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The unramified subquotient of the unramified principal series

Marcela Hanzer[1,*]

[1] *Department of Mathematics, University of Zagreb, Bijenička cesta 30, HR–10 000 Zagreb, Croatia*

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Abstract. We prove that, in a certain induced representation of p–adic symplectic group, the unramified subquotient appears as a subrepresentation. This result has not only local importance, but is also very useful in calculations with automorphic representations of the corresponding group over adeles, since for an irreducible automorphic representation, almost every local component representation is unramified.
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1. Introduction and preliminaries

Let F be a local non-archimedean field of characteristic zero, with ring of integers O_F. We are interested in the admissible representations of the symplectic group, which we realize as a matrix group in the following way:

$$Sp_{2n}(F) = \left\{ g \in GL_{2n}(F) : g \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} g^t = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} \right\}$$

where J_n is the $n \times n$ matrix defined by $J_n = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$.

We fix $K = Sp_{2n}(O_F)$ as a maximal compact subgroup of $Sp_{2n}(F)$. We say that an irreducible admissible representation of $Sp_{2n}(F)$ is unramified if it has a non-zero K-fixed vector (also is called K–spherical). Then, necessarily, this vector is unique, up to a scalar. We fix a Borel subgroup B_n of $Sp_{2n}(F)$ consisting of all the upper-triangular matrices in $Sp_{2n}(F)$, and according to that choice, block-upper-triangular matrices form the standard parabolic subgroups. Each such subgroup is uniquely determined by an ordered partition $(n_1, n_2, \ldots, n_k)$ of $m \le n$; in that case the corresponding standard parabolic subgroup, denoted by $P_{(n_1,\ldots,n_k)}$, has Levi subgroup isomorphic to $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times Sp_{2(n-m)}(F)$ (if $m = n$ the last factor in this product is not there). In this situation, for admissible representations ρ_i of $GL(n_i, F),\ i = 1, \ldots, k$ and an admissible representation σ of $Sp_{2(n-m)}(F)$, we denote the parabolically induced (normalized) representation $\mathrm{Ind}_{P_{(n_1,\ldots,n_k)}}^{Sp_{2n}(F)}(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \sigma)$ by $\rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \sigma$.

*Corresponding author. *Email address:* `hanmar@math.hr` (M. Hanzer)

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For a unitary character χ of $GL(1,F)$, and $\alpha, \beta \in \mathbb{R}$ such that $\beta+\alpha+1 \in \mathbb{Z}_{\geq 1}$, we denote by $\zeta(-\beta,\alpha;\chi)$ the unique irreducible subrepresentation of

$$\mathrm{Ind}_{B_n}^{GL(\beta+\alpha+1,F)}(\nu^{-\beta}\chi \otimes \cdots \otimes \nu^{\alpha}\chi)$$

(which is a character $\nu^{\frac{\alpha-\beta}{2}}\chi \circ \det$ of $GL(\beta+\alpha+1,F)$). Here ν denotes a non-archimedean absolute value on F.

In this paper we want to prove that in certain (parabolically induced) unramified principal series representations the (unique) irreducible unramified subquotient appears as a subrepresentation. This result has an immediate application in the global calculations with automorphic forms, since an automorphic representation of $Sp_{2n}(\mathbb{A})$ has, at almost every local place, an unramified representation as a local constituent (e.g., we use this result in [3]).

To be more precise, we prove the following (the notion of negative unramified representation of a symplectic p-adic group is defined below):

Theorem 1. *Let $\beta > \alpha > 0$ be integers, and χ an unramified character of F with $\chi^2 = 1$. Let π be a negative representation. Then, an irreducible unramified subquotient of $\zeta(-\beta,\alpha;\chi) \rtimes \pi$ is a subrepresentation; it is also a negative representation.*

We need the following simple, but important results:

Lemma 1. *Let π be an irreducible representation of a reductive p–adic group and let $P = MN$ be a parabolic subgroup of G. Suppose that M is a direct product of two reductive subgroups M_1 and M_2. Let τ_1 be an irreducible representation of M_1 and let τ_2 be a representation of M_2. Suppose that $\pi \hookrightarrow \mathrm{Ind}_P^G(\tau_1 \otimes \tau_2)$. Then there exists an irreducible representation τ_2' such that $\pi \hookrightarrow \mathrm{Ind}_P^G(\tau_1 \otimes \tau_2')$.*

Proof. This is Lemma 3.2 of [5]. □

Lemma 2. *Let $G = Sp_{2n}(F)$ and K as above. Assume that σ is K–spherical smooth representation of G, and σ is a subquotient of $\mathrm{Ind}_{MN}(\sigma' \otimes 1_N)$, for some smooth representation σ' of M. Then σ' is $M \cap K$–spherical.*

Proof. This is Lemma 1.1 (ii) of [7]. □

We also need this fundamental result (essentially due to Tadić [9]), cf. Theorem 3.1 of [7]. Here, for a smooth representation σ of $Sp_{2n}(F)$, the expression $\mu^*(\sigma)$ denotes (a semisimplification) of the sum of the Jacquet modules of σ with respect to all standard maximal parabolic subgroups of $Sp_{2n}(F)$.

Theorem 2. *Let σ be an irreducible admissible representation of $Sp_{2n}(F)$. We decompose into irreducible constituents in the appropriate Grothendieck group*

$$\mu^*(\sigma) = \sum_{\xi,\sigma'} \xi \otimes \sigma'.$$

Assume that $\alpha, \beta, \in \mathbb{R}$, $\alpha+\beta+1 \in \mathbb{Z}_{>0}$, *and* χ *is character of* F^*. *Then the following holds*

$$\mu^*(\zeta(-\beta,\alpha;\chi) \rtimes \sigma) = \sum_{\xi,\sigma'} \sum_{i=0}^{\alpha+\beta+1} \sum_{j=0}^{i} \zeta(-\alpha, \beta - i; \chi^{-1}) \times \zeta(-\beta, j-\beta-1; \chi) \times \xi \\ \otimes \zeta(j-\beta, i-\beta-1; \chi) \rtimes \sigma'.$$

For an irreducible representation π of $Sp_{2n}(F)$, by $r_{1,\ldots,1;0}(\pi)$ we denote the Jacquet module of that representation with respect to the Borel subgroup B_n. An irreducible admissible unramified representation π of $Sp_{2n}(F)$ is **strictly (or strongly) negative** if for every irreducible subquotient $\chi_1\nu^{s_1} \otimes \cdots \otimes \chi_n\nu^{s_n}$ of $r_{1,\ldots,1;0}(\pi)$ (where χ_i are unitary characters, $s_i \in \mathbb{R}$; $i = 1, \ldots, n$), the following holds

$$s_1 < 0, \tag{1}$$
$$s_1 + s_2 < 0, \tag{2}$$
$$\vdots \tag{3}$$
$$s_1 + s_2 + \cdots + s_n < 0. \tag{4}$$

We say that an unramified representation is **negative** if, in the situation as above, inequalities are not necessarily strict (i.e., $\leq$ holds).

Let χ_0 be the unique quadratic non–trivial unramified character of F.

The **Jordan blocks** are defined for an unramified strongly negative and negative representations of a symplectic group $Sp_{2n}(F)$ as follows. For an unramified strictly negative representation σ of $Sp_{2n}(F)$ there exists (a unique) set of positive odd rational integers $2m_1+1 < 2m_2+1 < \ldots < 2m_l+1$ and $2n_1+1 < 2n_2+1 < \ldots < 2n_k+1$ such that k is even and $2m_1+1+\cdots+2m_l+1+2n_1+1\cdots+2n_k+1 = 2n+1$ (so l is odd) such that

$$\sigma \hookrightarrow \zeta(-n_k, n_{k-1}; \chi_0) \times \cdots \times \zeta(-n_2, n_1; \chi_0) \\ \times \zeta(-m_l, m_{l-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \times \zeta(-m_1, -1; 1) \rtimes 1,$$

where, if $m_1 = 0$, there is no factor $\zeta(-m_1, -1; 1)$ (cf. [7], Lemma 5.5). Then, we define

$$\mathrm{Jord}(\sigma) = \{(\chi_0, 2n_1+1), \ldots, (\chi_0, 2n_k+1), (1, 2m_1+1), \ldots, (1, 2m_l+1)\}.$$

For a negative representation σ_{neg} there exist a unique strongly negative representation σ_{sn} and pairs $(\chi_1, l_1), \ldots, (\chi_j, l_j)$ ($l_i \in \mathbb{Z}_{l\geq 1}$, χ_i unramified unitary characters) unique up to a permutation and taking inverses of characters, such that (cf.[8] Theorem 0-3)

$$\sigma_{neg} \hookrightarrow \times_{i=1}^{j} \zeta(-\frac{l_i-1}{2}, \frac{l_i-1}{2}; \chi_i) \rtimes \sigma_{sn}.$$

Then we define a multiset $\mathrm{Jord}(\sigma_{neg}) = \mathrm{Jord}(\sigma_{sn}) + \sum_{i=1}^{k}\{(\chi_i, l_i), (\chi_i^{-1}, l_i)\}$. If χ is an unramified unitary character, and σ negative or strongly negative unramified representation, we denote $\mathrm{Jord}_\chi(\sigma) = \{a : (\chi, a) \in \mathrm{Jord}(\sigma)\}$.

2. The proof of the main theorem

As we can see from (5), Lemma 5.5 of [7] gives an embedding of strictly negative unramified representations in principal series representations (more precisely, an embedding in a representation which is a subrepresentation of principal series representations) but the exponents in this principal series are given in a precise order. But often, especially in global calculations with automorphic forms, this is not enough. We need to recognize the position of the unramified irreducible subquotient inside a more general induced representation. In this section we prove that we can embed strictly negative and negative representations in different ways in degenerate principal series representations, i.e., we prove, at the end, the Theorem mentioned in the Introduction. We prove it as a consequence of couple of propositions.

Let χ be an unramified character of F with $\chi^2 = 1$. Let π be a strongly negative representation of $Sp_{2n}(F)$, and α and β rational integers satisfying $\beta > \alpha \geq 0$.

Proposition 1. *If* $[2\alpha+1, 2\beta+1] \cap \mathrm{Jord}_\chi(\pi) = \emptyset$*, then, in the appropriate Grothendieck group, we have*

$$\zeta(-\beta, \alpha; \chi) \rtimes \pi = \pi_1 + \pi_2 + \Pi,$$

where π_1 *and* π_2 *are non-isomorphic irreducible subrepresentations of* $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ *(now we view it as a genuine representation, not in the semisimplification), one of them is strongly negative spherical; and* Π *is the unique irreducible quotient.*

Proof. We know that, up to a sign, Aubert duality ([1, 2]) (at the level of Grothendieck groups) applied to an irreducible representation π gives again an irreducible representation ([1], Corollaire 3.9). We denote this (genuine) representation by $\hat{\pi}$. We use the fact that, under the Aubert involution, the duals of strongly negative representations are square–integrable representations. In the case of the cuspidal support we have here, this involution is equivalent to Iwahori-Matsumoto involution. This means that $\hat{\pi}$ is a square–integrable representation with the same cuspidal support as π. Also, if $(\chi, a) \in \mathrm{Jord}(\hat{\pi})$, then $(\chi, a) \in \mathrm{Jord}(\pi)$ where now $\mathrm{Jord}(\hat{\pi})$ is Jordan block of square–integrable representation, as defined by Mœglin ([4, 5]). Indeed, assume that $(\chi, a) \in \mathrm{Jord}(\hat{\pi})$. Then $\delta([\nu^{-\frac{a-1}{2}}\chi, \nu^{\frac{a-1}{2}}\chi]) \rtimes \hat{\pi}$ is irreducible and a is odd (since $L(\chi, \Lambda^2\mathbb{C}, s) = 1$) (cf. for example, Section 2 of [5] where Jordan block of a square–integrable representation of a symplectic group is defined). But, according to [1], the representation $\delta([\nu^{-\frac{a-1}{2}}\chi, \nu^{\frac{a-1}{2}}\chi]) \rtimes \hat{\pi}$ is irreducible if and only if $\zeta(-\frac{a-1}{2}, \frac{a-1}{2}; \chi) \rtimes \pi$ is irreducible. So, if $a \in \mathrm{Jord}_\chi(\hat{\pi})$, the representation $\zeta(-\frac{a-1}{2}, \frac{a-1}{2}; \chi) \rtimes \pi$ is irreducible. But then, (if we also assume $a \geq 3$ if $\chi = 1$) by Corollary 5.1 of [7], $a \in \mathrm{Jord}_\chi(\pi)$. This means $\mathrm{Jord}(\hat{\pi}) \subset \mathrm{Jord}(\pi)$, unless $(1_{GL_1}, 1) \in \mathrm{Jord}(\hat{\pi})$; this case is a bit more subtle. Namely, if π is embedded in the induced representation as in (5) below, we want to prove that the irreducibility of $1_{GL_1} \rtimes \pi$ forces that the representation $\zeta(-m_1, -1; 1)$ does not appear in the induced representation, i.e., that $m_1 = 0$, and $(1_{GL_1}, 1) \in \mathrm{Jord}(\pi)$. If we assume the opposite, the irreducible representation $1_{GL_1} \rtimes \pi$ is a subrepresentation of $\Pi_1 = \zeta(-m_k, m_{k-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \rtimes \pi_0$, where π_0 is a unique irreducible spherical (negative) subrepresentation of $1_{GL_1} \times \zeta(-m_1, -1; 1) \rtimes 1$. This means that $1_{GL_1} \otimes \pi$ has to appear with multiplicity at least two in the appropriate Jacquet module of Π_1. But, if $\zeta(-m_1, -1; 1)$ really appears in Π_1, this is not the case, since then

$1_{GL_1} \otimes 1_{Sp_{2m_1}}$ would have to appear with multiplicity at least two in the appropriate Jacquet module of π_0 (as can be seen by application of Theorem 2), which is not the case. So, if $[2\alpha+1, 2\beta+1] \cap \mathrm{Jord}_\chi(\pi) = \emptyset$, then we also have $[2\alpha+1, 2\beta+1] \cap \mathrm{Jord}_\chi(\hat{\pi}) = \emptyset$. Now we apply Theorem 2.1 of [6], to obtain the claim in the dual situation. Namely, the two square–integrable subrepresentations of $\delta([\nu^{-\alpha}\chi, \nu^{\beta}\chi]) \rtimes \hat{\pi}$ described in Theorem 2.1 of [6], say σ_1 and σ_2, each have $\delta([\nu^{\alpha}\chi, \nu^{\beta}\chi]) \otimes \hat{\pi}$ in their appropriate Jacquet module (and Langlands quotient of $\delta([\nu^{\alpha}\chi, \nu^{\beta}\chi]) \rtimes \hat{\pi}$ does not have that representation in the Jacquet module). By the properties of the Aubert involution (Theorem 1.7 (ii) in [1]), the irreducible representation $\zeta(-\beta, \alpha; \chi) \otimes \pi$ is part of the appropriate Jacquet modules of $\hat{\sigma_i}$; $i = 1, 2$.

Another way to prove this Proposition is to directly transfer the results of Muić, but we would have to use the fact that Iwahori-Matsumoto involution (which is sufficient in this case) respects composition series. □

Proposition 2. *Assume that π is a strongly negative representation of $Sp_{2n}(F)$, χ an unramified character with $\chi^2 = 1$ and α, β rational integers satisfying $\beta > \alpha \geq 0$. The irreducible spherical subquotient of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is a subrepresentation of that representation.*

Proof. Let $\mathrm{Jord}(\pi) = \{(1, a_1), \ldots, (1, a_k), (\chi_0, b_1), \ldots, (\chi_0, b_l)\}$, with the notation explained in the Preliminaries, so that χ_0 denotes the unique unramified, non–trivial quadratic character of F. We note that $\sum_{(\chi,m)\in \mathrm{Jord}(\pi)} m = 2n+1$, a_i, $i = 1, 2, \ldots, k$ and b_j, $j = 1, 2, \ldots, l$ are odd integers with k odd and l even. We also impose that $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 \cdots < b_l$. We denote $a_i = 2m_i + 1$ and $b_j = 2n_j + 1$. Then

$$\begin{aligned}\pi \hookrightarrow \zeta(-m_k, m_{k-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \times \zeta(-n_l, n_{l-1}; \chi_0) \\ \times \cdots \times \zeta(-n_2, n_1; \chi_0) \times \zeta(-m_1, -1; 1) \rtimes 1.\end{aligned} \tag{5}$$

Here if $m_1 = 0$, i.e., $a_1 = 1$ there is no last factor in the previous expression. We prove the proposition by a case by case analysis. In Proposition 3 we prove that the irreducible spherical subquotient of

$$\zeta(-\beta, \alpha; 1) \times \zeta(-m_k, m_{k-1}; 1) \times \cdots \times \zeta(-m_3, m_2; 1) \times \zeta(-m_1, -1; 1) \rtimes 1 \tag{6}$$

is a subrepresentation there (if $\chi = 1$) or, symmetrically, analogously is proved (left to the reader) that the irreducible spherical subquotient of

$$\zeta(-\beta, \alpha; \chi_0) \times \zeta(-n_l, n_{l-1}; \chi_0) \times \cdots \times \zeta(-n_2, n_1; \chi_0) \rtimes 1 \tag{7}$$

is a subrepresentation.

We comment only on the reduction in the first case; i.e., how from Proposition 3 this Proposition follows. Assume that π_1 is the irreducible spherical subquotient of (6). We moreover prove in Proposition 3 that (since α, β satisfy conditions imposed above) π_1 is necessarily a strongly negative or negative representation. In the case π_1 is strongly negative, we have that $\mathrm{Jord}(\pi_1) = \{(a_1, 1), \ldots, (a_k, 1)\} \cup \{(2\alpha+1, 1), (2\beta+1, 1)\}$. We then denote

$$\zeta(\chi_0, \ldots) = \zeta(-n_l, n_{l-1}; \chi_0) \times \cdots \times \zeta(-n_2, n_1; \chi_0),$$

for the unramified representation of the appropriate general linear group appearing in the description of π. We then have

$$\zeta(-\beta,\alpha;1)\rtimes\pi\hookrightarrow\zeta(-\beta,\alpha;1)\times\zeta(\chi_0,\ldots)\times\zeta(-m_k,m_{k-1};1)\times\cdots\times\zeta(-m_3,m_2;1)\rtimes\zeta(-m_1,-1;1)\rtimes 1\cong\zeta(\chi_0,\ldots)\times\zeta(-\beta,\alpha;1)\times\zeta(-m_k,m_{k-1};1)\times\cdots\times\zeta(-m_3,m_2;1)\rtimes\zeta(-m_1,-1;1)\rtimes 1=\Pi$$

so that, assuming π_1 is a subrepresentation of (6) (by Proposition 3), $\zeta(\chi_0,\ldots)\rtimes\pi_1$ is a subrepresentation of Π. On the other hand, we can embed π_1 into a representation, say, Π' according to $\mathrm{Jord}(\pi_1)$ (in the same way we embedded π). But then $\zeta(\chi_0,\ldots)\rtimes\Pi'$ has a unique (strongly negative!) unramified subquotient, say π_0, which is a subrepresentation there (cf. introduction of [7]). Because of the multiplicity one result for unramified representations, this means that $\pi_0\hookrightarrow\zeta(\chi_0,\ldots)\rtimes\pi_1$. This means that $\pi_0\hookrightarrow\Pi$, but then another multiplicity one argument forces $\pi_0\hookrightarrow\zeta(-\alpha,\beta;1)\rtimes\pi$, and the claim is shown.

If π_1 is a negative representation, and we have that π_1 is a subrepresentation of (6), then $\pi_1\hookrightarrow\zeta(-\alpha,\alpha;1)\rtimes\sigma_{sn}$, or $\pi_1\hookrightarrow\zeta(-\alpha,\alpha;1)\times\zeta(-\beta,\beta;1)\rtimes\sigma_{sn}$, or $\pi_1\hookrightarrow\zeta(-\beta,\beta;1)\rtimes\sigma_{sn}$ for some strongly negative representation σ_{sn} (depending whether $\{(2\alpha+1,1),(2\beta+1,1)\}\cap\mathrm{Jord}(\pi)=\{(2\alpha+1,1)\}$, or $\{(2\alpha+1,1),(2\beta+1,1)\}$ or $\{(2\beta+1,1)\}$ respectively). But then again, if, for example, the first case occurs,

$$\zeta(\chi_0,\ldots)\rtimes\pi_1\hookrightarrow\zeta(\chi_0,\ldots)\times\zeta(-\alpha,\alpha;1)\rtimes\sigma_{sn}\cong\zeta(-\alpha,\alpha;1)\times\zeta(\chi_0,\ldots)\rtimes\sigma_{sn}$$

so that $\zeta(\chi_0,\ldots)\rtimes\sigma_{sn}$ has again a strongly negative subrepresentation, say σ'_{sn}, then $\zeta(-\alpha,\alpha;1)\rtimes\sigma'_{sn}$ has a (negative) unramified subrepresentation π_0 (Theorem 6.1 of [7]; note that [7] does not use the fact σ'_{sn} is unitary). This means (again by multiplicity one) that $\zeta(\chi_0,\ldots)\rtimes\pi_1$ has π_0 for a subrepresentation, and, in the same way as before, we have that $\pi_0\hookrightarrow\zeta(-\beta,\alpha;1)\rtimes\pi$. Other cases are treated similarly. □

So, to complete the proof of Proposition 2, we are left to prove the following statement. Keeping the notation from above, let π'_1 be an unramified strongly negative representation with $\mathrm{Jord}(\pi'_1)=\{(1,2a_1+1),\ldots,(1,2a_k+1)\}$ and $\beta>\alpha\geq 0$ integers.

Proposition 3. *The unramified subquotient of*

$$\zeta(-\beta,\alpha;1)\rtimes\pi'_1$$

is a negative subrepresentation; strongly negative only if $\mathrm{Jord}(\pi'_1)\cap\{1,2\alpha+1),(1,2\beta+1)\}=\emptyset$.

Proof. First, assume that $\mathrm{Jord}(\pi'_1)\cap\{(2\alpha+1,1),(2\beta+1,1)\}=\emptyset$. If $2\alpha+1$ is greater than every element in $\mathrm{Jord}(\pi'_1)$, the statement is just the canonical description of strongly negative representations, cf. Introduction of ([7]); the same thing goes if $2\beta+1$ is smaller than any element in $\mathrm{Jord}(\pi'_1)$. So let $|[2\alpha+1,2\beta+1]\cap\mathrm{Jord}(\pi'_1)|=l>0$, so that

$$a_{t-1}<\alpha<a_t<\cdots<a_{t+l-1}<\beta<a_{t+l},$$

where $\{2a_1+1<2a_2+1<\cdots<2a_t+1\cdots<2a_{t+l-1}+1<\cdots<2a_k+1\}=\mathrm{Jord}(\pi'_1)$. Now we divide our discussion into several cases:

(i) l even and t even,

(ii) l even and t odd,

(iii) l odd and t even,

(iv) l odd and t odd.

We now discuss the first case, so l and t are even. Then

$$\begin{aligned}\pi_1' \hookrightarrow\ & \zeta(-a_k, a_{k-1}; 1) \times \cdots \times \zeta(-a_{t+l+1}, a_{t+l}; 1) \\ & \times \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \\ & \times \zeta(a_{t-1}, a_{t-2}; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1.\end{aligned}$$

On the other hand, for the strongly negative representation π with $\mathrm{Jord}(\pi) = \mathrm{Jord}(\pi_1') \cup \{(2\alpha+1, 1), (2\beta+1, 1)\}$ the following holds:

$$\begin{aligned}\pi \hookrightarrow\ & \zeta(-a_k, a_{k-1}; 1) \times \cdots \times \zeta(-a_{t+l+1}, a_{t+l}; 1) \times \zeta(-\beta, a_{t+l-1}; 1) \\ & \times \zeta(-a_{t+l-2}, a_{t+l-3}; 1) \times \cdots \times \zeta(-a_{t+2}, a_{t+1}; 1) \times \zeta(-a_t, \alpha; 1) \\ & \times \zeta(-a_{t-1}, a_{t-2}; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1.\end{aligned}$$

Since the cuspidal support of $\zeta(-\beta, \alpha; 1) \rtimes \pi_1'$ coincides with the cuspidal support of π, we have $\pi \leq \zeta(-\beta, \alpha; 1) \rtimes \pi_1'$. Let $\tau_1 = \zeta(-a_k, a_{k-1}; 1) \times \cdots \times \zeta(-a_{t+l+1}, a_{t+l}; 1)$, and $\tau_2 = \zeta(-a_{t-1}, a_{t-2}; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1$ be unramified representations of the appropriate general linear and symplectic group, respectively. Then

$$\zeta(-\beta, \alpha; 1) \rtimes \pi_1 \hookrightarrow \zeta(-\beta, \alpha; 1) \times \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \times \tau_1 \rtimes \tau_2,$$

and

$$\begin{aligned}\pi \hookrightarrow\ & \zeta(-\beta, a_{t+l-1}; 1) \times \zeta(-a_{t+l-2}, a_{t+l-3}; 1) \times \cdots \times \zeta(-a_{t+2}, a_{t+1}; 1) \\ & \times \zeta(-a_t, \alpha; 1) \times \tau_1 \rtimes \tau_2.\end{aligned}$$

But this means that there exists an unramified irreducible representation $\pi' \leq \zeta(-a_{t+l-2}, a_{t+l-3}; 1) \times \cdots \times \zeta(-a_{t+2}, a_{t+1}; 1) \times \zeta(-a_t, \alpha; 1) \times \tau_1 \rtimes \tau_2$ such that

$$\pi \hookrightarrow \zeta(-\beta, a_{t+l-1}; 1) \rtimes \pi' \hookrightarrow \zeta(-\beta, \alpha; 1) \times \zeta(\alpha+1, a_{t+l-1}; 1) \rtimes \pi'.$$

Examining the cuspidal support, we see that π_1' is the unique irreducible unramified subquotient of $\zeta(\alpha+1, a_{t+l-1}; 1) \rtimes \pi'$, so that by Lemma 1 and Lemma 2 we have $\pi \hookrightarrow \zeta(-\beta, \alpha; 1) \rtimes \pi_1'$.

We now analyze the second case: l even and t odd. Let $\tau_1 = \zeta(-a_k, a_{k-1}; 1) \times \cdots \times \zeta(-a_{t+l+2}, a_{t+l+1}; 1)$, and $\tau_2 = \zeta(-a_{t-2}, a_{t-3}; 1) \times \cdots \times \zeta(-a_3, a_2; 1) \times \zeta(-a_1, -1; 1) \rtimes 1$. Then

$$\pi \hookrightarrow \tau_1 \times \zeta(-a_{t+l}, \beta; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-\alpha, a_{t-1}; 1) \rtimes \tau_2.$$

Let π_0 be the unique unramified (negative) subrepresentation of $\tau_1 \rtimes \tau_2$; then, obviously $\mathrm{Jord}(\pi_0) = \{2a_k + 1, \dots, 2a_{t+l+1}, a_{t-2}, \dots a_2, a_1\}$. We then have

$$\pi \hookrightarrow \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, \beta; 1) \times \zeta(-\alpha, a_{t-1}; 1) \rtimes \pi_0.$$

Let π_0' be a spherical subquotient (subrepresentation) of $\zeta(-a_{t+l}, \beta; 1) \times \zeta(-\alpha, a_{t-1}; 1) \rtimes \pi_0$, so that $\pi \hookrightarrow \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \rtimes \pi_0'$. By Proposition 1, we know that π_0' is also a subrepresentation of $\zeta(-\beta, \alpha; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0$. We therefore have a sequence of intertwinings induced by the intertwinings in the general linear groups

$$\pi \hookrightarrow \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-\beta, \alpha; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0 \to$$
$$\zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \cdots \times \zeta(-\beta, \alpha; 1) \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0 \to$$
$$\vdots$$
$$\zeta(-\beta, \alpha; 1) \times \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0.$$

Since the kernel of every of the above homomorphisms is not unramified, we still have

$$\pi \hookrightarrow \zeta(-\beta, \alpha; 1) \times \zeta(-a_{t+l-1}, a_{t+l-2}; 1) \times \zeta(-a_{t+1}, a_t; 1) \times \zeta(-a_{t+l}, a_{t-1}; 1) \rtimes \pi_0.$$

Now we easily see that $\pi \hookrightarrow \zeta(-\beta, \alpha; 1) \rtimes \pi_1'$.

The third and the fourth case are similar the second, and the first, respectively, and are left to the reader.

Special cases (i.e., when $t = 1$, or $t + l - 1 = k$ or $l = 0$) are handled in the same way as these situations above, just simpler (e.g., when $l = 0$ we immediately apply Proposition 1). □

We now prove our main result, which is the generalization of Proposition 2; and is also needed in the applications in the theory of automorphic forms (e.g., [3]).

Theorem 3. *Let $\beta > \alpha > 0$ be integers, and χ an unramified quadratic character of F. Let π be a negative representation. Then, an irreducible unramified subquotient of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is a subrepresentation; it is also a negative representation.*

Proof. Since the representation π is negative, there exist (by Theorem 6.1 of [7]) a unique strongly negative representation σ_{sn} and a sequence of unitary characters $\chi_1, \dots, \chi_k$ (also unramified), such that

$$\pi \hookrightarrow \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \sigma_{sn},$$

where $\alpha_1, \dots, \alpha_k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Since the cuspidal support of $\zeta(-\beta, \alpha; \chi) \rtimes \pi$ is the same as the cuspidal support of

$$\zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \zeta(-\beta, \alpha; \chi) \rtimes \sigma_{sn},$$

and this representation has a negative subrepresentation (this follows from the proof of Theorem 6.1 (ii) of [7] and Propositions 2 and 3 here), say π_2, we have that

$\pi_2 \leq \zeta(-\beta, \alpha; \chi) \rtimes \pi$. On the other hand, by Proposition 2, there is a negative representation, say π_3, which is a subrepresentation of $\zeta(-\beta, \alpha; \chi) \rtimes \sigma_{sn}$. Now, consider the following sequence of homomorphisms induced from the appropriate homomorphisms of general linear groups:

$$\begin{aligned}
&\pi_2 \hookrightarrow \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \pi_3 \hookrightarrow \\
&\zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \times \zeta(-\beta, \alpha; \chi) \rtimes \sigma_{sn} \to \\
&\zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \sigma_{sn} \to \\
&\zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_{k-1}, \alpha_{k-1}; \chi_{k-1}) \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \sigma_{sn} \to \\
&\qquad\qquad\vdots \\
&\zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \sigma_{sn}.
\end{aligned}$$

Each of this homomorphisms is either isomorphism (if $\zeta(-\alpha_i, \alpha_i; \chi_i) \times \zeta(-\beta, \alpha; \chi)$ is irreducible), or its kernel is not spherical. More precisely, if $\zeta(-\alpha_i, \alpha_i; \chi_i) \times \zeta(-\beta, \alpha; \chi)$ reduces, then either we have $-\alpha_i \leq -\beta - 1 \leq \alpha_i < \alpha$ or $-\beta \leq -\alpha_i - 1 \leq \alpha < \alpha_i$ (e.g.,[7], section 2). If the first possibility should occur, we would have $\beta + 1 \leq \alpha_i < \alpha$ which contradicts our assumptions on α and β. If the second possibility occurs, an irreducible unramified representation $\zeta(-\beta, \alpha_i; \chi_i) \times \zeta(-\alpha_i, \alpha; \chi_i)$ is a quotient of $\zeta(-\alpha_i, \alpha_i; \chi_i) \times \zeta(-\beta, \alpha; \chi)$, and is not in the kernel of the homomorphism

$$\zeta(-\alpha_i, \alpha_i; \chi_i) \times \zeta(-\beta, \alpha; \chi) \to \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_i, \alpha_i; \chi_i).$$

So we have

$$\pi_2 \hookrightarrow \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \sigma_{sn},$$

and also

$$\zeta(-\beta, \alpha; \chi) \rtimes \pi \hookrightarrow \zeta(-\beta, \alpha; \chi) \times \zeta(-\alpha_1, \alpha_1; \chi_1) \times \cdots \times \zeta(-\alpha_k, \alpha_k; \chi_k) \rtimes \sigma_{sn},$$

and the claim follows. □

Remark 1. *Although all the details are not checked, the author believes that similar results hold for other (quasi-split) classical p-adic groups. There is no real mathematical obstacle to the simultaneous treatment of all the classical groups, but the notational awkwardness-namely, for other classical groups the (degenerate) principal series representations in question might have half-integer exponents (which is the consequence of the situation with the rank-one reducibility) and this would somewhat notationally complicate the exposition.*

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