

## On some properties of the Lyapunov equation for damped systems\*

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**Abstract.** *We consider a damped linear vibrational system whose dampers depend linearly on the viscosity parameter  $v$ . We show that the trace of the corresponding Lyapunov solution can be represented as a rational function of  $v$  whose poles are the eigenvalues of a certain skew symmetric matrix. This makes it possible to derive an asymptotic expansion of the solution in the neighborhood of zero (small damping).*

**Key words:** *Damped-vibration, Lyapunov equation, optimization of dampers' viscosities*

**AMS subject classifications:** 70J10, 34D20, 34C10, 78M50

Received April 6, 2004

Accepted October 2, 2004

### 1. Introduction

We consider a damped linear vibrational system

$$M\ddot{x} + C\dot{x} + Kx = 0 \quad (1)$$

where the matrices  $M, C, K$  (mass, damping, stiffness) are symmetric,  $M, K$  are positive definite and  $C$  is positive semidefinite. If internal damping is neglected,  $C$  has often a small rank as it describes a few dampers built in to calm down dangerous vibrations. Often  $C$  has the form

$$C = vC_0$$

where  $v$  is variable viscosity and  $C_0$  describes the geometry of a damper.  $C_0$  will have rank 1, 2 or 3 according to whether the damper can exhibit linear, planar or spatial displacements.

An example is the so-called *n-mass oscillator* or *oscillator ladder* (Figure 1) where

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\*A part of this work was written while the first author was a visiting researcher in the Lehrgebiet Mathematische Physik, Fernuniversität, Hagen

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$$M = \text{diag}(m_1, m_2, \dots, m_n) \quad (2)$$

$$K = \begin{bmatrix} k_1 & -k_1 & & & & \\ -k_1 & k_1 + k_2 & -k_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-2} & k_{n-2} + k_{n-1} & -k_{n-1} & \\ & & & -k_{n-1} & k_{n-1} + k_n & \end{bmatrix}, \quad (3)$$

$$C = v e_1 e_1^T + v(e_3 - e_2)(e_3 - e_2)^T. \quad (4)$$

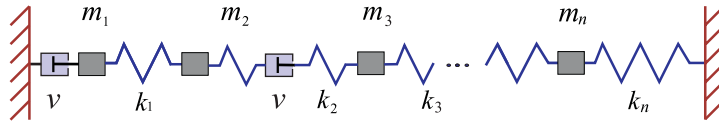


Figure 1. The  $n$ -mass oscillator with two dampers

Here  $m_i > 0$  are the masses,  $k_i > 0$  the spring constants or stiffnesses,  $e_i$  is the  $i$ -th canonical basis vector, and  $v$  is the viscosity of the damper applied on the  $i$ -th mass.

After the substitution

$$y_1 = \Omega \Phi^{-1} x, \quad y_2 = \Phi^{-1} \dot{x}, \quad (5)$$

where

$$K\Phi = M\Phi\Omega^2, \quad \Phi^T M\Phi = I \quad (6)$$

$$\Omega = \text{diag}(\omega_1, \dots, \omega_n), \quad \omega_1 < \dots < \omega_n \quad (7)$$

is the eigenreduction of the symmetric positive definite matrix pair  $K, M$ , the system (1) goes over into

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (8)$$

$$\mathbf{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -D \end{bmatrix}, \quad D = \Phi^T C \Phi. \quad (9)$$

Then

$$E = \frac{1}{2} \int_0^\infty (\dot{x}^T M \dot{x} + x^T K x) dt = \int_0^\infty \|\mathbf{y}\|^2 dt = \int_0^\infty \|e^{\mathbf{A}t} \mathbf{y}_0\|^2 dt, \quad (10)$$

where  $\mathbf{y}_0$  is the initial data. Thus,

$$E \equiv E(\mathbf{y}_0) = \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0,$$

where

$$\mathbf{X} = \int_0^{\infty} e^{\mathbf{A}^T t} e^{\mathbf{A} t} dt \quad (11)$$

solves the Lyapunov equation

$$\mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} = -\mathbf{I}. \quad (12)$$

Our penalty function is obtained by averaging  $E$  over all initial data with the equal energy, that is, we form

$$\bar{E} = \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 d\sigma \quad (13)$$

where  $d\sigma$  is a probability measure on the unit sphere in  $\mathbb{R}^{2n}$ . Since by the map

$$\mathbf{X} \mapsto \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 d\sigma$$

is given a linear functional on the space of the symmetric matrices, by the Riesz theorem there exists a symmetric matrix  $\mathbf{Z}$  such that

$$\mathbf{X} \mapsto \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 d\sigma = \text{Tr}(\mathbf{Z} \mathbf{X}), \text{ for all symmetric matrices } \mathbf{X}.$$

Let  $\mathbf{y} \in \mathbb{R}^{2n}$  be arbitrary. Set  $X = \mathbf{y} \mathbf{y}^T$ . Then

$$0 \leq \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 d\sigma = \text{Tr}(\mathbf{Z} \mathbf{X}) = \text{Tr}(\mathbf{Z} \mathbf{y} \mathbf{y}^T) = \text{Tr}(\mathbf{y}^T \mathbf{Z} \mathbf{y}),$$

hence  $\mathbf{Z}$  is always positive semi-definite.

For any given measure there is a unique positive semidefinite matrix  $\mathbf{Z}$  such that

$$\bar{E} = \text{Tr}(\mathbf{Z} \mathbf{X}). \quad (14)$$

For the measure  $\sigma$  generated by the Lebesgue measure (i.e. the usual surface measure) on  $\mathbb{R}^{2n}$ , we obtain  $\mathbf{Z} = \frac{1}{2n} \mathbf{I}$ . For the convenience of the reader, we give a sketch of the proof:

Recall,

$$\mathbf{Z}_{ij} = \int_S \mathbf{y}_i \mathbf{y}_j \sigma(d\mathbf{y}).$$

Using Minkowski formula (see [2]) one can easily see that

$$\mathbf{Z}_{ij} = \int_S \mathbf{y}_i \mathbf{y}_j \sigma(d\mathbf{y}) = \frac{1}{2\varepsilon} \lim_{\varepsilon \rightarrow 0} \int_{d(\mathbf{y}, S) \leq \varepsilon} \mathbf{y}_i \mathbf{y}_j \sigma(d\mathbf{y}),$$

here  $S$  denotes the unit sphere in  $\mathbb{R}^{2n}$  and  $d(\mathbf{y}, S)$  is a corresponding distance.

Obviously,  $\mathbf{Z}_{ij} = 0$  for  $i \neq j$  and  $\mathbf{Z}_{ii} = \mathbf{Z}_{jj}$ , for  $i, j \in \{1, 2, \dots, 2n\}$ . Since

$$\mathbf{Z}_{11} + \mathbf{Z}_{11} + \dots + \mathbf{Z}_{2n2n} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \text{vol}(\mathbf{y} \in \mathbb{R}^{2n} : d(\mathbf{y}, S) \leq \varepsilon) = 1,$$

it follows  $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$ . More details about the structure of the matrix  $\mathbf{Z}$  one can be found in [4].

We have shown that

$$\int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 \, d\sigma = \min \quad (15)$$

is equivalent to

$$\text{Tr}(\mathbf{Z}\mathbf{X}) = \min. \quad (16)$$

where  $\mathbf{Z}$  is a symmetric positive semidefinite matrix which may be normalized to have a unit trace.

If one is interested in damping a certain part of the spectrum of the matrix  $\mathbf{A}$  (which is very important in applications), then the matrix  $\mathbf{Z}$  will have a special structure. For example, let  $\sigma = \sigma_1 \times \sigma_2 \times \sigma_1 \times \sigma_2$ , where  $\sigma_1$  is a measure on the frequency subspace determined by  $\omega \leq \omega_{\max} \equiv \omega_s$  generated by the Lebesgue measure, that is,  $\sigma_1$  is a measure on the frequency subspace which corresponds to eigenfrequencies (defined by (7))  $\omega_1, \dots, \omega_s$  and  $\sigma_2$  is a Dirac measure on the complement. Then we obtain that the corresponding matrix  $Z$  has the form

$$\mathbf{Z}_s = \mathbf{Z} = \frac{1}{2s} \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (17)$$

where  $I_s$  is the identity matrix of the dimension  $s$  which is defined by  $\omega_{\max} = \omega_s$ . Here  $\omega_{\max} = \omega_s$  is a critical frequency with the property that the eigenfrequencies from (7) greater than  $\omega_s$  are not dangerous. Hence, we damp first  $s$  eigenfrequencies. The construction of  $\mathbf{Z}$  from (17) is a generalization of the simplest case  $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$ .

In [7] a simple solution of problem (14) has been presented for  $\mathbf{A}$  from (9) and  $\text{rank}(C) = 1$ . In particular,

$$\text{Tr}(\mathbf{Z}\mathbf{X}(v)) = \text{const} + \frac{a}{v} + bv, \quad a, b > 0, \quad (18)$$

which made it possible to find the minimum by a simple formula explicitly. The case  $\text{rank}(C) > 1$  seems to be essentially more difficult to handle.

The main result of this paper is the explicit formula:

$$\mathbf{X}(v) = \frac{\Psi_{-1}}{v} - \Psi_0 - v\widehat{\Psi}_1 + v \sum_{i=1}^s \frac{\lambda_i (\lambda_i \Phi_i - v\Upsilon_i)}{\lambda_i^2 + v^2}. \quad (19)$$

where  $\Psi_{-1}$ ,  $\Psi_0$ ,  $\widehat{\Psi}_1$ ,  $\Phi_i$  and  $\Upsilon_i$  are  $m \times m$  matrices and  $\lambda_i$  are eigenvalues of the pencil  $(\mathbb{A}_0, \mathbb{D})$  where  $\mathbb{A}_0$  and  $\mathbb{D}$  are matrices which correspond to the linear operators

$$\begin{aligned} \mathbf{X} &\mapsto -\mathbf{A}_0\mathbf{X} + \mathbf{X}\mathbf{A}_0, \\ \mathbf{X} &\mapsto \mathbf{D}\mathbf{X} + \mathbf{X}\mathbf{D}, \end{aligned}$$

respectively. Thus, technically, we have turned viscosity into the spectral parameter.

Further, we have obtained a simple formula for  $\Psi_{-1}$  in (19). This matrix is responsible for the behavior of the solution  $\mathbf{X}(v)$  in the neighborhood of zero (small damping):

$$\Psi_{-1} = \begin{bmatrix} \mathbf{D}_\Delta & 0 \\ 0 & \mathbf{D}_\Delta \end{bmatrix}, \quad (20)$$

where

$$\mathbf{D}_\Delta = (\text{diag}(\mathbf{D}))^{-1}. \quad (21)$$

We will use the following notation: matrices written in simple Roman fonts,  $M$ ,  $D$  or  $K$  for example will have  $n^2$  entries. Matrices written in mathematical bold fonts,  $\mathbf{A}$ ,  $\mathbf{B}$  will have  $m^2$  entries, where  $m = 2n$  (that is,  $\mathbf{A}$ ,  $\mathbf{B}$  are matrices defined on the  $2n$ -dimensional phase space). Finally, matrices written in Blackboard bold fonts  $\mathbb{A}$ , or  $\mathbb{D}$  will have more than  $m^2$  entries.

## 2. The main result

As we have said in the Introduction, our aim is to obtain the solution  $\mathbf{X}$  of the Lyapunov equation

$$\mathbf{A}(v)^T \mathbf{X} + \mathbf{X} \mathbf{A}(v) = -\mathbf{I}, \quad (1)$$

where  $\mathbf{Z}$  is defined by (17) and  $\mathbf{I}$  is an  $m \times m$  identity matrix.

From (9) it follows that  $\mathbf{A}(v)$  can be written as

$$\mathbf{A}(v) \equiv \mathbf{A}_0 - v \mathbf{D}, \text{ where } \mathbf{A}_0 = \begin{bmatrix} \Omega_1 & & & \\ & \Omega_2 & & \\ & & \ddots & \\ & & & \Omega_n \end{bmatrix}, \quad \Omega_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \quad (2)$$

and  $\mathbf{D} = \mathbf{D}_0 \mathbf{D}_0^T$ , where

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ d_{11} & d_{12} & \dots & d_{1r} \\ 0 & 0 & \dots & 0 \\ d_{21} & d_{22} & \dots & d_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ d_{n1} & d_{n2} & \dots & d_{nr} \end{bmatrix}, \quad (3)$$

$d_{ij}$  are entries of the matrix

$$\mathbf{D}_0 = \mathbf{P} \begin{bmatrix} 0 \\ L_C \end{bmatrix}, \quad C_0 = L_C L_C^T,$$

and  $\mathbf{P}$  is the ‘‘perfect shuffling’’ permutation.

Now, we proceed with solving equation (1). As it is well known, Lyapunov equation (1) is equivalent to ([3, Theorem 12.3.1])

$$(\mathbf{I} \otimes (\mathbf{A}_0 - v \mathbf{D})^T + (\mathbf{A}_0 - v \mathbf{D})^T \otimes \mathbf{I}) \cdot \text{vec}(\mathbf{X}) = -\text{vec}(\mathbf{I}), \quad (4)$$

where  $\mathbf{L} \otimes \mathbf{T}$  denotes the Kronecker product of  $\mathbf{L}$  and  $\mathbf{T}$ , and  $\text{vec}(\mathbf{I})$  is the vector formed by "stacking" the columns of  $\mathbf{I}$  into one long vector.

Further, we will need the following two  $m^2 \times m^2$  matrices defined by

$$\mathbb{A}_0 = \mathbf{I} \otimes \mathbf{A}_0^T + \mathbf{A}_0^T \otimes \mathbf{I}, \quad \mathbb{D} = \mathbf{I} \otimes \mathbf{D}_0 \mathbf{D}_0^T + \mathbf{D}_0 \mathbf{D}_0^T \otimes \mathbf{I}. \quad (5)$$

It is easy to show that  $\mathbb{D} = \mathbb{D}_F \mathbb{D}_F^T$ , where

$$\mathbb{D}_F = [I \otimes \mathbf{D}_0 \quad \mathbf{D}_0 \otimes I]. \quad (6)$$

Now, using (6) and (5) it follows that solution  $\text{vec}(\mathbf{X})$  of equation (4) can be written as

$$\text{vec}(\mathbf{X}) = -(\mathbb{A}_0 - v \mathbb{D}_F \mathbb{D}_F^T)^{-1} \text{vec}(\mathbf{I}). \quad (7)$$

Obviously, there exists a unitary matrix  $\mathbb{U}$  such that

$$\mathbb{U}^T \mathbb{A}_0 \mathbb{U} = \begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix}, \quad (8)$$

where  $\mathbb{A}_0$  is the skew-symmetric matrix, corresponds with the linear operator defined in (5) and  $\widehat{\mathbb{A}}_0$  is a non-singular block diagonal matrix defined by

$$\widehat{\mathbb{A}}_0 = \text{diag}(\Xi_1, \dots, \Xi_{m_2}) \quad \text{where} \quad \Xi_i = \begin{bmatrix} 0 & -\mu_i \\ \mu_i & 0 \end{bmatrix} \quad (9)$$

where  $\pm i\mu_i$ ,  $i = 1, \dots, m_2$  are non-zero eigenvalues of matrix  $\mathbb{A}_0$ , that is,  $0 \neq \mu_i = (\pm\omega_i) - (\pm\omega_j)$  for  $i, j = 1, \dots, n$  (see [3, Corollary 12.2.2]). Note that  $m_2 = (m^2 - m)/2$ . Set

$$\begin{bmatrix} \mathbb{D}_1 \\ \mathbb{D}_2 \end{bmatrix} = \mathbb{U}^T \mathbb{D}_F, \quad (10)$$

where  $\mathbb{D}_F$  is defined in (6).

Now,

$$\mathbb{A}_0 - v \mathbb{D}_F \mathbb{D}_F^T = \mathbb{U} \left( \begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix} - v \begin{bmatrix} \mathbb{D}_1 \mathbb{D}_1^T & \mathbb{D}_1 \mathbb{D}_2^T \\ \mathbb{D}_2 \mathbb{D}_1^T & \mathbb{D}_2 \mathbb{D}_2^T \end{bmatrix} \right) \mathbb{U}^T. \quad (11)$$

Taking

$$\mathbb{G} = \begin{bmatrix} I & \\ -\mathbb{D}_2 \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} & I \end{bmatrix} \quad (12)$$

we obtain

$$\mathbb{A}_0 - v \mathbb{D}_F \mathbb{D}_F^T = \mathbb{U} \mathbb{G}^{-1} \begin{bmatrix} -v \mathbb{D}_1 \mathbb{D}_1^T & \\ & \widetilde{\mathbb{A}} \end{bmatrix} \mathbb{G}^{-T} \mathbb{U}^T, \quad (13)$$

where

$$\widetilde{\mathbb{A}} = \widehat{\mathbb{A}}_0 - v \mathbb{D}_2 (I - \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} \mathbb{D}_1) \mathbb{D}_2^T. \quad (14)$$

Note that we can write

$$\mathbb{D}_2 (I - \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} \mathbb{D}_1) \mathbb{D}_2^T = \mathbb{F} \mathbb{F}^T. \quad (15)$$

Further, we have to find the inverse of  $\tilde{\mathbb{A}} = \hat{\mathbb{A}}_0 - v\mathbb{F}\mathbb{F}^T$ . This is obtained by using the *Sherman-Morrison-Woodbury formula* ([1, (2.1.4), p.51]), that is,

$$(\hat{\mathbb{A}}_0 - v\mathbb{F}\mathbb{F}^T)^{-1} = \hat{\mathbb{A}}_0^{-1} + v\hat{\mathbb{A}}_0^{-1}\mathbb{F} \left( I - v\mathbb{F}^T\hat{\mathbb{A}}_0^{-1}\mathbb{F} \right)^{-1} \mathbb{F}^T\hat{\mathbb{A}}_0^{-1}. \quad (16)$$

The inverse of  $I - v\mathbb{F}^T\hat{\mathbb{A}}_0^{-1}\mathbb{F}$  remains to be found. Let

$$\Lambda = \text{diag} \left( \begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \lambda_s \\ -\lambda_s & 0 \end{bmatrix} \right), \quad (17)$$

where  $\pm i\lambda_1, \pm i\lambda_2, \dots, \pm i\lambda_s$  are non-vanishing finite eigenvalues of the problem

$$(\mathbb{A}_0 - \lambda\mathbb{D}) \text{vec}(\mathbf{Y}) = 0. \quad (18)$$

Since  $\mathbb{F}^T\hat{\mathbb{A}}_0^{-1}\mathbb{F}$  is skew-symmetric, then there exists an orthogonal matrix  $U_S$  of order  $2(r-1)m$  such that

$$U_S^T \mathbb{F}^T \hat{\mathbb{A}}_0^{-1} \mathbb{F} U_S = \begin{bmatrix} 0 & 0 \\ 0 & \Gamma \end{bmatrix}, \quad (19)$$

where  $\Gamma = \Lambda^{-1}$ .

Using (13), (16), (19) and (7) it follows

$$\text{vec}(\mathbf{X}) = -\mathbb{U}\mathbb{G}^T \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \mathbb{G}\mathbb{U}^T \text{vec}(\mathbf{I}), \quad (20)$$

where

$$\Delta_1 = -\frac{(\mathbb{D}_1\mathbb{D}_1^T)^{-1}}{v}, \quad \Delta_2 = \hat{\mathbb{A}}_0^{-1} + v\hat{\mathbb{A}}_0^{-1}\mathbb{F}U_S \begin{bmatrix} I & 0 \\ (I - v\Gamma)^{-1} \end{bmatrix} U_S^T \mathbb{F}^T \hat{\mathbb{A}}_0^{-1}. \quad (21)$$

Since  $\Gamma$  is block diagonal we have

$$(I - v\Gamma)^{-1} = \text{diag} \left( \frac{1}{\lambda_1^2 + v^2} \begin{bmatrix} \lambda_1^2 & -v\lambda_1 \\ v\lambda_1 & \lambda_1^2 \end{bmatrix}, \dots, \frac{1}{\lambda_s^2 + v^2} \begin{bmatrix} \lambda_s^2 & -v\lambda_s \\ v\lambda_s & \lambda_s^2 \end{bmatrix} \right). \quad (22)$$

Using (21) and (20) it follows

$$\text{vec}(\mathbf{X}) = \left( \frac{\mathbb{V}_{-1}}{v} - \mathbb{V}_0 - v\hat{\mathbb{V}}_1 + v \sum_{i=1}^s \frac{\lambda_i (\lambda_i \mathbb{W}_i - v\mathbb{Z}_i)}{\lambda_i^2 + v^2} \right) \text{vec}(\mathbf{I}), \quad (23)$$

where matrices  $\mathbb{V}_{-1}, \mathbb{V}_0, \hat{\mathbb{V}}_1, \mathbb{W}_i$  and  $\mathbb{Z}_i$  are constructed using (12), (21) and (20).

By "reshaping" vectors in (23) back into  $m \times m$  matrices we obtain the solution of equation (1)

$$\mathbf{X} = \frac{\Psi_{-1}}{v} - \Psi_0 - v\hat{\Psi}_1 + v \sum_{i=1}^s \frac{\lambda_i (\lambda_i \Phi_i - v\Upsilon_i)}{\lambda_i^2 + v^2}. \quad (24)$$

If one is interested in deriving an optimal damping, then according to (14) one has to minimize the function  $f(v) = \text{tr}(\mathbf{Z}\mathbf{X}(v))$ , where

$$\text{tr}(\mathbf{Z}\mathbf{X}(v)) = \text{vec}(\mathbf{Z})^T \text{vec}(\mathbf{X}) = -\text{vec}(\mathbf{Z})^T (\mathbb{A}_0 - v\mathbb{D}_F\mathbb{D}_F^T)^{-1} \text{vec}(\mathbf{I}). \quad (25)$$

This gives

$$\text{Tr}(\mathbf{Z}\mathbf{X}) = \frac{X_{-1}}{v} - X_0 - v\hat{X}_1 + v \sum_{i=1}^s \frac{\lambda_i (\lambda_i X_i - v Y_i)}{\lambda_i^2 + v^2}, \quad (26)$$

where

$$\begin{aligned} X_{-1} &= \text{vec}(\mathbf{Z})^T \mathbb{V}_{-1} \text{vec}(\mathbf{I}), \\ X_0 &= \text{vec}(\mathbf{Z})^T \mathbb{V}_0 \text{vec}(\mathbf{I}), \\ \hat{X}_1 &= \text{vec}(\mathbf{Z})^T \hat{\mathbb{V}}_{-1} \text{vec}(\mathbf{I}), \end{aligned} \quad (27)$$

and

$$X_i = \text{vec}(\mathbf{Z})^T \mathbb{W}_i \text{vec}(\mathbf{I}), \quad Y_i = \text{vec}(\mathbf{Z})^T \mathbb{Z}_i \text{vec}(\mathbf{I}). \quad (28)$$

Note that the function  $\text{Tr}(\mathbf{Z}\mathbf{X})$  from (26) is a generalization of the function  $\text{Tr}(\mathbf{X})$  defined in (18).

**Remark 1.** *In the case when  $\mathbf{Z}$  is a diagonal matrix, the function  $\text{Tr}(\mathbf{Z}\mathbf{X})$  from (26) has the following simpler form:*

$$\text{Tr}(\mathbf{Z}\mathbf{X}) = \frac{X_{-1}}{v} - v\hat{X}_1 + v \sum_{i=1}^s \frac{\lambda_i^2 X_i}{\lambda_i^2 + v^2}.$$

Finally, we derive an explicit formula for the matrix  $\Psi_{-1}$ . Let  $\mathbb{U}_0$  be that part of  $\mathbb{U}$  corresponding to the null-space of  $\mathbb{A}_0$ . From (23) it follows that

$$\text{vec}(\Psi_{-1}) = \mathbb{V}_{-1} \text{vec}(\mathbf{I}) \equiv -\mathbb{U}_0(\mathbb{D}_1\mathbb{D}_1^T)^{-1}\mathbb{U}_0^T \text{vec}(\mathbf{I}). \quad (29)$$

It is easily seen that the  $i$ -th column of the matrix  $\mathbb{U}_0$  can be written as  $\text{vec}(\mathbf{O}_i)$ , where  $\mathbf{O}_i$  is a block-diagonal matrix and

$$\begin{aligned} \mathbf{O}_{2i-1} &= \text{diag}(0, \dots, 0, O_{ii}, 0, \dots, 0) \\ \mathbf{O}_{2i} &= \text{diag}(0, \dots, 0, \hat{O}_{ii}, 0, \dots, 0) \end{aligned}$$

where  $O_{ii}$  and  $\hat{O}_{ii}$  are "orthogonal" solutions of

$$-\Omega_i O_{ij} + O_{ij} \Omega_j = 0 \quad i, j = 1, \dots, n, \quad (30)$$

that is

$$O_{ii} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{and} \quad \hat{O}_{ii} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \quad (31)$$

Let  $\langle A, B \rangle = \text{Tr}(A^T B)$  be the usual Frobenius scalar product. Then, we can write the  $(p, q)$ -th element of  $\mathbb{U}_0^T \mathbb{D}_F \mathbb{D}_F^T \mathbb{U}_0$  as

$$(\mathbb{U}_0)_{(:,p)}^T \mathbb{D}_F \mathbb{D}_F^T (\mathbb{U}_0)_{(:,q)} = \langle \mathbf{D}_{pq} \mathbf{O}_p + \mathbf{O}_p \mathbf{D}_{pq}, \mathbf{O}_q \rangle \quad p, q = 1, \dots, m, \quad (32)$$



where  $\mathbf{D}_{pq} = (D_{pq})$  and

$$D_{pq} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}_{(p,:)} \mathbf{D}_{(q,:)}^T \end{bmatrix}. \quad (33)$$

The orthonormality property

$$\langle \mathbf{O}_p, \mathbf{O}_q \rangle = \delta_{pq},$$

implies

$$(\mathbf{U}_0^T \mathbb{D}_F \mathbb{D}_F^T \mathbf{U}_0)^{-1} = \text{diag}(1/(\mathbf{D})_{22}, 1/(\mathbf{D})_{22}, \dots, 1/(\mathbf{D})_{mm}, 1/(\mathbf{D})_{mm}). \quad (34)$$

Using the fact that  $\mathbf{U}_0 \mathbf{U}_0^T \text{vec}(\mathbf{I}) = \text{vec}(\mathbf{I})$ , from (29) it follows that

$$\Psi_{-1} = \text{diag} \left( \frac{1}{\mathbf{D}_{22}}, \frac{1}{\mathbf{D}_{22}}, \dots, \frac{1}{\mathbf{D}_{mm}}, \frac{1}{\mathbf{D}_{mm}} \right). \quad (35)$$

After applying a perfect shuffle permutation we have

$$\Psi_{-1} = \begin{bmatrix} \mathbf{D}_\Delta & 0 \\ 0 & \mathbf{D}_\Delta \end{bmatrix}, \quad (36)$$

where

$$\mathbf{D}_\Delta = (\text{diag}(\mathbf{D}))^{-1}. \quad (37)$$

The explicitness of the obtained formulas is attractive for possible numerical computation. Our first attempts to perform this task did not succeed due to unexpected complexity problems. We will come back to this issue in our future research.

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