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L'Hôpital's rule without derivatives*

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Abstract. Quotients of derivatives of continuously differentiable functions of the single variable are characterized without using differentiation. This yields augmentations of L'Hôpital's rule, for an indeterminate form of type 0/0, and reformulations of the theorem of Lagrange.

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1. Introduction

This paper is motivated by L'Hôpital's rule, one of the oldest results in differential calculus [4, 6]. The rule essentially says that the limit of a quotient of functions:

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} \tag{1}$$

where $\lim_{x \to x^*} f(x) = 0$ and $\lim_{x \to x^*} g(x) = 0$ is equal to the limit of the quotient of their derivatives:

$$\lim_{x \to x^*} \frac{f'(x)}{g'(x)} \tag{2}$$

provided that (2) exists. In general, there are situations where (2) exists but not (1), e. g., if $f(x) = 1$, $g(x) = x$ and $x^* = 0$. For the special case where $f(x^*) = g(x^*) = 0$, f' and g' are continuous and $g'(x^*) \neq 0$, it is easy to see why L'Hôpital's rule works, e. g., [3] or [6, p. 385].

Although the study of functions of the single variable is relatively simple [6], one has to be careful when working with ratios of functions and derivatives such as L'Hôpital's rule because the rule is "capable of yielding spurious results" [1] and counter examples [1, 3, 7]. In order to understand the rule our first objective is to characterize the quotient $f'(x^*)/g'(x^*)$. This is done in Section 2 using a characterization of zero-derivative points from [8]. The characterization was introduced there for continuously differentiable functions in n variables with a globally Lipschitz

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derivative. For functions of the single variable it says that at an arbitrary interior point x^* of an interval $I = [a, b]$, $f'(x^*) = 0$ if and only if, $|f(x) - f(x^*)|$ is bounded on I by a parabola with the apex at x^*. In symbols:

$$f'(x^*) = 0 \Leftrightarrow \exists \Lambda \geq 0 : |f(x) - f(x^*)| \leq \Lambda \cdot (x - x^*)^2, x \in I. \tag{3}$$

We use the results on the quotient of derivatives from Section 2 to augment L'Hôpital's rule in Section 3. They are also used to reformulate the theorem of Lagrange for functions in several variables and a single constraint in Section 4. For an application to Fermat's extreme value theorem see [10].

2. Characterizing quotients of derivatives

Following [8, 9], consider a scalar function of the single variable f defined on an open set containing the interval $I = [a, b]$ where $a < b$. Let us assume that f is continuously differentiable on I and that its derivative satisfies the global Lipschitz condition on I. This means that there is a number $L \geq 0$ such that

$$|f'(s) - f'(t)| \leq L \cdot |s - t|, \quad \text{for } s \text{ and } t \text{ in } I.$$

In particular, the first derivatives of twice differentiable functions [2, 5] have this property. Number L is called a Lipschitz constant of the derivative on I and we note that it is not uniquely determined. We will also use the notion of a "uniformly bounded function" on $I \backslash \{x^*\}$:

Definition 1. *Consider an interval $I = [a, b]$ and its point x^*. We say that a function $\Phi(x)$ is uniformly bounded on $I \backslash \{x^*\}$ if $\Phi(x)$ is defined on $I \backslash \{x^*\}$ and if there is a constant $c = c(x^*)$ such that $|\Phi(x)| \leq c$ for $x \in I \backslash \{x^*\}$.*

Example 1. *Consider $I = [-1, 1]$ and $x^* = 0$. Then $\Phi(x) = x^2/x$ is uniformly bounded on $I \backslash \{x^*\}$, but x/x^2 is not. Function $\Phi(x) = 0$, if $x \neq 0$, and $\Phi(0) = 1$ is uniformly bounded on $I \backslash \{x^*\}$ for every $x^* \in I$. One can specify $c(x^*) = 1$, if $x^* \neq 0$, and $c(0) = 0$. Another illustration is given in Example 5.*

Theorem 1 (Characterizing quotients of derivatives)**.** *Consider continuously differentiable functions f and g defined on an open set containing the interval $I = [a, b]$ where $g'(x) \neq 0$ for $x \in I$. Assume that the derivatives of f and g satisfy the global Lipschitz condition on I and consider an interior point x^* of I. If $r = r(x^*)$ is an arbitrary number then the following statements are equivalent:*

(i) $r = f'(x^*)/g'(x^*)$;

(ii) $\big|f(x) - f(x^*) - r \cdot \big(g(x) - g(x^*)\big)\big| \leq \Lambda \cdot (x - x^*)^2$ *for some* $\Lambda \geq 0$ *and for every* x *in* I;

(iii) $(1/\Lambda) \cdot \big|f(x) - f(x^*) - r \cdot \big(g(x) - g(x^*)\big)\big| \leq (x - x^*)^2$ *for every* $\Lambda > 0$ *sufficiently large and for every* x *in* I;

(iv) The ratio function $\big|f(x) - f(x^*) - r \cdot \big(g(x) - g(x^*)\big)\big|/(x - x^*)^2$ *is uniformly bounded on* $I \backslash \{x^*\}$.

Proof. Only the equivalence of (i) and (ii) needs to be proved. Assume that an arbitrary number $r = r(x^*)$ satisfies (i). Consider the "transfer function" $T(x, x^*) = f(x) - \big(f'(x^*)/g'(x^*)\big) \cdot g(x)$. After differentiation and substitution $x = x^*$, we have $T'(x^*, x^*) = 0$. This means that x^* is a zero derivative point of $T(x, x^*)$. Therefore $|T(x, x^*) - T(x^*, x^*)| \leq \Lambda \cdot (x - x^*)^2$ for some $\Lambda \geq 0$ and for every x in I, by (3). After a back substitution and rearrangement we have (ii). On the other hand, when (ii) holds for some r then we use the function $T(x, x^*) = f(x) - r \cdot g(x)$ and note that $|T(x, x^*) - T(x^*, x^*)| \leq \Lambda \cdot (x - x^*)^2$. This implies $T'(x^*, x^*) = 0$, again by (3). Therefore $f'(x^*) = r \cdot g'(x^*)$. Hence $r = f'(x^*)/g'(x^*)$ which is (i). □

Example 2. *Let $f(x) = 1$ and $g(x) = x - x^*$ where x^* is an arbitrary fixed number. Using Theorem 1(iv), a number $r = r(x^*)$ is the quotient of their derivatives at x^* if and only if the ratio $|r/(x - x^*)|$ is uniformly bounded around x^*. This is true if and only if $r = 0$.*

The requirement that the derivatives in Theorem 1 have the global Lipschitz property cannot be omitted as the example below shows. This is not a serious problem for the applied mathematician since, loosely speaking, "almost all" continuously differentiable functions that are used in modelling of real life processes have a global Lipschitz derivative on a compact interval. Otherwise "the rate of change of the rate of change" would be unbounded at some interior point of the interval. In the context of, say, Newtonian mechanics this would require an unlimited force.

Example 3. *Consider $f(x) = |x|^{3/2}$ and $g(x) = x$. Note that f is continuously differentiable but it does not have a global Lipschitz derivative on $I = [-1, 1]$. Choose $x^* = 0$. Then $r = 0$ by (i) but, for the same r, the ratio in (iv) yields $|x|^{3/2}/x^2 \to \infty$ as $x \to x^*$. We have a contradiction because the ratio is not uniformly bounded.*

Theorem 1 recovers differentiation-free characterizations of the derivative from [9]. Indeed, the derivative of f at x^* is

$$f'(x^*) = \lim_{x \to x*} \frac{f(x) - f(x^*)}{x - x^*}.$$

This is the limit (1) of indeterminate type 0/0 with $f(x)$ being replaced by $f(x) - f(x^*)$ and $g(x)$ by $x - x^*$. In this situation $f'(x^*)/g'(x^*) = f'(x^*)$.

Theorem 2 (Alternative characterizations of the derivative)**.** *Consider a continuously differentiable function f defined on an open set containing the interval $I = [a, b]$. Assume that the derivative of f satisfies the global Lipschitz condition on I and consider an interior point x^* of I. If $r = r(x^*)$ is an arbitrary number then the following statements are equivalent:*

(i) *$r = f'(x^*)$;*

(ii) *$|f(x) - f(x^*) - r \cdot (x - x^*)| \leq \Lambda \cdot (x - x^*)^2$ for some $\Lambda \geq 0$ and for every x in I;*

(iii) *$(1/\Lambda) \cdot |f(x) - f(x^*) - r \cdot (x - x^*)| \leq (x - x^*)^2$ for every $\Lambda > 0$ sufficiently large and for every x in I;*

(*iv*) *The ratio function* $|f(x) - f(x^*) - r \cdot (x - x^*)|/(x - x^*)^2$ *is uniformly bounded on* $I \backslash \{x^*\}$.

Applications of the derivative abound in almost every science. Let us give an application of Theorem 2 in mechanics. The instantaneous velocity of a moving particle, along a trajectory described by $f(x)$, is defined in the literature as the derivative $f'(x^*)$ [6]. This means that Theorem 2 gives equivalent formulations of the derivative without using differentiation. These formulations are introduced here globally over an interval I around x^* rather than locally at x^*.

Example 4. *A freely falling body follows the trajectory* $f(t) = -\frac{1}{2}\, g\, t^2$ *in time* t, *where* g *is the constant of gravity. (This neglects air resistance). We wish to determine its instantaneous velocity* $r = r(t^*)$ *at a time* t^* *during the fall using the statement* (*iv*). *A number* r *is the instantaneous velocity if and only if*

$$\begin{aligned}
&|f(t) - f(t^*) - r \cdot (t - t^*)|/(t - t^*)^2 \\
&\qquad = \big|\big(-\tfrac{1}{2}\, g(t + t^*) - r\big)/(t - t^*)\big| \text{ is uniformly bounded along } t \neq t^* \\
&\qquad = \big|\big(-\tfrac{1}{2}\, g(t + t^*) + g\, t\big)/(t - t^*)\big|,\ t \neq t^*, \text{ after substitution } r = -g\, t \\
&\qquad = \tfrac{1}{2}\, g.
\end{aligned}$$

The ratio is uniformly bounded which means that the number $r(t^*) = -g\, t^*$, *and only this number, is the instantaneous velocity at* t^*.

3. Augmentations of L'Hôpital's rule

L'Hôpital's rule for indeterminate form of type 0/0 says that (1) implies (2) if (2) exists. If f and g are continuously differentiable with $g'(x^*) \neq 0$ then (2) does not require limits since

$$\lim_{x \to x^*} \frac{f'(x)}{g'(x)} = \frac{f'(x^*)}{g'(x^*)}.$$

This means that L'Hôpital's rule reduces to checking the quotient of derivatives

$$\lim_{x \to x^*} \frac{f(x)}{g(x)} = \frac{f'(x^*)}{g'(x^*)}.$$

Characterizations of such quotients are given in Theorem 1. Hence we have the following result.

Theorem 3 (Differentiation-free augmentations of L'Hôpital's rule for indeterminate form of type 0/0)**.** *Consider continuously differentiable functions* f *and* g *defined on an open set containing the interval* $I = [a, b]$ *where* $g'(x) \neq 0$ *for* $x \in I$. *Assume that the derivatives of* f *and* g *satisfy the global Lipschitz condition on* I *and consider an interior point* x^* *of* I. *Also assume that* $f(x^*) = 0$ *and* $g(x^*) = 0$. *If* $r = r(x^*)$ *is an arbitrary number then the four statements:*

(*i*) $r = \lim_{x \to x^*} \dfrac{f(x)}{g(x)}$

(ii), (iii), and (iv) given in Theorem 1

are equivalent, i.e., any of them implies the other three.

We refer to *(ii)* as the "quadratic envelope formulation of L'Hôpital's rule", *(iii)* is its "normalized formulation", and *(iv)* is its "uniform bound formulation".

Example 5. *Consider $f(x) = \sin x$ and $g(x) = x$ on an interval I with $x^* = 0$ in its interior. We wish to determine*

$$r = \lim_{x \to x^*} \frac{f(x)}{g(x)}$$

without using differentiation. The formulations (ii)–(iv) are, respectively

$|\sin x - rx| \leq \Lambda x^2$ *for some* $\Lambda \geq 0$ *and for every* x *in* I;

$(1/\Lambda)|\sin x - rx| \leq x^2$ *for every* $\Lambda > 0$ *sufficiently large and for every* x *in* I;

function $\Phi(x) = |\sin x - rx|/x^2$ *is uniformly bounded on* $I \backslash \{0\}$.

These statements are satisfied if and only if, $r = 1$. Formulations (ii) and (iv) are depicted by Figs. 1–3 with the choice $\Lambda = 1/2$ in Figs. 1–2. Fig. 2 shows two violations for the incorrect limits $r = 0$ and $r = 2$. For these values of r, both

$$\Phi(x) = |\sin x|/x^2 \to \infty$$

and

$$\Phi(x) = |\sin x - 2x|/x^2 \to \infty$$

as $x \to x^*$. Therefore these two functions are not uniformly bounded on $I \backslash \{x^*\}$.

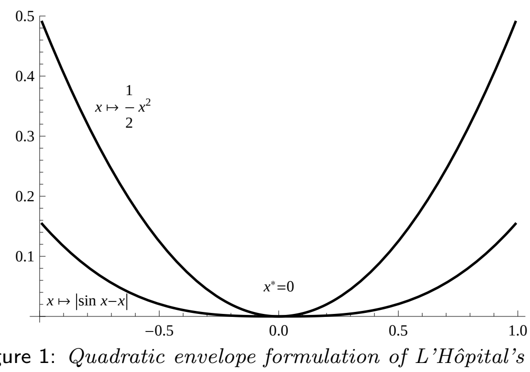

Figure 1: *Quadratic envelope formulation of L'Hôpital's rule*

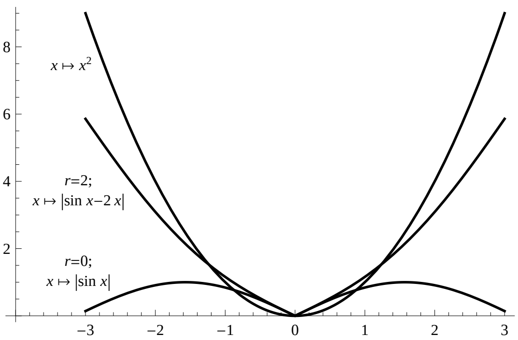

Figure 2: *Violations of the quadratic envelope formulation of L'Hôpital's rule*

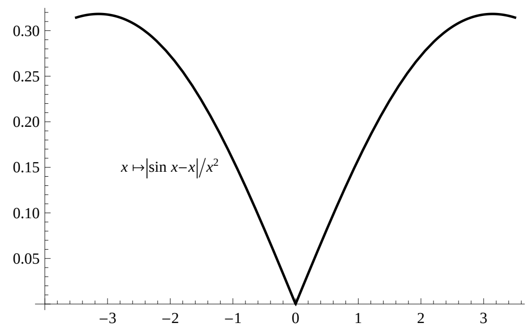

Figure 3: *Uniform bound formulation of L'Hôpital's rule*

4. Theorem of Lagrange

Consider two continuously differentiable scalar functions f and h defined on an open set in $\mathbb{R}^n$ containing a compact convex set K. Let us study the optimization problem

$$\text{Opt}\, f(x) \text{ subject to } h(x) = 0. \tag{4}$$

Assume that $x^* = (x_i^*)$ is a locally optimal solution of (4). Then the theorem of Lagrange says that there is a number λ such that $\nabla f(x^*) = \lambda \nabla h(x^*)$. Using partial derivatives this is

$$\frac{\partial f}{\partial x_i}(x^*) = \lambda\, \frac{\partial h}{\partial x_i}(x^*), \quad i = 1, \ldots, n.$$

Theorem 1 can be used to obtain additional information on the behaviour of f and h around x^* without using differentiation. First we "freeze" $n-1$ variables of f and h at x^* and look at these functions only in the remaining variable x_i. This introduces the following $2n$ functions of the single variable $\xi = x_i$:

$$f_i^*(\xi) = f(x_1^*, \ldots, x_{i-1}^*, \xi, x_{i+1}^*, \ldots, x_n^*), \quad i = 1, \ldots, n$$
$$h_i^*(\xi) = h(x_1^*, \ldots, x_{i-1}^*, \xi, x_{i+1}^*, \ldots, x_n^*), \quad i = 1, \ldots, n.$$

We consider only those partial derivatives of h for which $[h_i^*(\xi)]'(x_i^*) \neq 0$. Denote by N the index set $N = \{1, 2, \ldots, n\}$ and then $N^* = \{i \in N : [h_i^*(\xi)]'(x_i^*) \neq 0\}$.

Finally, assume that the derivatives of $f_i^*(\xi)$ and $h_i^*(\xi)$, $i \in N^*$, satisfy the global Lipschitz condition on some compact intervals in

$$K_i^* = \{\xi : (x_1^*, \dots, x_{i-1}^*, \xi, x_{i+1}^*, \dots, x_n^*)^{\mathrm{T}}\} \cap K$$

each containing x_i^* in its relative interior, $i \in N^*$. One can now use any of the four statements of Theorem 1 to reformulate the theorem of Lagrange. Let us use the statement (*iv*).

Theorem 4 (Theorem of Lagrange for a single constraint)**.** *Consider two continuously differentiable scalar functions f and h defined on an open set containing a compact convex set K in $\mathbb{R}^n$. Assume that K has a point $x^* = (x_i^*)$ in its topological interior where $\nabla h(x^*) \neq 0$. Also assume that the derivatives of f_i^* and h_i^*, $i \in N^*$, satisfy the global Lipschitz condition on some compact intervals K_i^* containing x_i^* in their interiors, $i \in N^*$. If x^* is a locally optimal solution of* (4) *then there is a number λ such that the ratio functions*

$$\left|f_i^*(\xi) - f(x_i^*) - \lambda\, h_i^*(\xi)\right| / (\xi - x_i^*)^2$$

are uniformly bounded on $K_i^* \backslash \{x_i^*\}$, $i \in N^*$.

Example 6. *Let us check, using Theorem 4, whether $x^* = (5,5)^T$ is a candidate for local optimality of $f(x) = x_1 \cdot x_2$ on the feasible set determined by $h(x) = x_1 + x_2 - 10 = 0$. We can specify, e. g., $K = \{(x_1,x_2)^T : -1 \le x_1, x_2 \le 6\}$. Here $N^* = \{1,2\}$, $f_1^*(\xi) = f_2^*(\xi) = 5\xi$, $h_1^*(\xi) = h_2^*(\xi) = \xi - 5$, $K_1^* = \{(\xi,5)^T : -1 \le \xi \le 6\}$ and $K_2^* = \{(5,\xi)^T : -1 \le \xi \le 6\}$. The theorem says that if x^* is a local optimum then there must exist a λ such that $|5\xi - 25 - \lambda(\xi-5)|/(\xi-5)^2$, i. e., $|\lambda - 5|/|\xi - 5|$, $\xi \neq 5$, is uniformly bounded. This is true if and only if $\lambda = 5$. We have found the Lagrange multiplier without using differentiation and we conclude that x^* is a candidate for local optimality. The uniform bound requirement is violated at, e. g., $x^* = (0,10)^T$ because in this case $10/|\xi| \to \infty$ as $\xi \to 0$. This point cannot be a local optimum.*

5. Conclusion

Using a differentiation-free characterization of zero derivative points we have characterized quotients of derivatives of continuously differentiable functions of the single variable on a compact interval. Our results recover alternative formulations of the derivative and yield augmentations of L'Hôpital's rule for an indeterminate form 0/0. We have also obtained differentiation-free reformulations of the theorem of Lagrange for functions in several variables with a single constraint.

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