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Some relations between rectifying and normal curves in Minkowski 3-space

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Abstract. In this paper, we obtain explicit parameter equations of spacelike rectifying curves in E_1^3 whose projection onto spacelike, timelike and lightlike plane of E_1^3 is a normal curve. We also obtain explicit parameter equations of spacelike normal curves in E_1^3 whose projection onto lightlike plane of E_1^3, with respect to a chosen screen distribution, is a rectifying W-curve.

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1. Introduction

In the Euclidean space $\mathbb{E}^3$ there exist three classes of curves, so-called *rectifying, normal and osculating curves* satisfying Cesaro's fixed point condition ([10]) meaning that rectifying, normal and osculating planes of such curves always contain a particular point. If all normal or osculating planes of a curve in $\mathbb{E}^3$ pass through a particular point, then the curve is spherical or planar, respectively. It is also known that if all rectifying planes of a non-planar curve in $\mathbb{E}^3$ pass through a particular point, then the ratio of torsion and curvature of such curve is a non-constant linear function ([1]). Some characterizations of rectifying curves in Minkowski 3-space E_1^3 are given in [7]. In particular, there exists a simple relationship between rectifying curves and Darboux vectors (centrodes), which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession ([2]). Normal curves in Minkowski 3-space are characterized in [5,6]. Spacelike and timelike normal curves in E_1^3 always lie in some quadric and null normal curves in E_1^3 are the null straight lines.

It is a quite interesting problem to obtain explicit parameter equations of rectifying and normal curves in Minkowski 3-space. In order to obtain such equations, it is natural to impose some extra condition on the corresponding curve. In this paper, we obtain explicit parameter equation of spacelike rectifying curve in E_1^3 assuming that orthogonal projection of such curve onto spacelike or timelike plane of E_1^3 is a normal curve. We prove that the straight lines are the only rectifying curves in E_1^3 whose projection onto lightlike plane with respect to a chosen screen distribution is

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a normal curve. Also, we find explicit parameter equation of non-planar spacelike normal curve in E_1^3 assuming that the projection of such curve onto lightlike plane of E_1^3 (with respect to a chosen screen distribution) is a rectifying W-curve.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with the indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that an arbitrary vector $v \in \mathbb{E}_1^3$ can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$ ([3,9]). The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$ and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively ([9]). A spacelike or timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Arbitrary curve in E_1^3 is called *W-curve*, if all its curvature functions are constant.

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^3$, consisting of the tangent, principal normal and binormal vector field, respectively. If α is a non-null curve with non-null principal normal N, the Frenet equations read ([8])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 \\ -\epsilon_1\kappa_1 & 0 & -\epsilon_1\epsilon_2\kappa_2 \\ 0 & -\epsilon_2\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where $\kappa_1(s)$, $\kappa_2(s)$ are the first and second curvature of the curve, $\epsilon_1 = g(T,T) = \pm 1$, $\epsilon_2 = g(N,N) = \pm 1$ and $g(B,B) = -\epsilon_1\epsilon_2$, respectively.

If α is a spacelike curve with null principal normal N, it is called *pseudo null* curve and the Frenet equations have the form ([11])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ 0 & \kappa_2 & 0 \\ -\kappa_1 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where the first curvature $\kappa_1(s) = 0$ if α is a straight line, or $\kappa_1(s) = 1$ in all other cases. In this case, the following conditions hold

$$g(T,T) = 1, g(N,N) = g(B,B) = 0, g(T,N) = g(T,B) = 0, g(N,B) = 1.$$

Recall that the *normal* and *rectifying* curves in E_1^3 are defined in [6] and [7] as the curves whose the position vector α (with respect to some chosen origin) always lies in its normal plane $T^\perp$ and rectifying plane $N^\perp$, respectively. Therefore, the position vector of normal or rectifying curve α in E_1^3 satisfy the equations $g(\alpha, T) = 0$, $g(\alpha, N) = 0$, respectively.

Let Ω be the lightlike plane of E_1^3. Denote by $T\Omega$ the tangent bundle subspace of Ω in E_1^3. Since $g|_\Omega$ is degenerate on Ω, the *radical* or the *null space* of $T_p\Omega$ at each point $p \in \Omega$, is a subspace $Rad\, T_p\Omega$ defined by ([3])

$$Rad\, T_p\Omega = \{Y \in T_p\Omega \,|\, g_p(X,Y) = 0, \forall X \in T_p\Omega\}.$$

For the radical space there holds

$$Rad\, T_p\Omega = T_p\Omega \cap T_p\Omega^{\perp}.$$

Moreover, since Ω is lightlike plane in E^3_1, it follows that $\dim(T_p\Omega^{\perp}) = 1$ and therefore $\dim(Rad\, T_p\Omega) = 1$ and $Rad\, T_p\Omega = T_p\Omega^{\perp}$. The vector bundle $Rad\, T\Omega$ is called the *radical (null) distribution* of Ω.

Denote by $S(T\Omega)$ a complementary vector bundle of $Rad\, T\Omega$ in $T\Omega$. This means that

$$T\Omega = Rad\, T\Omega \oplus S(T\Omega).$$

Vector bundle $S(T\Omega)$ is called a *screen distribution* on Ω. In particular, for a given screen distribution $S(T\Omega)$ there exists a unique complementary vector bundle $ltr(T\Omega)$ to $T\Omega$ in $TE^3_1|_\Omega$. The vector bundle $ltr(T\Omega)$ is called the *lightlike transversal vector bundle* of Ω. Consequently, the tangent bundle TE^3_1 splits into the following three non-intersecting complementary (but not orthogonal) vector bundles ([3]):

$$TE^3_1|_\Omega = Rad\, T\Omega \oplus S(T\Omega) \oplus ltr(T\Omega).$$

Let us consider arbitrary curve α lying fully in E^3_1 (and not lying in its lightlike plane) and its projection $\beta = P^*(\alpha)$ onto lightlike plane Ω of E^3_1, where $P^* : E^3_1 \to \Omega$ is the projection map. Then the projection map P^* is not orthogonal and unique ([4]). In particular, the curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ and the projected curve β are related by

$$\alpha(s) = \beta(s) + g(\alpha(s), A)B, \tag{3}$$

where A and B are null vectors satisfying the conditions

$$\begin{aligned} &Rad\, T\Omega = span\{A\}, \quad ltr(T\Omega) = span\{B\},\\ &g(A,B) = 1, \quad g(B,W) = 0, \quad \forall\, W \in S(T\Omega). \end{aligned}$$

Next we show that by choosing different screen distributions on Ω, different parameter equations of the projected curves $\beta = P^*(\alpha)$ can be obtained.

Up to isometries of E^3_1, assume that Ω has the equation $x_1 = x_2$. Then the radical distribution $Rad\, T\Omega$ is spanned by

$$A = (1, 1, 0). \tag{4}$$

Let $Q = (0, 1, 0)$ be locally defined nowhere zero section defined on the open subset $U \subset E^3_1$. Then $g(Q, A) = 1$, $g(Q, Q) = 1$ and therefore the lightlike transversal vector bundle is spanned by ([3])

$$B = \frac{1}{g(A,Q)}\{Q - \frac{g(Q,Q)}{2g(A,Q)}A\} = (-\frac{1}{2}, \frac{1}{2}, 0). \tag{5}$$

It follows that the corresponding screen distribution $S(T\Omega)$ is spanned by

$$W = (0, 0, 1). \tag{6}$$

Substituting (4) and (5) in (3), we obtain that the projected curve β has parameter equation of the form

$$\beta(s) = (\frac{\alpha_1(s)+\alpha_2(s)}{2}, \frac{\alpha_1(s)+\alpha_2(s)}{2}, \alpha_3(s)). \tag{7}$$

Let $Q = (0,1,1)$ be the next locally defined nowhere zero section defined on the open subset $U \subset E^3_1$. Then $g(Q,A) = 1$, $g(Q,Q) = 2$, so relation (5) implies that the lightlike transversal vector bundle is spanned by

$$B = (-1, 0, 1). \tag{8}$$

It follows that the corresponding screen distribution $S(T\Omega)$ is spanned by

$$V = (1, 1, -1). \tag{9}$$

Substituting (4) and (8) in (3), we find that the projected curve β is given by

$$\beta(s) = (\alpha_2(s), \alpha_2(s), \alpha_3(s) + \alpha_1(s) - \alpha_2(s)). \tag{10}$$

Therefore, different choices of screen distributions on Ω provide different parameter equations of the projected curves (for more details, see [4]).

3. Some relations between spacelike rectifying curves and planar normal curves in $\mathbb{E}^3_1$

In this section we obtain some explicit parameter equations of spacelike rectifying curves in E^3_1. Recall that up to a parametrization, every spacelike rectifying curve with non-null principal normal in E^3_1 has one of the following parameter equations ([7]):

$$\alpha(t) = \frac{a}{\cos t}x(t), \quad \beta(t) = \frac{b}{\sinh t}y(t), \quad \gamma(t) = \frac{c}{\cosh t}z(t),$$

where a, b, $c \in R^+_0$, $x(t)$ is some unit speed spacelike curve lying in the pseudosphere $S^2_1(1)$, $y(t)$ is some unit speed timelike curve lying in the pseudosphere $S^2_1(1)$ and $z(t)$ is some unit speed spacelike curve lying in the pseudohyperbolic space $H^2_0(1)$. In particular, rectifying curve α has spacelike rectifying plane, while the rectifying curves β and γ both have timelike rectifying planes and spacelike and timelike position vectors, respectively.

In the theorems which follow, we will determine explicit parameter equations of the curves $x(t)$, $y(t)$ and $z(t)$ by imposing extra condition on the rectifying curve. Namely, we will assume that its orthogonal projection onto non-degenerate plane of E^3_1 is a normal curve.

Theorem 1. *Let α be a spacelike rectifying curve in E^3_1 and β the orthogonal projection of α onto spacelike plane of E^3_1.*

(i) If α has non-null principal normal and β is normal curve, then up to a parametrization α is given by

$$\alpha(u) = (c \cos u, \cos(cu), \sin(cu)), \tag{11}$$

where $c \in R^+_0$, $c \neq 1$;

(*ii*) *If α has null principal normal and β is normal curve, then up to a parametrization α is given by*

$$\alpha(u) = (\cos u, \cos u, \sin u). \tag{12}$$

Proof. (i) Let β be the orthogonal projection of α onto spacelike plane of E_1^3. Then α is given by

$$\alpha(s) = \beta(s) + \mu(s)\mathrm{v}, \tag{13}$$

where s is arclength parameter of α, $\mu(s)$ is some non-constant differentiable function and v$= (1, 0, 0)$ is a unit timelike vector.

Assume that α has non-null principal normal and that β is a normal curve. Then the equation of β reads

$$\beta(t) = (0, \cos t, \sin t) \tag{14}$$

where t is arclenght parameter of β. Differentiating relation (13) with respect to s and using the condition $g(\alpha'(s), \alpha'(s)) = 1$, we get

$$1 + \mu'^2(s) = t'^2(s). \tag{15}$$

Differentiating relation (14) with respect to s we find

$$g(\beta(s), \beta''(s)) = -t'^2(s), \tag{16}$$

which together with the condition $g(\alpha(s), \alpha''(s)) = 0$ yield

$$t'^2(s) + \mu''(s)\mu(s) = 0. \tag{17}$$

Relations (15) and (17) imply second order differential equation

$$\mu''(s)\mu(s) + \mu'^2(s) + 1 = 0 \tag{18}$$

whose general solution reads

$$\mu(s) = \sqrt{c^2 - (s + c_1)^2}, \tag{19}$$

where $c \in R_0^+$, $c_1 \in R$ and $|c| > |s + c_1|$. The arclength parameter of β is given by

$$t(s) = \int_0^s ||\beta'(u)||\, du.$$

By using relations (13) and (19) we obtain

$$||\beta'(s)|| = \frac{1}{\sqrt{1 - (\dfrac{s + c_1}{c})^2}},$$

and therefore

$$t(s) = c \arcsin(\frac{s + c_1}{c}). \tag{20}$$

Relations (14) and (20) imply

$$\beta(s) = (0, \cos(c \arcsin(\frac{s + c_1}{c}), \sin(c \arcsin(\frac{s + c_1}{c})). \tag{21}$$

Substituting (19) and (21) in (13) and using reparametrization

$$u = \arcsin(\frac{s+c_1}{c}), \tag{22}$$

we obtain that α is given by (11). In particular, by using (13),(19) and (21) we get

$$g(\alpha''(s),\alpha''(s)) = \frac{c^2[(1-c^2)(s+c_1^2)^2 + c^2(c^2-1)]}{c^2-(s+c_1)^2)^3}. \tag{23}$$

Since $g(\alpha''(s),\alpha''(s)) \neq 0$ it follows that $c \neq 1$ which proves statement (i).

(ii) Assume that α has null principal normal and that β is a normal curve. As in the proof of statement (i) we easily obtain that α is given by (11). In particular, by using (23) it follows that $g(\alpha'',\alpha'') = 0$ if $c = 1$. Finally, putting $c = 1$ in (11) we get that α is given by (12), which proves statement (ii). $\square$

Remark 1. *By reparametrization* $u = \arcsin(\frac{a}{c}\tan z)$, $a^2 = 1 - c^2$, $c \in R_0^+$, $c \neq 1$, *parameter equation (11) can also be written as*

$$\alpha(z) = \frac{a}{\cos z}x(z),$$

where

$$x(z) = \frac{1}{a}(\sqrt{\cos^2 z - a^2}, \cos(c\arcsin(\frac{a}{c}\tan z))\cos z, \sin(c\arcsin(\frac{a}{c}\tan z))\cos z)$$

is a unit speed spacelike curve lying in a pseudosphere $S_1^2(1)$.

Theorem 2. *Let* α *be spacelike rectifying curve in* E_1^3 *and* β *the orthogonal projection of* α *onto timelike plane of* E_1^3.

(*a*) *If* α *has non-null principal normal and* β *is spacelike or timelike normal curve respectively, then up to a parametrization* α *is given by*

$$\alpha(u) = (\cosh(cu), \sinh(cu), c\cosh u),$$

or by

$$\alpha(u) = (\sinh(cu), \cosh(cu), c\sinh u),$$

where $c \in R_0^+$, $c \neq 1$.

(*b*) *If* α *has null principal normal and* β *is spacelike or timelike normal curve respectively, then up to a parametrization* α *is given by*

$$\alpha(u) = (\cosh u, \sinh u, \cosh u),$$

or by

$$\alpha(u) = (\sinh u, \cosh u, \sinh u).$$

(*c*) *If* β *is null normal curve, then* α *is the straight line.*

Proof. In cases (a) and (b) the proof is similar to the proof of Theorem 1, so we omit it and give the proof only for the case (c). Assume that β is the orthogonal projection of α onto timelike plane of E^3_1. Then α is given by

$$\alpha(s) = \beta(s) + \mu(s)\mathrm{v}, \tag{24}$$

where s is arclength parameter of α, $\mu(s)$ is some non-constant differentiable function and $\mathrm{v} = (0, 0, 1)$ is a unit spacelike vector. Assume that β is a null normal curve. Then up to isometries of E^3_1, β is the null straight line given by $\beta(t) = t(1, 1, 0)$. Differentiating relation (24) with respect to s and using the condition $g(\alpha'(s), \alpha'(s)) = 1$ we obtain $\mu(s) = s$. Hence $\alpha(s) = s(1, 1, 1)$ which means that α is the straight line. $\square$

In the next theorem we prove that the straight lines are the only rectifying curves in E^3_1 whose projection onto lightlike plane (with respect to a chosen screen distribution) is a normal curve.

Theorem 3. *Let α be a spacelike rectifying curve with non-null principal normal in E^3_1 and β the projection of α onto lightlike plane with the equation $x_1 = x_2$. If β is a normal curve and the screen distribution is given by (6) or (9), then α is the straight line.*

Proof. Assume that β is a normal curve and that the screen distribution is given by (6). By using relation (7) and the condition $g(\beta(s), \beta'(s)) = 0$ we find $\alpha_3(s)\alpha_3'(s) = 0$. It follows that α is a planar curve and therefore the straight line.

Next, assume that the screen distribution is given by (9). By using relation (10) and the condition $g(\beta(s), \beta'(s)) = 0$ we obtain

$$(\alpha_3(s) + \alpha_1(s) - \alpha_2(s))(\alpha_3(s) + \alpha_1(s) - \alpha_2(s))' = 0,$$

where s is arclength parameter of α. We distinguish two cases:

(1) $\alpha_3(s) + \alpha_1(s) - \alpha_2(s) = 0$;

(2) $(\alpha_3(s) + \alpha_1(s) - \alpha_2(s))' = 0$.

(1): $\alpha_3 + \alpha_1 - \alpha_2 = 0$. Then the curve α is given by

$$\alpha(s) = (\alpha_1(s), \alpha_1(s) + \alpha_3(s), \alpha_3(s)), \tag{25}$$

so the condition $g(\alpha'(s), \alpha'(s)) = 1$ implies

$$\alpha_1(s) = \frac{1}{2}\int \frac{ds}{\alpha_3'(s)} - \alpha_3(s). \tag{26}$$

From the relation $g(\alpha(s), \alpha''(s)) = 0$ we find

$$\alpha_1''\alpha_3 + \alpha_1\alpha_3'' + 2\alpha_3\alpha_3'' = 0. \tag{27}$$

Relations (26) and (27) imply

$$\alpha_3''(s)\Big(-\frac{\alpha_3(s)}{\alpha_3'^2(s)} + \int \frac{ds}{\alpha_3'(s)}\Big) = 0.$$

We distinguish two subcases:

(1.1) $\alpha_3''(s) = 0$;

(1.2) $-\frac{\alpha_3(s)}{\alpha_3'^2(s)} + \int \frac{ds}{\alpha_3'(s)}$=0.

(1.1): If $\alpha_3''(s) = 0$, by using (25) and (26) we obtain that α has parameter equation

$$\alpha(s) = (\frac{s - 2a^2 s - 2ab}{2a}, \frac{s}{2a}, as + b), \quad a \in R_0, b \in R$$

and thus it is the straight line.

(1.2): If

$$-\frac{\alpha_3(s)}{\alpha_3'^2(s)} + \int \frac{ds}{\alpha_3'(s)} = 0,$$

differentiating the last equation with respect to s we find $\alpha_3(s)\alpha_3''(s) = 0$. Consequently, this subcase reduces to the subcase (1.1), which means that α is the straight line.

(2): $(\alpha_3(s) + \alpha_1(s) - \alpha_2(s))' = 0$. Then α is given by

$$\alpha(s) = (\alpha_1(s), \alpha_1(s) + \alpha_3(s) - c, \alpha_3(s)), \quad c \in R,$$

and therefore it is congruent to the curve given by (25). Hence α is the straight line, which proves the theorem. ☐

Remark 2. *The projected curve β in Theorem 3 is the null straight line.*

4. Some relations between non-planar spacelike normal curves and planar rectifying curves in $\mathbb{E}_1^3$

It is known that every non-planar spacelike normal curve in E_1^3 lies in some quadric ([5]). In this section, we obtain some explicit parameter equations of non-planar spacelike normal curves in E_1^3 assuming that their projection onto lightlike plane of E_1^3, with respect to a chosen screen distribution, is a rectifying W-curve. In Euclidean 3-space, it is known that for rectifying curves the curvature ratio $\kappa_2(s)/\kappa_1(s)$ is a non-constant linear function ([1]). The same property hold for timelike and spacelike rectifying curves with non-null principal normal in Minkowski 3-space ([7]). On the other hand, rectifying curves with null principal normal in E_1^3 are planar curves lying fully in the lightlike plane $\{T, N\}$. Therefore, every rectifying curve with null principal normal in E_1^3 is a pseudo null curve. The following theorem is given in [11] for pseudo null W-curves.

Theorem 4 ([11, Theorem A]). *All pseudo null spacelike curves in E_1^3 with constant curvatures can be classified as:*

(1) $\kappa_1(s) = 0$ *if and only if β is a part of a spacelike straight line;*

(2) $\kappa_1(s) = 1$ *and* $\kappa_2(s) = 0$ *if and only if β is a part of a planar curve with parametrization*

$$\beta(s) = (\frac{s^2}{2}, \frac{s^2}{2}, s); \tag{28}$$

(3) $\kappa_1(s) = 1$ *and* $\kappa_2(s) = c_2 =$ *constant* $\neq 0$ *if and only if* β *is a part of a planar curve with parametrization*

$$\beta(s) = \frac{1}{c_2^2}(\cosh(c_2 s) + \sinh(c_2 s), \cosh(c_2 s) + \sinh(c_2 s), c_2^2 s). \tag{29}$$

In the next theorem, we obtain explicit parameter equation of non-planar spacelike normal curve in E_1^3 whose projection onto lightlike plane of E_1^3 (with respect to a chosen screen distribution) is a rectifying W-curve.

Theorem 5. *Let* α *be a non-planar spacelike normal curve in* E_1^3 *and* β *the projection of* α *onto lightlike plane with the equation* $x_1 = x_2$ *and the screen distribution given by (6). If* β *is a rectifying W-curve with parameter equation (28) or (29) respectively, then* α *is given by*

$$\alpha(s) = (-\frac{c}{s^2} + \frac{s^2}{2} + \frac{1}{2}, \frac{c}{s^2} + \frac{s^2}{2} - \frac{1}{2}, s), \quad s^2 > 8c, \quad c \in R, \tag{30}$$

or else by

$$\alpha(s) = (\frac{2 - c_2}{c_2^2} e^{c_2 s} - c e^{-c_2 s} + \frac{c_2^2}{4} s^2 e^{-c_2 s}, \frac{1}{c_2} e^{c_2 s} + c e^{-c_2 s} - \frac{c_2^2}{4} s^2 e^{-c_2 s}, s),$$

where $c_2 \in R_0, c \in R$.

Proof. First assume that β is given by (28). From relations (7) and (28) we obtain that α is given by

$$\alpha(s) = (s^2 - \alpha_2(s), \alpha_2(s), s), \tag{31}$$

where $\alpha_2(s)$ is some differentiable function and s is arclength parameter of β. By using the condition $g(\alpha(s), \alpha'(s)) = 0$ we obtain linear differential equation

$$\alpha_2'(s) + \frac{2}{s}\alpha_2(s) = 2s - \frac{1}{s}, \quad s \neq 0$$

whose general solution reads

$$\alpha_2(s) = \frac{c}{s^2} + \frac{s^2}{2} - \frac{1}{2}, \quad c \in R.$$

Substituting this in (31) we find that α has parameter equation (30).

If β is given by (29), the proof is analogous. □

Remark 3. *The curve given by (30) lies in pseudosphere* $-x^2 + y^2 + z^2 = 2c$ *if* $c > 0$, *lightcone* $-x^2 + y^2 + z^2 = 0$ *if* $c = 0$ *and pseudohyperbolic space* $-x^2 + y^2 + z^2 = 2c$ *if* $c < 0$.

The last theorem can be proved in a similar way.

Theorem 6. *Let α be a non-planar spacelike normal curve in E_1^3 and β the projection of α onto lightlike plane with the equation $x_1 = x_2$ and the screen distribution given by (9). If β is a rectifying W-curve with parameter equation (28) or (29) respectively, then α is given by*

$$\alpha(s) = (\frac{1}{s(1+\frac{s}{2})}(c + \frac{s^4}{4} + \frac{s^3}{2} + \frac{s^2}{2}), \frac{s^2}{2}, s(1+\frac{s}{2}) - \frac{1}{s(1+\frac{s}{2})}(c + \frac{s^4}{4} + \frac{s^3}{2} + \frac{s^2}{2}))$$

where $c \in R$, or else by

$$\begin{aligned}\alpha(s) = (&\frac{1}{s+\frac{1}{c_2}e^{c_2 s}}(c + \frac{1}{c_2^2}e^{2c_2 s} + \frac{s^2}{2} + \frac{s}{c_2}e^{c_2 s}), \frac{1}{c_2}e^{c_2 s}, s + \frac{1}{c_2}e^{c_2 s} \\ &- \frac{1}{s+\frac{1}{c_2}e^{c_2 s}}(c + \frac{1}{c_2^2}e^{2c_2 s} + \frac{s^2}{2} + \frac{s}{c_2}e^{c_2 s})), \quad c_2 \in R_0, c \in R.\end{aligned}$$

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