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Diophantine approximation for the cubic root of polynomials of $\mathbb{F}_2[X]$

Khalil Ayadi[1], Mohamed Hbaib[1] and Faiza Mahjoub[1,*]

[1] *Département de mathématiques, Faculté des sciences, Université de Sfax, BP 802, 3038 Sfax, Tunisie*

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Abstract. In this paper, with different approaches we study rational approximation for the algebraic formal power series in $\mathbb{F}_2((X^{-1}))$ solving the irreducible equation

$$\alpha^3 = R,$$

where R is a polynomial of $\mathbb{F}_2[X]$. Moreover, for some polynomials R, we give explicitly the continued fraction expansion of the root of this equation.
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1. Introduction

1.1. Continued fraction algorithm in the field $\mathbb{F}_q((X^{-1}))$

Let p be a given prime number and $\mathbb{F}_q$ the finite field of characteristic p having q elements ($q = p^s,\ s \geq 1$). We consider the ring of polynomials $\mathbb{F}_q[X]$ and the field of rational function $\mathbb{F}_q(X)$, in an indeterminate X with coefficients in $\mathbb{F}_q$. Let $\mathbb{F}_q((X^{-1}))$ be the field of formal power series

$$\mathbb{F}_q((X^{-1})) = \{\alpha = \sum_{i\geq n_0} \alpha_i X^{-i}, \alpha_i \in \mathbb{F}_q, n_0 \in \mathbb{Z}\}.$$

If $\alpha = \sum_{i\geq n_0} \alpha_i X^{-i}$ is an element of $\mathbb{F}_q((X^{-1}))$ with $\alpha_{n_0} \neq 0$, we introduce the absolute value of α by $|\alpha| = q^{-n_0}$ which is not archimedian and $-n_0 = \deg\alpha$. This field $\mathbb{F}_q((X^{-1}))$ can be identified with the completion of $\mathbb{F}_q(X)$ for this absolute value. As in the classical case of the real numbers we have a continued fraction theory with polynomials in X playing the role of integers. For each $\alpha \in \mathbb{F}_q((X^{-1}))$ we write $[\alpha]$ for the polynomial part of α and $\{\alpha\} = \alpha - [\alpha]$ for the fractional part of α. For any $\alpha \in \mathbb{F}_q((X^{-1}))$, let $a_0 = [\alpha]$, we have :

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

*Corresponding author. *Email addresses:* ayedikhalil@yahoo.fr (K. Ayadi), mohamed.hbaib@fss.rnu.tn (M. Hbaib), faiza.mahjoub@yahoo.fr (F. Mahjoub)

and we denote

$$\alpha = [a_0, \ldots, a_n, \ldots],$$

where $(a_k)_{k\geq 1}$ are polynomials of degree ≥ 1. The later expression is called continued fraction expansion of α and the sequence $(a_k)_{k\geq 0}$ is called the sequence of partial quotients of α. Furthermore, the rational $\frac{P_n}{Q_n} = [a_0, \ldots, a_n]$ converges toα. For $n \geq 0$ the sequence $(\frac{P_n}{Q_n})_{n\geq 0}$ is defined by:

$$P_{n+1} = a_{n+1}P_n + P_{n-1},$$
$$Q_{n+1} = a_{n+1}Q_n + Q_{n-1},$$

where $Q_0 = 1, Q_1 = a_1, P_0 = a_0$ and $P_1 = a_0a_1 + 1$.

We have the following important equality :

$$|\alpha - \frac{P_n}{Q_n}| = |a_{n+1}|^{-1}|Q_n|^{-2} \quad for \;\; n \geq 0.$$

We say that a formal power series has bounded partial quotients if the polynomial $(a_k)_{k\geq 0}$ are bounded in degrees.

Baum and Sweet have given in [1] the following lemmas:

Lemma 1. *Let* $P, Q \in \mathbb{F}_2[X]$ *and* $\alpha \in \mathbb{F}_2((X^{-1}))$*. If* $|Q\alpha - P| = 2^{-(d+\deg Q)}$, $\gcd(P, Q) = 1$*,* $d > 0$*, then for some* n*,* $Q = Q_n$*,* $P = P_n$ *and* $\deg a_{n+1} = d$*.*

Lemma 2. *If α is an algebraic formal power series with unbounded (or bounded) partial quotients, and if, in the continued fraction for α, we replace X by any polynomial $p(X)$ of positive degree in X, then we obtain another algebraic continued fraction with unbounded (or bounded) partial quotients.*

Khintchine [4] conjectured that if x is an algebraic number with degree ≥ 3 then x has a continued fraction expansion with unbounded sequence of partial quotients. More is known in the case of algebraic formal power series over a finite field.

In 1976, Baum and Sweet [1] showed that the unique solution in $\mathbb{F}_2((X^{-1}))$ of the cubic equation

$$X\alpha^3 + \alpha + X = 0 \tag{1}$$

has a continued fraction expansion with partial quotients of bounded degree. Their proof does not yield a description of those partial quotients. Later Mills and Robbins [8] succeeded in giving a complete description of the sequence of partial quotients of the solution of the equation (1). They also provided some examples in higher characteristic. Robbins [11] has also given a family of cubic formal power series with bounded partial quotients. Nevertheless it appears that there is still very little known about the nature of continued fractions of algebraic power series. In particular, even though there seem to be many examples with bounded partial quotients, for any particular example, it may be difficult or impossible to provide a proof. Baum and Sweet [1] also gave some simple examples with unbounded partial quotients.

Algebraic formal power series with unbounded partial quotients can also be quite complicated even when the partial quotient sequence is recognizable. Such formal power series were studied by Mills and Robbins [8] and Lasjaunias in [6], [5]. In [9], Mkaouar described an algorithm to compute the partial quotients of continued fraction expansion for certain algebraic formal power series. He gave the following result

Theorem 1. *Let $P(Y) = A_nY^n + A_{n-1}Y^{n-1} + \ldots + A_0$ with $A_i \in \mathbb{F}_q[X]$ and $|A_{n-1}| > |A_i|$, for all $i \neq n-1$. Then P has a unique root $\alpha \in \mathbb{F}_q((X^{-1}))$ satisfying $|\alpha| > 1$. Moreover, $[\alpha] = -[\frac{A_{n-1}}{A_n}]$ and the formal power series $\beta = \frac{1}{(\alpha-[\alpha])}$ has the same property as α.*

1.2. The approximation exponent

We study the approximation of the elements of $\mathbb{F}_q((X^{-1}))$ by the elements of $\mathbb{F}_q(X)$. In particular, we consider this approximation for the elements of $\mathbb{F}_q((X^{-1}))$ which are algebraic over $\mathbb{F}_q(X)$. In order to measure the quality of rational approximation, we introduce the following notation and definition. Let α be an irrational element of $\mathbb{F}_q((X^{-1}))$.
For all real numbers μ, we define

$$B(\alpha,\mu) = \liminf_{|Q|\to\infty} |Q|^{\mu}|Q\alpha - P|$$

where P and Q run over $\mathbb{F}_q[X]$ with $Q \neq 0$. Now the approximation exponent of α is defined by

$$\nu(\alpha) = \sup\{\mu \in \mathbb{R} : B(\alpha,\mu) < \infty\}.$$

It is clear that we have
$B(\alpha,\mu) = \infty$ if $\mu > \nu(\alpha)$, $B(\alpha,\mu) = 0$ if $\mu < \nu(\alpha)$ and $0 \leq B(\alpha,\nu(\alpha)) \leq \infty$.
We recall that if $\dfrac{P_n}{Q_n}$ is a convergent to α, we have

$$|Q_n\alpha - P_n| = |Q_n|^{-\frac{\deg Q_{n+1}}{\deg Q_n}}.$$

Since the best rational approximation to α are its convergents in the above notation we have

$$\nu(\alpha) = \limsup(\frac{\deg Q_{k+1}}{\deg Q_k}) = 1 + \limsup(\frac{deg a_{k+1}}{\sum\limits_{1\leq i\leq k} \deg a_i}).$$

It is clear that the approximation exponent can be determined when the continued fraction of the element is explicitly known. Since $|Q_n\alpha - P_n| \leq \frac{1}{|Q_n|}$, for all irrational $\alpha \in \mathbb{F}_q((X^{-1}))$ we have $\nu(\alpha) \geq 1$. Furthermore Mahler's version of Liouville's theorem says that if $\alpha \in \mathbb{F}_q((X^{-1}))$ is algebraic over $\mathbb{F}_q(X)$ of degree $n > 1$ then $B(\alpha, n-1) > 0$. Consequently, for $\alpha \in \mathbb{F}_q((X^{-1}))$ algebraic over $\mathbb{F}_q(X)$ of degree $n > 1$ we have $\nu(\alpha) \in [1, n-1]$.

We now use the following vocabulary: If $\alpha \in \mathbb{F}_q((X^{-1}))$, we say that
• α is badly approximable if $\nu(\alpha) = 1$ and $B(\alpha, 1) > 0$. This is equivalent to saying

that the partial quotients in the continued fraction expansion for α are bounded.
• α is normally approximable if $\nu(\alpha) = 1$ and $B(\alpha, 1) = 0$.
• α is well approximable if $\nu(\alpha) > 1$.

Many examples can be studied. A famous example in $\mathbb{F}_p((X^{-1}))$, which was given by Mahler in 1949 [7], satisfies the algebraic equation $\alpha = X^{-1}+\alpha^p$. For this element α, algebraic of degree p, the approximation exponent is maximal ($\nu(\alpha) = p-1$). One is then led to wonder whether this example is exceptional. In 1975, another example was given by Osgood [10]. Nevertheless, there exist in $\mathbb{F}_q((X^{-1}))$ algebraic elements α of degree > 2 for which $\nu(\alpha) = 1$. The first example was obtained by Baum and Sweet [1] : in $\mathbb{F}_2((X^{-1}))$, α satisfying (1) is a cubic element for which $\nu(\alpha) = 1$ (and $B(\alpha, 1) > 0$). Other examples were given by Mills and Robbins[8], for other characteristics, and they noticed that most of these elements belong to a special class of formal power series, the class of hyperquadratic which we will call $\mathcal{H}$, of the irrational elements in $\mathbb{F}_q((X^{-1}))$ ($q = p^s,\ s \geq 1$), satisfying an algebraic equation of the form

$$\alpha = \frac{A\alpha^{p^t} + B}{C\alpha^{p^t} + D},$$

where $A, B, C, D \in \mathbb{F}_q[X]$ and $t \geq 0$. It gradually became apparent that the elements of class $\mathcal{H}$ deserve special consideration. Later, the rational approximation of these elements were studied by Voloch [13] and more deeply by de Mathan [3]. They could show that there are no normally approximable elements of class $\mathcal{H}$. By the work of de Mathan [3], we know moreover that for elements of class $\mathcal{H}$, the approximation exponent $\nu(\alpha)$ is a rational number and $B(\alpha, \nu(\alpha)) \neq 0, \infty$. Many elements of class $\mathcal{H}$ are well approximable, but the question of determining those which are badly approximable remains open.

Remark 1. *If α and β are in $\mathbb{F}_p((X^{-1}))$ and $\alpha = f(\beta^r)$, where f is a linear fractional transformation with polynomial coefficients and r is a power of the characteristic p (we may have $r = p^0 = 1$), then we have $\nu(\alpha) = \nu(\beta)$ and $B(\alpha, \nu(\alpha)) = C \cdot B(\beta, \nu(\beta))$ where $C > 0$ is real number. In particular, for all $A, B, C, D \in \mathbb{F}_p[X]$ with $AD - BC \neq 0$ and $t \geq 0$, we have*

$$\nu(\alpha) = \nu(\frac{A\alpha^{p^t} + B}{C\alpha^{p^t} + D}).$$

The proof of this Remark can be found in Schmidt's article [12].
We consider now the case of an equation

$$\alpha^n = R \tag{2}$$

in the field $\mathbb{F}_q((X^{-1}))$, where n divides $p^t - 1$, and $R \in \mathbb{F}_p(X)$ such that $R \notin (\mathbb{F}_p(X))^n$. It is clear that any root of (2) belongs to the class $\mathcal{H}$ because it is equivalent to the equation

$$R^{\frac{(p^t-1)}{n}}\alpha = \alpha^{p^t}.$$

For instance, there are no examples of an algebraic formal power series solution for this equation (where α is not quadratic) that is known to be badly approximable.

It was proved by Osgood [10]that if $\gcd(n,p) = 1$ then the n^{th} root of $1 + \frac{1}{X}$ in $\mathbb{F}_p((X^{-1}))$ has an approximation exponent equal to $n-1$.

At the end of his paper, de Mathan [3] asked whether for an element α satisfying the equation (2) one may have $\nu(\alpha) = 1$ (where α is not quadratic). He gave the approximation exponent of some examples of these elements for the case $n = 3$ and $p = 2$. He proved the following result

Proposition 1. *Let α, β, γ be elements of $\mathbb{F}_2((X^{-1}))$ such that*

$$\alpha^3 = \frac{(X^3 + X + 1)}{X^3}, \quad \beta^3 = \frac{(X^4 + X^2 + X + 1)}{X^4}, \quad \gamma^3 = \frac{(X^4 + X + 1)}{X^4}.$$

One has $\nu(\alpha) = \dfrac{3}{2}, B(\alpha, \dfrac{3}{2}) = 1, \nu(\beta) = \dfrac{4}{3}, B(\beta, \dfrac{4}{3}) = 1, \nu(\gamma) = \dfrac{5}{4}, B(\gamma, \dfrac{5}{4}) = \dfrac{1}{8}.$

Lasjaunias [6] gave, depending on R, the approximation exponent of elements satisfying the equation (2). In particular, for the case $n = 3$ and $p = 2$, he gave the following Theorem

Theorem 2. *Let $P, Q \in \mathbb{F}_2[X]$ be coprime and of same degree. Assume that $\frac{P}{Q} \notin (\mathbb{F}_2(X))^3$. Let $\lambda = \frac{\deg(P-Q)}{\deg Q}$. Then the equation $y^3 = \frac{P}{Q}$ has a unique root $\alpha \in \mathbb{F}_2((X^{-1}))$ with $|\alpha - 1| < 1$. If $\lambda < \frac{(2-\sqrt{2})}{3}$, then $\nu(\alpha) = 2 - 3\lambda$. Moreover, we have $\nu(\alpha) = 2$ if and only if there exist $P_0, Q_0 \in \mathbb{F}_2[X]$ and $C \in \mathbb{F}_2[X]$ such that $\frac{P}{Q} = (\frac{P_0}{Q_0})^3(1 + \frac{1}{C})$.*

We now come to present our work. We will show that the irrational root of the equation

$$\alpha^3 = R,$$

where $R \in \mathbb{F}_2[X]$ and $\deg R \in 3\mathbb{N}$ is well approximable for almost all R (note that the approximation exponent of this root belongs to $[1, 2]$). This result is based on a specific decomposition of the polynomial R. This decomposition will allow us to determinate the approximation exponent of the solution of this equation, furthermore, it will help us in some cases to find an explicit continued fraction of this solution. As an application, we finish by studying the continued fraction expansion and the approximation exponent of the cubic root of polynomials of degree ≤ 6.

2. Results

Lemma 3. *Let $R \in \mathbb{F}_2[X]$ be such that the degree of R is a multiple of 3. Then there is a unique $\alpha \in \mathbb{F}_2((X^{-1}))$ such that $\alpha^3 = R$.*

Proof. Uniqueness is obvious. Let $R = X^m + a_{m-1}X^{m-1} + \cdots + a_0$ where $m \in 3\mathbb{N}$ and $Q = X^m$. We have $|\dfrac{R}{Q}| = 1$. Let

$$\gamma_n = \prod_{i=0}^{n} (\frac{Q}{R})^{4^i}.$$

Then

$$\gamma_{n+1}+\gamma_n=\gamma_n(1+(\frac{Q}{R})^{4^{n+1}}).$$

So

$$|\gamma_{n+1}-\gamma_n|=|1+(\frac{Q}{R})^{4^{n+1}}|=|1+\frac{Q}{R}|^{4^{n+1}}.$$

Hence $\lim|\gamma_{n+1}-\gamma_n|=0$, then $\lim\gamma_n=\gamma\in\mathbb{F}_2((X^{-1}))$ and

$$\gamma=\prod_{i=0}^{\infty}(\frac{Q}{R})^{4^i}=\frac{Q}{R}\gamma^4.$$

Therefore $\gamma^3=\dfrac{R}{Q}$ and so $(\gamma X^{\frac{m}{3}})^3=R$. □

Lemma 4. *Let $R\in\mathbb{F}_2[X]$ be such that $\deg R\equiv 0(mod\ 3)$. Then the polynomial R is uniquely expressible as S^3+T, where $S,\ T\in\mathbb{F}_2[X]$ such that $\deg T<2\deg S$.*

Proof. We have

$$R=\sum_{0\le i\le 3m} r_iX^i\in\mathbb{F}_2[X]$$

with $r_{3m}=1$. We search

$$S=\sum_{0\le i\le m} s_iX^i\in\mathbb{F}_2[X]$$

with $s_m=1$ such that $\deg(R+S^3)<2m$. Hence the first upper $m+1$ coefficients of S^3 and of R must be equal. This is equivalent to solving the following system of m equations

$$s_ms_{m-i}+s_{m-1}s_{m-i+2}+\cdots=r_{3m-i}\quad for\quad 1\le i\le m.$$

This system is easily solved by induction starting from the first equation $s_ms_{m-1}=r_{3m-1}$. Setting $T=R+S^3$ we have the desired statement. □

We can write the equation $\alpha^3=R\neq 0$ as $\alpha=\frac{R}{\alpha^2}$. In relation with the equation defining elements of $\mathcal{H}$, we have here $|\Delta|=|(AD-BC)|=|R|$. Consequently, we have the following Lemma (see [5, Lemma 3]).

Lemma 5. *Let α be an irrational power series in $\mathbb{F}_2((X^{-1}))$ satisfying*

$$\alpha^3=R, \tag{3}$$

where $R\in\mathbb{F}_2[X]$ and $\deg R\in 3\mathbb{N}$. Let us suppose that there exist a partial quotient a of α such that $|a|>|R|$. Then the sequence of partial quotients of the continued fraction expansion for α is unbounded.

To compute the continued fraction expansion of a formal power series satisfying (3), with $R=S^3+T$ and $\deg T<2\deg S$, we can use the algorithm given in

Theorem (1). In fact, if we substitute $\alpha = S + \frac{1}{\beta}$ in (3) where $\beta \in \mathbb{F}_2((X^{-1}))$ with $|\beta| > 1$, we obtain that β is a solution of the algebraic equation

$$T\beta^3 + S^2\beta^2 + S\beta + 1 = 0 \tag{4}$$

with $\deg T < 2\deg S$. So if we determine the explicit continued fraction of β, we deduce that of α. We will present the steps of the algorithm, derived from Theorem (1), which allows us to determine the partial quotients of β in the following Lemma

Lemma 6. *By* $[a_0, \ldots, a_n, \ldots]$ *we denoted the continued fraction expansion of the irrational solution* β *of the equation* (4) *and* $\beta_m = [a_m, a_{m+1}, \ldots]$. *If we take* $A_0 = T$, $B_0 = S^2$, $C_0 = S$ *and* $D_0 = 1$ *then* $\deg B_0 > \deg A_0, C_0, D_0$.
As $\beta_m = a_m + \frac{1}{\beta_{m+1}}$, *then, by iteration on* m, β_m *satisfies an equation of the form*

$$A_m\beta_m^3 + B_m\beta_m^2 + C_m\beta_m + D_m = 0$$

with

$$\begin{aligned} A_{m+1} &= A_m a_m^3 + B_m a_m^2 + C_m a_m + D_m, \\ B_{m+1} &= A_m a_m^2 + C_m, \\ C_{m+1} &= A_m a_m + B_m, \\ D_{m+1} &= A_m, \end{aligned}$$

and we have $a_0 = [\dfrac{S^2}{T}]$ *and* $a_{m+1} = [\dfrac{B_{m+1}}{A_{m+1}}]$ *for all* $m \geq 0$.

Note that if A_m divides B_m, then $C_{m+1} = 0$, $B_{m+2} = A_{m+1}a_{m+1}^2$ and $A_{m+2} = A_m$ for all $m \geq 0$.

Remark 2. *We write below the few lines of program (using Maple) to obtain the first fifty partial quotients of the continued fraction expansion of the solution of the equation* (3).

```
p:=2:n:=50:e=T
a:=array(1..n):b:=array(1..n):c:array(1..n)d:array(1..n)
:a[1]=e:b[1]=S ^2:c[1]=S:d[1]=1:q[1]=quo(b[1],a[1],X)mod 2:
for i from 2 to n do
a[i]:=simplify(a[i-1]*q[i-1] ^3+b[i-1]*q[i-1] ^2+ c[i-1]*q[i-1]+d[i-1])
mod p:
b[i]:=simplify(a[i-1]*q[i-1] ^2+c[i-1]) mod p:
c[i]:=simplify(a[i-1]*q[i-1]+b[i-1]) mod p:
d[i]:=simplify(a[i-1]) mod p:
q[i]:=quo(b[i],a[i],X) mod p:od:print(q);
```

Using the previous algorithm, we give now theoretically the explicit continued fraction expansion of cubic root of some polynomials $R \in \mathbb{F}_2[X]$ and their corresponding values of approximation exponent.

Theorem 3. *If T divides S, i.e. there exists $K \in \mathbb{F}_2[X]$ such that $S = KT$, then the continued fraction expansion of the irrational solution of the equation*

$$\alpha^3 = S^3 + T$$

in the field $\mathbb{F}_2((X^{-1}))$ is $\alpha = [KT, K^2T, KT, \gamma]$, where $\gamma = [a_0, \dots, a_n, \dots] \in \mathbb{F}_2((X^{-1}))$ such that $a_n = K^{2^{n+1}} T^{u_{n+1}} (K^3T^2 + 1)^{u_n}$ with

$$u_n = \frac{2^{n+1} + (-1)^n}{3},$$

for all $n \geq 0$. Moreover, we have $\nu(\alpha) = 2$.

Proof. In this case, we have $\beta = \dfrac{1}{\alpha + S}$ is a solution of the following equation:

$$T\beta^3 + K^2T^2\beta^2 + KT\beta + 1 = 0.$$

We write $\alpha = KT + \frac{1}{\beta}$ and $\beta = [b_0, b_1, \cdots]$. Using the formulas from the Lemma 6 we get $b_0 = K^2T$ and $b_1 = KT$. From there on we denote $b_n = a_{n-2}$. We have $\alpha = [KT, b_0, b_1, \gamma]$ where $\gamma = [a_0, a_1, a_2, \cdots]$. Then by induction we obtain

$$a_n = K^{2^n} T^{u_{n+1}} (K^3T^2 + 1)^{u_n}, \text{ where } u_n = \frac{2^{n+1} + (-1)^n}{3}, \text{ for } n \geq 0.$$

Consequently, there are two rationals λ and μ such that $\deg(a_n) = 2^n\lambda + (-1)^n\mu$ for $n \geq 0$ where $\lambda = 4\deg(K) + \frac{8}{3}\deg(T)$ and $\mu = \deg(K) + \frac{1}{3}\deg(T)$. Therefore,

$$\frac{\deg(a_{n+1})}{\sum_{0 \leq i \leq n} \deg(a_i)}$$

tends to 1 as n tends to infinity. So $\nu(\gamma) = 2$ and by Remark (1) $\nu(\alpha) = \nu(\gamma)$. $\square$

Theorem 4. *If T divides S^2 and not S, i.e. there exists $K \in \mathbb{F}_2[X]$ such that $S^2 = KT$, then the continued fraction expansion for the irrational solution of the equation*

$$\alpha^3 = S^3 + T$$

in the field $\mathbb{F}_2((X^{-1}))$ is $\alpha = [S, K, S, \gamma]$, where $\gamma = [a_0, \dots, a_n, \dots] \in \mathbb{F}_2((X^{-1}))$ such that $a_n = K^{2^{n+1}} T^{v_n} (KS + 1)^{v_{n+1}}$ and

$$v_n = \frac{2^n + (-1)^{n+1}}{3},$$

for all $n \geq 0$. Moreover, we have $\nu(\alpha) = 2$.

Proof. With the same method used in the last proof we compute the partial quotients of α. Further, there exist two constants λ' and μ' such that $\deg(a_n) = 2^n\lambda' + (-1)^n\mu'$ for $n \geq 0$. The constants λ' and μ' can also be explicited: $\lambda' = 2\deg(K) + \frac{2}{3}\deg(T)$ and $\mu' = \frac{1}{2}\deg(K) - \frac{1}{6}\deg(T)$. Therefore,

$$\frac{\deg(a_{n+1})}{\sum_{0 \leq i \leq n} \deg(a_i)}$$

tends to 1 as n tends to infinity. So $\nu(\gamma) = 2$ and by Remark (1) $\nu(\alpha) = \nu(\gamma)$. $\square$

We provide now some remarks as an immediate consequence of Remark 1.

Remark 3.

(*i*) *If* $\gcd(S,T) = K^3$, *then there exist* $S' \in \mathbb{F}_2[X]$ *and* $T' \in \mathbb{F}_2[X]$ *such that* $S = K^3S'$, $T = K^3T'$ *and* $\gcd(S',T') = 1$. *So, we can study the equation* $\beta^3 = K^6S'^3 + T'$ *and deduce the behavior of the partial quotients of* α *from* β *such that* $\alpha = K\beta$.

(*ii*) *Suppose that there exists* $K \in \mathbb{F}_2[X]$ *such that* $S = KS'$, $T = K^3T'$ *where* S' *and* T' $\in \mathbb{F}_2[X]$ *then* $\alpha^3 = K^3S'^3 + K^3T' = K^3(S'^3 + T')$. *So we study the properties of the formal power series* $\gamma^3 = S'^3 + T'$ *and since* $\alpha = \frac{\gamma}{K}$, *we deduce the properties of continued fraction for* α.

(*iii*) *If* $\alpha^3 = K^{2^n}, n \geq 0$ *and* $K \in \mathbb{F}_2[X]$, *then we survey the equation* $\gamma^3 = K$. α *and* γ *have the same rational approximation properties since* $\alpha = \gamma^{2^n}$.

Now we give results concerning the approximation exponent of the cubic root of a polynomial. First, following the result given by Lasjaunias in Theorem (2), and using a method introduced by de Mathan in [3] to calculate the approximation exponent $\nu(\alpha)$ when $\nu(\alpha)$ is large enough, we add the following lemma

Lemma 7. *Let* $P, Q \in \mathbb{F}_2[X]$ *be a coprime and of same degree. Assume that* $\frac{P}{Q} \notin (\mathbb{F}_2(X))^3$. *Let*

$$\lambda = \frac{\deg(P - Q)}{\deg Q}.$$

Then the equation $y^3 = \frac{P}{Q}$ *has a unique root* $\alpha \in \mathbb{F}_2((X^{-1}))$ *with* $|\alpha - 1| < 1$. *Moreover, if* $\lambda < \frac{1}{3}$, *then* $\nu(\alpha) \geq 2 - 3\lambda$.

Proof. We follow the proof of Theorem (2) fairly closely. We prove the existence and the uniqueness of the irrational solution α with the same reasoning as the proof of Lemma 3.

Let us consider P_0, $Q_0 \in \mathbb{F}_p[X]$ such that $\gcd(P_0, Q_0) = 1$ and

$$\left|\frac{P_0}{Q_0} - 1\right| < 1.$$

We define a sequence of $\mathbb{F}_2(X)$ by:

$$\frac{P_{i+1}}{Q_{i+1}} = \left(\frac{P}{Q}\right)\left(\frac{Q_i}{P_i}\right)^2 \text{ for } i \geq 0 \text{ and } \gcd(P_i, Q_i) = 1. \qquad (S_1)$$

Then (S_1) is equivalent to

$$\frac{P_i}{Q_i} = \left[\left(\frac{P_0}{Q_0}\right)^{2^i}\left(\frac{Q}{P}\right)^{\frac{2^i-(-1)^i}{3}}\right]^{(-1)^i} \text{ for } i \geq 0 \text{ and } \gcd(P_i, Q_i) = 1. \qquad (S_2)$$

Since $|\alpha| = |\frac{P_i}{Q_i}| = 1$ for all $i \geq 0$ and

$$\alpha - \frac{P_{i+1}}{Q_{i+1}} = \frac{P}{Q}\frac{1}{\alpha^2} - \frac{P}{Q}(\frac{Q_i}{P_i})^2$$

then for all $i \geq 0$

$$\left|\alpha - \frac{P_{i+1}}{Q_{i+1}}\right| = \left|\frac{1}{\alpha} - \frac{Q_i}{P_i}\right|^2 = \left|\alpha - \frac{P_i}{Q_i}\right|^2.$$

Thus for all $i \geq 0$

$$\left|\alpha - \frac{P_i}{Q_i}\right| = \left|\alpha - \frac{P_0}{Q_0}\right|^{2^i}.$$

We consider the sequence starting from $\frac{P_0}{Q_0} = 1$ with $P_0 = Q_0 = 1$. Then

$$\left|\alpha - \frac{P_i}{Q_i}\right| = |\alpha - 1|^{2^i}.$$

Since $(\alpha - 1)(\alpha^2 + \alpha + 1) = \alpha^3 - 1$ and $|\alpha| = 1$, then $|\alpha^2 + \alpha + 1| = 1$ and so $|\alpha - 1| = |\alpha^3 - 1|$. Then for $i \geq 0$ we have

$$\left|\alpha - \frac{P_i}{Q_i}\right| = |\alpha^3 - 1|^{2^i} = \left|\frac{P}{Q} - 1\right|^{2^i},$$

so

$$\left|\alpha - \frac{P_i}{Q_i}\right| = |Q|^{-2^i(1 - \frac{\deg(P+Q)}{\deg Q})}. \tag{5}$$

Set

$$\mu = 1 - \frac{\deg(P+Q)}{\deg Q} = 1 - \lambda.$$

From (S_2) we have

$$|Q_i| = |Q|^{\frac{2^i - (-1)^i}{3}}.$$

Then

$$|Q_i|^{-3\mu} = |Q|^{-\mu 2^i + \mu(-1)^i}.$$

From the equality (5) we have

$$\left|\alpha - \frac{P_i}{Q_i}\right| = |Q|^{-\mu 2^i},$$

then

$$|\alpha - \frac{P_i}{Q_i}| = |Q_i|^{-3\mu}|Q|^{\mu(-1)^{i+1}}. \tag{6}$$

Equality (6) yields that $\nu(\alpha) \geq 3\mu - 1 = 2 - 3\lambda$. $\square$

Theorem 5. *Let S, $T \in \mathbb{F}_2[X]$ with $\deg T < 2\deg S$. Let α be the irrational solution of the equation $y^3 = S^3 + T$.*

(i) If T divides S^2, then $\nu(\alpha) = 2$.

(ii) If T does not divide S and $\deg T < \deg S$, then $\nu(\alpha) \geq 2 - \dfrac{\deg T}{\deg S}$.

(iii) If $\gcd(S,T) = 1$ and $0 \leq \deg T < (2-\sqrt{2})\deg S$, then $\nu(\alpha) = 2 - \dfrac{\deg T}{\deg S}$.

Proof. (i): Let $\beta = \dfrac{\alpha}{S}$. Then

$$\beta^3 = 1 + \frac{T}{S^3} = 1 + \frac{1}{H}.$$

So by Theorem 2 $\nu(\beta) = 2$. Hence $\nu(\dfrac{\alpha}{S}) = 2$ and by Remark 1 we deduce that $\nu(\alpha) = 2$.

(ii): Let $\beta = \dfrac{\alpha}{S}$. Then

$$\beta^3 = \frac{R}{R+T}.$$

Suppose that $\gcd(R,T) = K \in \mathbb{F}_2[X]$, then there exist U,$V \in \mathbb{F}_2[X]$ such that $\gcd(U,V) = 1$ such that $R = KU$ and $T = KV$. So

$$\frac{R}{R+T} = \frac{U}{U+V}.$$

We have

$$0 \leq \frac{\deg V}{\deg U} < \frac{\deg T}{\deg R} < \frac{1}{3}.$$

Then from Lemma 7

$$\nu(\beta) \geq 2 - 3\frac{\deg V}{\deg U} > 2 - \frac{\deg T}{\deg S}.$$

Suppose that $K = 1$. Then

$$\nu(\beta) \geq 2 - 3\frac{\deg T}{\deg R} \geq 2 - \frac{\deg T}{\deg S}.$$

We conclude that

$$\nu(\beta) \geq 2 - \frac{\deg T}{\deg S}.$$

So by Remark 1 we deduce that

$$\nu(\alpha) \geq 2 - \frac{\deg T}{\deg S}.$$

(iii): We prove this case by using Theorem 2 and with the same reasoning as ii). $\square$

We see that $\nu(\alpha) > 1$ in the two cases $i)$ and $ii)$, then we deduce from this Theorem that if T divides S or T does not divide S and $\deg T < \deg S$ then α has a continued fraction expansion with unbounded partial quotients.

As an application of previous results, we will study the continued fraction expansion and the approximation exponent of the cubic root of polynomials of degree ≤ 6. This study yields the following result.

Corollary 1. *The continued fraction expansion of the cubic root of a polynomial of degree ≤ 6 is unbounded.*

Proof. The following table presents the study of the irrational root of the equation $\alpha^3 = R \in \mathbb{F}_2[X]$ where R is a polynomial of degree 3. □

Equations	Method of determination of partial quotients or approximation exponent of α	$\nu(\alpha)$
$\alpha^3 = X^3 + 1$	$\alpha = [X, X^2, X, b_0, \cdots, b_n, \cdots]$, $b_n = X^{2^{n+1}}(X^3+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^3 + X^2 + X = (X+1)^3 + 1$	substituting X by $X+1$ Lemma 2	$\nu(\alpha) = 2$
$\alpha^3 = X^3 + X$	$\alpha = [X, X, X, b_0, \cdots, b_n, \cdots]$, $b_n = X^{\frac{2^{n+1}+(-1)^{n+1}}{3}}(X^2+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^3 + X^2 = (X+1)^3 + X + 1$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^3 + X + 1$	$\frac{\alpha^3}{X^3} = \frac{X^3+X+1}{X^3}$ Corollary 1, Remark 1	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^3 + X^2 + 1$	substituting X by $X+1$	$\nu(\alpha) = \frac{3}{2}$

Remark 4. *For some equations, we will compute the approximation exponent of their solutions in the same way as the Proposition 1.*

We also remark that

$$\alpha^3 = \frac{X^2+X+1}{X^2} = \frac{X^3+X^2+X}{X^3}.$$

Then $(X\alpha)^3 = X^3 + X^2 + X = (X+1)^3 + 1, \nu(X\alpha) = 2$ *and then* $\nu(\alpha) = 2$.

We will give now the table which presents the study of the irrational root of the equation $\alpha^3 = R \in \mathbb{F}_2[X]$ where R is a polynomial of degree 6. Before this, we introduce the following remarks.

Equations	**Method of determination of partial quotients or approximation exponent of α**	$\nu(\alpha)$
$\alpha^3 = X^6+1$ $= (X^2)^3+1$	$\alpha = [X^2, X^4, X^2, b_0, \cdots, b_n, \cdots]$, $b_n = X^{2^{n+2}}(X^6+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^4+X^2$ $= (X+1)^6+1$	substituting X by $X+1$ Lemma 2	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X$ $= (X^2)^3+X$	$\alpha = [X^2, X^4, X^2, b_0, \cdots, b_n, \cdots]$, $b_n = X^{2^{n+1}} X^{\frac{2^{n+2}+(-1)^{n+1}}{3}} (X^5+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^4+X^2+X$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X+1$ $= (X^2)^3+X+1$	Theorem 5	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6+X^4+X^2+X+1$	substituting X by $X+1$	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6+X^2$ $= (X^2)^3+X^2$	$\alpha = [X^2, X^4, X^2, b_0, \ldots, b_n, \ldots]$, $b_n = (X^2)^{\frac{2^{n+2}+(-1)^{n+1}}{3}} (X^4+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^4$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^2+1$	$\alpha^3 = (X^3+X+1)^2$ Remark 1, 3 and Proposition 1	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6+X^4+1$	substituting X by $X+1$	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6+X^2+X$	$\frac{\alpha^3}{X^6} = \frac{X^6+X^2+X}{X^6} = \frac{X^5+X+1}{X^5}$ Theorem 2,Remark 1	$\nu(\alpha) \geq \frac{7}{5}$
$\alpha^3 = X^6+X^4+X+1$	substituting X by $X+1$	$\nu(\alpha) \geq \frac{7}{5}$
$\alpha^3 = X^6+X^2+X+1$ $= (X^2)^3+X^2+X+1$	$dega_{38} = 12$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6+X^4+X$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6+X^3$ $= (X^2)^3+X^3$	$\alpha = [X^2, X, X^2, b_0, \ldots, b_n, \ldots]$, $b_n = X^{2^{n+1}}(X^3)^{\frac{2^{n+2}+(-1)^{n+1}}{3}} (X^3+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 4	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^4+X^3+X$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^3+1$	$\frac{\alpha^3}{X^6} = \frac{X^6+X^3+1}{X^6}$ Remark 4, 1	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^4+X^3+X+1$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6+X^3+X$ $= (X^2)^3+X^3+X$	$dega_{45} = 8$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6+X^4+X^3+1$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6+X^4+X^3$	$\frac{\alpha^3}{X^3} = X^3+X+1$ Remark 1	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6+X^3+X+1$	substituting X by $X+1$	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6+X^3+X^2$	$\frac{\alpha^3}{X^6} = \frac{X^6+X^3+X^2}{X^6} = \frac{X^4+X+1}{X^4}$ Proposition 1, Remark 1	$\nu(\alpha) = \frac{5}{4}$
$\alpha^3 = X^6+X^4+X^3$ $+X^2+X+1$	substituting X by $X+1$	$\nu(\alpha) = \frac{5}{4}$
$\alpha^3 = X^6+X^4+X^3+X^2+X$	$\frac{\alpha^3}{X^6} = \frac{X^6+X^3+X^2+X}{X^6} = \frac{X^5+X^2+X+1}{X^5}$ Remark 4, 1	$\nu(\alpha) \geq \frac{4}{3}$
$\alpha^3 = X^6+X^3+X^2+1$	substituting X by $X+1$	$\nu(\alpha) \geq \frac{4}{3}$
$\alpha^3 = X^6+X^4+X^3+X^2$	$\frac{\alpha^3}{X^6} = \frac{X^6+X^4+X^3+X^2}{X^6}$ $= \frac{X^4+X^2+X+1}{X^4}$ Proposition 1, Remark 1	$\nu(\alpha) = \frac{4}{3}$

$\alpha^3 = X^6 + X^3 + X^2 + X$	substituting X by $X+1$	$\nu(\alpha) = \frac{4}{3}$
$\alpha^3 = X^6 + X^3 + X^2 + X + 1$ $= (X^2)^3 + X^3 + X^2 + X + 1$	$dega_{50} = 18$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^4 + X^3 + X^2 + 1$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5$	$\frac{\alpha^3}{X^3} = X^3 + X^2$ Remark 1	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^2 + X$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + 1$	$dega_{53} = 10$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^2 + X + 1$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X$	$\frac{\alpha^3}{X^6} = \frac{X^5 + X^4 + 1}{X^5} = \frac{(X+1)^3}{X^3}\frac{X^2+1}{X^2}$ Theorem 2, Remark 1	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^2 + 1$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^2$	$dega_{13} = 10$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X + 1$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^3$	$\frac{\alpha^3}{X^3} = X^3 + X^2 + 1$ Remark 1	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6 + X^5 + X^3 + 1$	substituting X by $X+1$	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6 + X^5 + X^3 + X^2 + 1$	$\frac{\alpha^3}{(X^2+X)^3} = \frac{(X^2+X)^3 + X^2 + X + 1}{(X^2+X)^3}$ Lemma 2 Remark 1	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6 + X^5 + X^3 + X$	Theorem 4	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^3 + X^2 + X + 1$	$dega_{28} = 18$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^3 + X^2 + X$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^4$	$\frac{\alpha^3}{X^3} = X^3 + X^2 + X$ Remark 1, 4	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^2 + X + 1$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + 1$	$dega_{18} = 10$ Lemma 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^4 + X^2 + X$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^4 + X$ $= (X^2+X)^3 + X^3 + X$	$\alpha = [X^2+X, X, X^2+X, b_0, \dots, b_n, \dots]$, $b_n = X^{2^{n+1}}(X^3+X)^{\frac{2^{n+2}+(-1)^{n+1}}{3}}$ $\times (X^3+X^2+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 4	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^2$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X + 1$	$dega_{13} = 10$ Theorem 5, Remark 2	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^4 + X^2 + 1$	substituting X by $X+1$	$\nu(\alpha) > 1$
$\alpha^3 = X^6 + X^5 + X^4 + X^3 + 1$ $= (X^2+X)^3 + 1$	Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^3$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^3 + X$ $= (X^2+X)^3 + X$	Theorem 3	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^3 + X + 1$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^3 + X^2$ $= (X^2+X)^3 + X^2$	$\alpha = [X^2+X, X^2+1, X^2+X, b_0, \dots, b_n, \dots]$, $b_n = (X^2+1)^{2^{n+1}}(X^2)^{\frac{2^{n+2}+(-1)^{n+1}}{3}}$ $\times (X^4+X^3+X^2+X+1)^{\frac{2^{n+1}+(-1)^n}{3}}$ Theorem 4	$\nu(\alpha) = 2$

$\alpha^3 = X^6 + X^5 + X^4 + X^3 + X^2 + 1$	substituting X by $X+1$	$\nu(\alpha) = 2$
$\alpha^3 = X^6 + X^5 + X^4 + X^3$ $+X^2 + X + 1$	$\frac{\alpha^3}{(X^2+X+1)^3} = \frac{(X^2+X+1)^3 + X^2 + X + 1 + 1}{(X^2+X+1)^3}$ Proposition 1 Remark 1	$\nu(\alpha) = \frac{3}{2}$
$\alpha^3 = X^6 + X^5 + X^4 + X^3 + X^2 + X$ $= (X^2+X)^3 + X^2 + X$	Theorem 3	$\nu(\alpha) = 2$

In conclusion, we note that throughout this work we give results for the approximation exponent to the solution of strictly positive degree of the equation (4). Indeed, by the change of variable $\alpha = S + \frac{1}{\beta}$ we get that the equations (4) and (3) are equivalent. According to Remark 1, we have $\nu(\alpha) = \nu(\beta)$ and therefore we obtain the approximation exponent of β.

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