{1}{\rho(A^{-1},\lambda)}\hat{\Phi}\left(\frac{g(A^{-1})}{\rho(A^{-1},\lambda)}\right),$$

where

$$g^2(A^{-1}) = N_2^2(V_{-1}) = [N_2^2(A^{-1}) - \sum_{k=1}^{\infty}|\lambda_k(A^{-1})|^2\ ]^{1/2}.$$

Here V_{-1} is the nilpotent part of A^{-1}. That is, V_{-1} a Volterra (a qusinilpotent Hilbert-Schmidt operator), such that

$$A^{-1} = V_{-1} + D^{-1},$$

where D is a normal operator with $\sigma(A^{-1}) = \sigma(D^{-1})$. In addition, V_{-1} and D^{-1} have the same invariant subspaces. For more details see [12, Theorem 6.3.4].

The following properties of $g(.)$ are valid, cf [12, Section 6.4]

$$g^2(A^{-1}) \le N_2^2(A^{-1}) - |Trace\ (A^{-1})^2|.$$

If A^{-1} is a normal Hilbert-Schmidt operator, then $g(A^{-1}) = 0$. Moreover,

$$g^2(A^{-1}) \le 2N_2^2((A^{-1})_I) \text{ where } (A^{-1})_I := (A^{-1} - (A^{-1})^*)/2i.$$

Now Lemmas 4 and 5 imply

Theorem 2. *Let A satisfy conditions* (1) *and $\tilde{A}$ be a linear operator in H, such that*

$$\nu(A,\tilde{A}) := \|(A - \tilde{A})A^{-1}\| < 1.$$

Then $\tilde{A}$ is invertible,

$$\|\tilde{A}^{-1}\| \le \frac{\|A^{-1}\|}{1 - \nu(A,\tilde{A})} \ \ \textit{and}\ \ \left\|\tilde{A}^{-1} - A^{-1}\right\| \le \zeta := \frac{\|A^{-1}\|\nu(A,\tilde{A})}{1 - \nu(A,\tilde{A})}.$$

Moreover, $rsv_A(\tilde{A}) \leq z_0(\zeta)$, where $z_0(\zeta)$ is the unique positive root of the equation

$$\frac{\zeta}{x}\hat{\Phi}(g(A^{-1})/x) = 1. \tag{11}$$

Furthermore, substitute the equality $x = yg(A^{-1})$ into (11) and apply [12, Lemma 8.3.2]. Then we can assert that $z(\zeta) \leq \delta(\zeta)$, where

$$\delta(\zeta) := \begin{cases} e\zeta & \text{if } g(A^{-1}) \leq e\zeta, \\ g(A^{-1})[ln\ (g(A^{-1})/\zeta)]^{-1/2} & \text{if } g(A^{-1}) > e\zeta \end{cases}$$

Thus

$$rsv_A(\tilde{A}) \leq \delta(\zeta). \tag{12}$$

4. Fractional powers and logarithm

In this section it is assumed that

$$\beta(A) := \inf\ Re\ \sigma(A) > 0. \tag{13}$$

Obviously,

$$\rho(A, -t) = \inf_{s\in\sigma(A)} |s + t| \geq t + \beta(A) > 0 \quad (t \geq 0).$$

In addition, with $s = x + iy$, we have

$$\left|1 + \frac{t}{s}\right|^2 = \left|1 + \frac{t\bar{s}}{|s|^2}\right|^2 = \left|1 + \frac{t(x - iy)}{|s|^2}\right|^2 = (1 + tx|s|^{-2})^2 + (yt)^2|s|^{-4}$$
$$\geq 1\ \ (t \geq 0; x \geq \beta(A)).$$

Thus

$$\psi(A, -t) = \inf_{s\in\sigma(A)} |1 + t/s| \geq 1.$$

Consequently, by Theorem 1,

$$\|(A + tI)^{-1}\| \leq \frac{\Phi(\tau(A))}{t + \beta(A)}\ \ (t \geq 0). \tag{14}$$

Define the function

$$h(A) = \int_0^\infty f(t)(A + tI)^{-1}dt, \tag{15}$$

where the scalar function f satisfies the condition

$$\int_0^\infty |f(t)|(1 + t)^{-1}dt < \infty. \tag{16}$$

The formula (15) is a particular case of the Hirsch calculus, cf. [18]. Thus we have proved the following

Theorem 3. *Let conditions* (1) *and* (13) *hold. Let* $h(A)$ *be defined by* (15). *Then*

$$\|h(A)\| \leq \Phi(\tau(A)) \int_0^\infty \frac{|f(t)|}{t+\beta(A)} dt.$$

In Section 7 we present an example of the operator illustrating Theorem 3.

Recall that the fractional power of A can be defined by the formula

$$A^{-\nu} = \frac{sin\ (\pi\nu)}{\pi} \int_0^\infty t^{-\nu}(A+It)^{-1}dt \quad (0<\nu<1), \tag{17}$$

provided (13) holds, cf. [16, Section I.5.2, formula (5.8)]. Now the previous theorem implies the inequality

$$\|A^{-\nu}\| \leq \frac{sin\ (\pi\nu)\ \Phi(\tau(A))}{\pi} \int_0^\infty \frac{dt}{t^\nu(t+\beta(A))} \quad (0<\nu<1). \tag{18}$$

Recently, many interesting papers are devoted to fractional powers of linear operators, cf. [2, 3, 8, 9, 10, 19, 20] and references therein. In particular, in [22], the authors use the means of positive operators to establish Furuta-type operator monotonicity results for negative powers. Let A, B be bounded linear operators on a Hilbert space satisfying $0 \leq B \leq A$. Furuta showed the operator inequality $(A^rB^pA^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$ as long as positive real numbers p, q, r satisfy $p+2r \leq (1+2r)q$ and $1 \leq q$. In the paper [21], the author shows that this inequality is valid if negative real numbers p, q, r satisfy a certain condition. Of course we could not survey the whole subject here and refer the reader to the excellent book [18] as well as to the above listed papers and references therein.

The operator logarithm arises in numerous applications, in particular, its importance can be ascribed to it being the inverse function of the matrix exponential. Moreover, if we consider a vector differential equation with a T-periodic matrix, then according to the Floquet theory, its Cauchy operator $U(t)$ is equal to $V(t)e^{\Gamma t}$ where $V(t)$ is a T-periodic matrix and $\Gamma = \frac{1}{T} ln\ U(T)$, cf. [7]. Put

$$ln(A) := (A-I) \int_0^\infty (tI+A)^{-1} \frac{dt}{1+t}.$$

see [18, Theorem 10.1.3]. Now the previous theorem implies the inequality

$$\|ln(A)x\| \leq \Phi(\tau(A)) \int_0^\infty \frac{dt}{(t+\beta(A))(1+t)} \|(A-I)x\| \ \ (x \in Dom(A)). \tag{19}$$

About the interesting recent results on the operator logarithm see [6, 15, 14] and references given therein.

5. The Lyapunov norm

In this section we establish an estimate for the semigroup e^{-At} generated by $-A$. Will say that an operator $-A$ is a L^2-stable, if

$$l(A) := 2 \int_0^\infty \|e^{-At}\|^2 dt < \infty. \tag{20}$$

Thanks to the Parseval equality

$$l(A)=\frac{1}{\pi}\int_{-\infty}^{\infty}\|(A+iyI)^{-1}\|^2dy.$$

Put

$$W=2\int_0^{\infty}e^{-A^*s}e^{-As}ds. \tag{21}$$

Recall the generalized Lyapunov theorem [7, Theorem I.5.1]: in order for the spectrum of a bounded operator A to lie in the interior of the left half-plane, it is necessary and sufficient that there exists a positive definite Hermitian operator W, such that

$$WA+A^*W=-2I. \tag{22}$$

Besides W is defined by (21). Obviously,

$$\|W\|\le l(A). \tag{23}$$

Define the scalar product

$$(x,y)_W=(Wx,y) \text{ and the norm } \|x\|_W=\sqrt{(x,x)_W}.$$

Recall that $(.,.)$ and $\|.\|$ are the scalar product and norm in H, respectively; so $\|h\|_W\le\sqrt{l(A)}\|h\|$ $(h\in H)$.

Furthermore, with $s=x+iy$, $t\in\mathbb{R}$ and $a=\|A_I\|$, under (13), we have

$$|s+it|^2=|x+i(y+t)|^2=x^2+(y+t)^2\ge\beta^2(A)>0\quad(|t|\le a)$$

and $|s+it|^2\ge\beta^2(A)+(|t|-a)^2>0\quad(|t|\ge a)$. Put

$$\xi(t)=\begin{cases}(\sqrt{\beta^2(A)+(|t|-\|A_I\|)^2} & \text{if } |t|\ge\|A_I\|,\\ \beta(A) & \text{if } |t|\le\|A_I\|\end{cases}.$$

Then $\rho(A,-it)\ge\xi(t), t\in\mathbb{R}$. In addition,

$$|1+\frac{it}{s}|^2=\frac{|s+it|^2}{|s|^2}=\frac{|x+i(y+t)|^2}{|s|^2}=\frac{x^2+(y+t)^2}{x^2+y^2}\ge\frac{x^2}{x^2+y^2}$$
$$\ge\vartheta^2:=\frac{\beta^2(A)}{\beta^2(A)+\|A_I\|^2}\quad(x\ge\beta(A);\ |y|\le\|A_I\|,t\in\mathbb{R}).$$

Thus $\psi(A,-it)\ge\vartheta, t\in\mathbb{R}$, and by Theorem 1,

$$\|(A+it)^{-1}\|\le\frac{\Phi(\tau(A)/\vartheta)}{\xi(t)}\quad(t\in\mathbb{R}).$$

Hence, $l(A)\le\tilde{l}(A)$, where

$$\tilde{l}(A):=\frac{\Phi^2(\tau(A)/\vartheta)}{\pi}\int_{-\infty}^{\infty}\frac{dt}{\xi^2(t)}.$$

As it is well-known, [7, Section I.5]

$$\|e^{-At}x\|_W \leq exp\left[-\frac{t}{\|W\|}\right]\|x\|_W \quad (x \in H;\ t \geq 0).$$

Thus we arrive at the following result.

Theorem 4. *Let A satisfy conditions* (1) *and* (13) *and W be defined by* (21). *Then*

$$\|e^{-At}x\|_W \leq exp\left[-\frac{t}{\tilde{l}(A)}\right]\|x\|_W \quad (x \in H;\ t \geq 0). \tag{24}$$

6. The multiplicative representations for the resolvent

Since $A^{-1} \in HS_2$, and has the Schur basis, we have

$$A^{-1} = D^{-1} + V_{-1} \quad (\sigma(A^{-1}) = \sigma(D^{-1})),$$

where V_{-1} is a quasinilpotent operator. In addition, from [12, Theorem 10.3.1] it follows that

$$(\lambda I - A^{-1})^{-1} = (\lambda I - D^{-1})^{-1} \overrightarrow{\prod_{1\leq k\leq\infty}} \left(I + \frac{V_{-1}\Delta P_k}{\lambda - \lambda_k(A^{-1})}\right) \quad (\lambda \notin \sigma(A^{-1})). \tag{25}$$

Here the arrow means that the indexes increase from left to right. The product in (25) is the limit in the operator norm of the products

$$\overrightarrow{\prod_{1\leq k\leq n}} \left(I + \frac{V_{-1}\Delta P_k}{\lambda - \lambda_k(A^{-1})}\right)$$

as $n \to \infty$. Now taking into account that $(A - zI)^{-1} = A^{-1}(I - zA^{-1})^{-1}$, we arrive at the following result.

Theorem 5. *Under the hypothesis of Theorem 1 we have*

$$(A - Iz)^{-1} = A^{-1}(I - zD^{-1})^{-1} \overrightarrow{\prod_{1\leq j\leq\infty}} \left(I + \frac{\lambda_j(A)V_{-1}\Delta P_j}{\lambda_j(A) - z}\right) \quad (z \notin \sigma(A)) \tag{26}$$

Since, $\Delta P_k V_{-1}\Delta P_j = 0, j \leq k$, we can rewrite (26) as

$$(A - I\lambda)^{-1} = A^{-1}\sum_{j=1}^{\infty} \frac{\lambda_j(A)\Delta P_j}{\lambda_j(A) - z} \overrightarrow{\prod_{j+1\leq k\leq\infty}} \left(I + \frac{\lambda_k(A)V_{-1}\Delta P_k}{\lambda_k(A) - z}\right) \quad (z \notin \sigma(A)) \tag{27}$$

7. An example and concluding remarks

Let $L^2(0,1)$ be the complex Hilbert space of scalar functions with the scalar product

$$(f,h)=\int_0^1 f(x)\overline{h}(x)dx \ \ (f,h\in L^2(0,1)).$$

On the set

$$Dom(A)=\{u\in L^2(0,1): u^{(k)}\in L^2(0,1) \ (k=1,2);\ u(0)=u(1)=0\}$$

define the operator A by

$$(Aw)(x)=-\frac{d^2w(x)}{dx^2}+\int_0^1 K(x,s)w(s)ds \ \ (w\in \ Dom(A)),$$

where $K(x,s)$ is a scalar real kernel, such that the operator $\hat{K}$ defined by

$$(\hat{K}w)(x)=\int_0^1 K(x,s)w(s)ds$$

is bounded in $L^2(0,1)$. In addition, $K(x,s)=-K(s,x)$. So $A=E+\hat{K}$, where E is defined on $Dom(A)$ by $(Ew)(x):=-\frac{d^2w(x)}{dx^2}$. Since E is self-adjoint, we have $A_I=\hat{K}_I$, where

$$(\hat{K}_I w)(x)=\frac{1}{2i}\int_0^1 (K(x,s)+K(s,x))w(s)ds.$$

The Green function $G(t,s)$ of E is

$$G(x,s)=\begin{cases} x(1-s) & \text{if } 0\le x\le s\le 1,\\ s(1-x) & \text{if } 0\le s\le x\le 1\end{cases},$$

cf. [1]. Besides, the inverse operator

$$(E^{-1}f)(x)=\int_0^1 G(x,s)f(s)ds$$

is a Hilbert-Schmidt one and

$$N_2^2(\hat{K}E^{-1})=\int_0^1\int_0^1|\int_0^1 K(x,x_1)G(x_1,s)dx_1|^2 ds\, dx\le N_2^2(E^{-1})\|\hat{K}\|^2.$$

Assume that $N_2(\hat{K}E^{-1})<1$. Since $A=(I+\hat{K}E^{-1})E$, we obtain

$$N_2(A^{-1})=N_2(E^{-1}(I+\hat{K}E^{-1})^{-1})\le\frac{N_2(E^{-1})}{1-N_2(BE^{-1})}.$$

Now one can apply Theorem 1.

To apply Theorem 3 to the considered operator we use the matrix representation of A. Let $e_k(x) = \frac{1}{\sqrt{2}} sin\ (\pi k x)$ be the normalized eigenfunctions of E. In addition, let $\hat{K}$ be represented in $\{e_k(x)\}_{k=1}^\infty$ by a (real) matrix $(b_{jk})_{j,k=1}^\infty$. So A is represented by the matrix (a_{jk}) with $a_{jj} = j^2 + b_{jj}$ and $a_{jk} = b_{jk}$ $(j \neq k)$. We will consider A as a perturbation of the operator T represented by the triangular matrix $T = (t_{jk})_{j,k=1}^\infty$, where $t_{jk} = a_{jk}$ $(j \leq k)$ and $t_{jk} = 0$ $(j > k)$. Then, with the notation $q_T = \|A - T\|$, we have

$$q_T^2 \leq N_2^2(A-T) = \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} |b_{jk}|^2.$$

Put $D_T = diag(k^2 + b_{kk})$, $V_T = T - D_T$ and

$$\tau_T := N_2(V_T D_T^{-1}) = \left[\sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{|b_{jk}|^2}{|k^2 + b_{kk}|^2} \right]^{1/2}.$$

Since the inverse to T is compact, the eigenvalues of T are the diagonal values of the triangular representation. So $\lambda_k(T) = k^2 + b_{kk}$. Besides, $\rho(T, \lambda) = \inf_k |k^2 + b_{kk} - \lambda|$ and $\psi(A, \lambda) = \inf_k |1 - \frac{k^2 + b_{kk}}{\lambda}|$. By Lemma 3,

$$\|(T - \lambda)^{-1}\| \leq \frac{1}{\rho(T, \lambda)} \Phi(\tau_T / \psi(T, \lambda)). \tag{28}$$

Assume that

$$\beta(T) := \inf_k k^2 + b_{kk} > 0.$$

Then

$$\rho(T, -\lambda) \geq \beta(T) + Re\ \lambda, \psi(T, -\lambda) \geq \inf_k |1 + \frac{k^2 + b_{kk}}{\lambda}| \geq 1 \ (Re\ \lambda > 0).$$

Obviously, any λ satisfying the inequality $q_T \|(T - \lambda)^{-1}\| < 1$ is regular for A. Hence, according to (28), if $q_T \Phi\left(\frac{\tau_T}{\psi(T,\lambda)}\right) < \rho(T, \lambda)$ then $\lambda \notin \sigma(A)$. Suppose $\beta(T) < q_T \Phi(\tau_T)$. If $Re\ \lambda + \beta(T) > q_T \Phi(\tau_T)$, then $-\lambda$ is regular for A. Hence, $\beta(A) \geq \beta(T) - q_T \Phi(\tau_T)$. Now we can apply Theorem 3.

It should be noted that $\psi(A, \lambda)$ essentially depends on $\sigma(A)$. Indeed, for example, if $\sigma(A)$ is real and positive, then

$$\psi(A, i\omega) = \inf_{s \in \sigma(A)} \left| 1 - \frac{i\omega}{s} \right| = \left(1 + \frac{\omega^2}{\beta^2(A)} \right)^{1/2},$$

for $\omega \in \mathbb{R}$. Moreover, $\psi(A, \omega) = 1 + \frac{|\omega|}{\beta(A)}$ for $\omega \in (-\infty, 0)$, and $\psi(A, \omega) = 0$ for $\omega \in \sigma(A)$.

Finally, note that to the best of our knowledge, estimates for the resolvent of a non-selfadjoint operator under condition (1) in the available literature are unknown. As it was above shown, in the selfadjoint case Theorem 1 gives us the equality. In addition, Theorem 3 allows us, in particular, to estimate the fractional powers of nonselfajoint operators, which play an essential role in the theory of differential equations. Observe also that (24) gives us the estimate for a semigroup via $\tilde{l}(A)$ which is calculated by Theorem 1. Besides, the semigroup does not need to use inequality (24).

In conclusion, I would like to thank the referees for their really helpful remarks.

References

[1] R. P. Agarwal, D. O'Regan, *Ordinary and partial differential equations*, Universitext, Springer, New York, 2009.

[2] A. Bátkai, *On the domain characterization of fractional powers of certain operator matrices generating analytic semigroups*, Stud. Sci. Math. Hung. **40**(2003), 327–340.

[3] A. Bendikov, P. Maheux, *Nash type inequalities for fractional powers of non-negative self-adjoint operators*, Trans. Am. Math. Soc. **359**(2007), 3085–3097.

[4] D. Butnariu, S. Reich, S. Sabach, *A strong convergence theorem for resolvents of monotone operators*, J. Convex Anal. **17**(2010), 991–1006.

[5] C. Chen, M. Li *On fractional resolvent operator functions*, Semigroup Forum **80**(2010), 121–142.

[6] S. Clark, Sums of operator logarithms, Q. J. Math. **60**(2009), 413–427.

[7] Yu L. Daleckii, M. G. Krein, *Stability of solutions of differential equations in Banach space*, Amer. Math. Soc., Providence, 1971.

[8] R. DeLaubenfels, F. Yao, S. Wang, *Fractional powers of operators of regularized type*, J. Math. Anal. Appl. **199**(1996), 910–933.

[9] T. Diagana, *Fractional powers of the algebraic sum of normal operators*, Proc. Am. Math. Soc. **134**(2006), 1777–1782.

[10] T. Diagana, *Fractional powers of hyponormal operators of Putnam type*, Int. J. Math. Math. Sci. (2005), 1925–1932.

[11] I. M. Gel'fand, G. E. Shilov, *Some questions of theory of differential equations*, Nauka, Moscow, 1958, in Russian.

[12] M. I. Gil', *Operator functions and localization of spectra*, Lecture notes in mathematics, Vol. 1830, Springer–Verlag, Berlin, 2003.

[13] I. C. Gohberg, M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Trans. Mathem. Monographs, Vol. 18, Amer. Math. Soc., Providence, 1969.

[14] M. Haase, *Spectral properties of operator logarithms*, Math. Z. **245**(2003), 761-779.

[15] M. Hasse, *The functional calculus for sectorial operators*, Birkhäuser, Boston, 2006.

[16] S. G. Krein, *Linear differential equations in a Banach space*, Transl. Mathem. Monogr, vol. 29, Amer. Math. Soc., Providence, 1971.

[17] J. Locker, *The eigenvalues and completeness for regular and simply irrregular two-point differential operator*, Memories of American Math. Sci. **195**(2008), Providence.

[18] C. Martínez, M. Sanz, *The theory of fractional powers of operators*, Elsevier, Amsterdam, 2001.

[19] C. Martínez, M. Sanz, A. Redondo, *Fractional powers of almost non-negative operators*, Fract. Calc. Appl. Anal. **8**(2005), 201–230.

[20] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Commun. Pure Appl. Math. **60**(2007), 67–112.

[21] Tanahashi, Kôtaro, *The Furuta inequality with negative powers*, Proc. Am. Math. Soc. **127**(1999), 1683–1692.

[22] C. Yang, H. Zuo, *A monotone operator function via Furuta-type inequality with negative powers*, Math. Inequal. Appl. **6**(2003), 303–308.