TOLERANCE SENSITIVITY ANALYSIS: THIRTY YEARS LATER

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Abstract

Tolerance sensitivity analysis was conceived in 1980 as a pragmatic approach to effectively characterize a parametric region over which objective function coefficients and right-hand-side terms in linear programming could vary simultaneously and independently while maintaining the same optimal basis. As originally proposed, the tolerance region corresponds to the maximum percentage by which coefficients or terms could vary from their estimated values. Over the last thirty years the original results have been extended in a number of ways and applied in a variety of applications. This paper is a critical review of tolerance sensitivity analysis, including extensions and applications.

Key words: *Optimization, Sensitivity, Tolerance*

1. INTRODUCTION

Sensitivity analysis is an important element in making decisions and research on it continues to be very active (e.g., see Filippi [2005], Gal and Greenberg [1997], Hadigheh and Terlaky [2006], Hladik [2008a, 2008b, 2008c, 2008d, 2010a, 2010b], Jansen, Jong and Terlaky [1997], Koltai and Terlaky [2000], Wendell [2004] and Zlobec [2001a, 2001b]). Herein we explore one stream of research called tolerance sensitivity analysis, which was conceived in 1980 and proposed in the early to mid 1980s by Wendell [1982, 1984a, 1985]. As originally developed, the approach computes a maximum percentage, called the maximum tolerance percentage, within which coefficients and terms of a linear program may vary simultaneously and independently over an a priori set while preserving the same optimal basis. This is illustrated in the following example from Wendell [1984a]:

Example:

$$
\begin{aligned}
&\text{Max: } 12x_1 + 20x_2 + 18x_3 + 40x_4 \\
&\text{s.t.} \\
&4x_1 + 9x_2 + 7x_3 + 10x_4 + x_5 = 6000 \\
&x_1 + x_2 + 3x_3 + 4x_4 + x_6 = 4000 \\
&x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

The optimal basic feasible solution is: x_1*=4000/3, x_4*=200/3, x_2*=x_3*=x_5*=x_6*=0.

Some tolerance results are:

(i) With no a priori limits on the variability of the objective function coefficients, the maximum tolerance is 8.47%; thus each coefficient of objective function can vary simultaneously and independently up to 8.47% of its estimated value while still retaining the same optimal basis.

(ii) If a decision maker specifies that the coefficients of x_1 will not vary outside the interval [11.5, 12.5], then the maximum tolerance is 12.93%.

(iii) Alternatively, if a decision maker can specify that the coefficient of x_1 is precisely 12, then the maximum tolerance equals 17.24%.

Since publication of the original tolerance work, a number of papers have been written extending and applying the approach. Herein we review this work using a general framework to help understand the interrelationships among their contributions. For simplicity, we primarily focus on variations in the objective function coefficients.

The paper proceeds as follows: section 2 proposes a general framework for tolerance analysis of objective function coefficients. Then section 3 reviews results based on the original tolerance formulations. Section 4 considers alternative tolerance formulations and extensions of the original tolerance approach for linear problems. Section 5 concludes with a brief discussion of other work, including variations in the constraints, multi-objective problems, applications and extensions to nonlinear problems.

2. A GENERAL TOLERANCE FRAMEWORK

In order to understand the interrelationships of much of the work in tolerance sensitivity, we now present a general tolerance framework for perturbations in the objective function.

Consider the optimization problem:

$$Max\{f(x,\gamma): x \in X\} \tag{1}$$

where $X \subseteq R^n$ is a given constraint set and $f(x,\gamma)$ is a function on X with parameters $\gamma \in R^s$.

Let Γ, called an a priori parametric region, denote a set within y may vary. We assume that $0 \in \Gamma$ and let x^* denote a feasible solution to (1). Suppose that when $\gamma = 0$ the solution x^* satisfies some specified optimality condition, often characterized as a set of equations and/or inequalities in x and γ. For example, when (1) is a linear programming problem and x^* is a basic feasible solution, the optimality condition could be that "$c_j - z_j \leq 0$" holds for the nonbasic variables of x^* when $\gamma = 0$. Of course, such a condition may hold for γ beyond $\gamma = 0$. Let $P(x^*)$, called a critical region, denote the set of γ for which the specified optimality condition holds. A basic question in sensitivity analysis is: How much can γ deviate from 0 within Γ and still remain in the critical region $P(x^*)$? Tolerance sensitivity analysis is an approach to answer this question in the case when "how much" is determined by the size of a hyperbox within which the parameters γ can vary simultaneously and independently over Γ.

To characterize this approach, let $u \in R^{2s}$ be nonnegative ($u \geq 0$) and consider a hyperbox $B(u)$ defined as

$$B(u) = \{\gamma : -u_i \leq \gamma_i \leq u_{s+i}, \quad for\ i = 1,2,\cdots,s\}$$

Let $\rho(u)$ denote a function that characterizes the size of the hyperbox. Given a function $\rho(u)$, the tolerance problem can be posed as:

$$Max\{\rho(u) : \gamma \in \Gamma \cap B(u) \Rightarrow \gamma \in P(x^*)\} \tag{2}$$

where "$\Rightarrow$" denotes a "if-then" implication. If u denotes a feasible solution to (2), observe that the parameters γ may vary simultaneously and independently within the region $B(u) \cap \Gamma$ while preserving the specified optimality condition for x^*.

3. ORIGINAL LP TOLERANCE RESULTS

Wendell [1982, 1984a, 1985] considered problem (1) when $f(x,\gamma) = (\hat{c} + \gamma G)x$, where $\hat{c}$ denotes the estimated value of the coefficients and G is a s x n matrix. In particular, the papers considered the case when G is a n x n diagonal matrix including the following two special cases:

Additive Perturbations – when for each j either $G_{jj} = 1$ or 0;

Multiplicative Perturbations – when for each j either $G_{jj} = |\hat{c}_j|$ or 0.

Observe that multiplicative perturbations enable us to express variations from estimated coefficients $\hat{c}$ as a percentage. Using such percentage deviations is intuitively appealing in practice. Also note that having $G_{jj} = 0$ is a convenient way to characterize cases when no variations are being considered.

Let X = $\{ x : Ax = b, x \geq 0 \}$ and assume that X is nonempty and bounded with A being full row rank.

Let x^* denote an optimal basic feasible solution when $\gamma = 0$ with B_0 denoting the corresponding basis.

Let J_0 and N_0 denote the corresponding index set of the basic and non-basic variables respectively. Using the index sets, we can partition vectors and matrices by writing c_{J_0}, c_{N_0}, A_{N_0}, etc. To simplify notation, let $H_j^{\le} = \{\gamma : \hat{c}_j + \gamma G_j - \hat{c}_{j_0} B_0{}^{-1} A_j - \gamma_{j_0} G B_0{}^{-1} A_j \le 0\}$ and let $H_j^{=}$ denote the corresponding set when "≤" is replaced by "=".

Given the optimality condition for x^* that the basis B_0 is optimal, we have $P(x^*) = \bigcap_{j \in N_0} H_j^{\le}$, corresponding to the condition "$c_j - z_j \le 0$" for $j \in N_0$. Observe that $P(x^*)$ is polyhedral. The original tolerance approach can be viewed as setting all components of u equal to one number, which we call a tolerance and denote as τ. In other words, the hyperbox can be characterized as the set $\{\gamma : \|\gamma\|_\infty \le \tau\}$, where $\|\cdot\|_\infty$ is the Tchebycheff norm and the tolerance τ denotes the size of the hyperbox. Problem (2) corresponds to finding the largest tolerance τ, denoted as τ*, where the optimality condition $\gamma \in P(x^*)$ holds for all $\gamma \in \Gamma$ when $\|\gamma\|_\infty \le \tau$. Thus, the maximum tolerance corresponds to a symmetric tolerance hyperbox B($\bar{u}$) where $\bar{u}_i = \tau^*$ for all i. A key part in the approach for solving (2) in this case is to explore the fact that $P(x^*)$ is polyhedral. Specifically, the approach decomposes (2) into separate, simpler problems of considering one halfspace at a time. That is, for each $j \in N_0$, the approach computes

$$\tau_j = \sup\{\tau : \|\gamma\|_\infty \le \tau, \gamma \in \Gamma \Rightarrow \gamma \in H_j^{\le}\}.$$

The following theorem from Wendell [1997] summarizes the original tolerance results in Wendell [1982,1984a, 1985].

Theorem

(i) If N_0 is empty then $\tau^* = \infty$. Otherwise, the number τ^* (possibly infinity) equals $\mathrm{Min}_{j \in N_0} \tau_j$

(ii) The number τ_j for $j \in N_0$ is infinite iff $\sup\{\gamma G_{\cdot j} - \gamma_{J_0} G_{B_0^{-1}} \hat{A}_{\cdot j} : \gamma \in \Gamma\} \le \hat{c}_{J_0} B_0^{-1} \hat{A}_{\cdot j} - \hat{c}_j$

(iii) If $j \in N_0$ & τ_j is finite then $\tau_j = Min\{\|\gamma\|_\infty : \gamma \in \Gamma \cap H_j^{=}\}$

(iv) If $j \in N_0$ & $\Gamma = R^n$ and if G is diagonal with $G_{jj} = c'_j$ then

$$\tau_j = \frac{\hat{c}_{J_0} B_0^{-1} \hat{A}_{\cdot j} - \hat{c}_j}{|c'_j| + \sum_{i=1}^{m} |c'_{j_i} B_{\cdot i}^{-1} \hat{A}_{\cdot j}|}$$

where $B_{i\bullet}^{-1}$ denotes the i[th] row of B_0^{-1}.

Observe that the supremum in (ii) is a linear program. In some important special cases, testing the condition (ii) is easy (see Wendell [1984a]). In particular, when $\Gamma = R^s$ the condition (ii) holds iff the coefficients of γ in the supremum are all zero. Note that (iii) is an optimization problem which (by using standard absolute value tricks, etc.) can be reformulated as a linear program. Also, in some important special cases solving (iii) is easy. In particular, when Γ is a hyperbox in that it simply consists of lower and upper bounds on each component of γ, then τ_j can be computed using a simple relaxation procedure (see Wendell [1984a]). Observe that in the important special case given in part (iv) of the above theorem, the value of τ_j can be written in closed form.

An advantage of the original tolerance approach is the ease with which a decision-maker can interpret its results. Namely, the permitted variability of the coefficients from their respective estimated values while retaining the same optimal basis can be expressed as a single percentage. Furthermore, the approach can exploit information about allowable ranges of variations in the coefficients to yield larger maximum tolerance percentages. The approach also gives the values of γ at which the maximum tolerance τ^* is attained. Obtaining a priori information that precludes such values from occurring can yield a larger maximum tolerance, as illustrated in the example.

A disadvantage with the approach is that for moderate or large size problems the maximum tolerance may often be at or near zero. The reason for this is the tendency for such problems to have alternative or near-alternative optimal bases. We will see how the original tolerance approach has been extended to address this issue.

4. SUBSEQUENT TOLERANCE RESULTS IN LP

One way to deal with the issue of having the maximum tolerance at/near zero is to extend the hyperbox B($\bar{u}$) by permitting it to be nonsymmetrical. Arsham and Oblak [1990], Wondolowski [1991] and Wendell [1992] proposed a procedure for doing this in the case when there is no a priori information on γ. This procedure can be viewed as solving (2) when $\rho(u) = min\{u_1, u_2, \cdots, u_{2s}\}$. It amounts to solving (2) for each $i = 1, 2, \ldots, 2s$ with $\rho(u) = u_i$ where $u_{i+s} = 0$ when $i = 1, 2, \ldots, s$ and where $u_{i-s} = 0$ when $i = s+1, s+2, \ldots, 2s$. The resultant limits are denoted by u' which characterizes a hyperbox B(u'). Since these limits are calculated independently, the numbers u' are sometimes called individual tolerances. The hyperbox B(u') is an extension of B($\bar{u}$) in that $u' \geq \bar{u}$. This extension is illustrated in Figure 1. As noted in Wendell [1992], the extended hyperbox can be viewed as an extension of ordinary sensitivity analysis.

Also indicated in Figure 1 is the fact that the hyperbox B(u') can sometimes be further extended within *P(x*)*. An efficient algorithm for doing this was recently proposed by Filippi [2005] in the case when the a priori set Γ is a hyperbox. Filippi's algorithm, which is an extension of the relaxation procedure in Wendell [1984], grows the hyperbox by interactively fixing those values of u that hit a hyperplane $H_j^=$ or a bound of Γ and then lets the others extend further until they hit a hyperplane or a bound of Γ (or, of course, until they extend to infinity). This can be viewed as solving a nested set of problems (2) as follows.

Let S = {1, 2, …, 2s} and let B(u) be a symmetric hyperbox. Now let h denote the index of u_i such that u_h^* is an optimal solution to (2) when $\rho(u) = Min\{u_i : i \in S\}$. Now iteratively add the condition to B(u) that $u_h = u_h^*$, delete h from the set S, and solve (2) as above. Filippi proves that the algorithm yields a hyperbox B(u*) where $u^* \geq u'$ and that is maximal (in that no other hyperbox preserving the condition in (2) contains it as a proper subset).

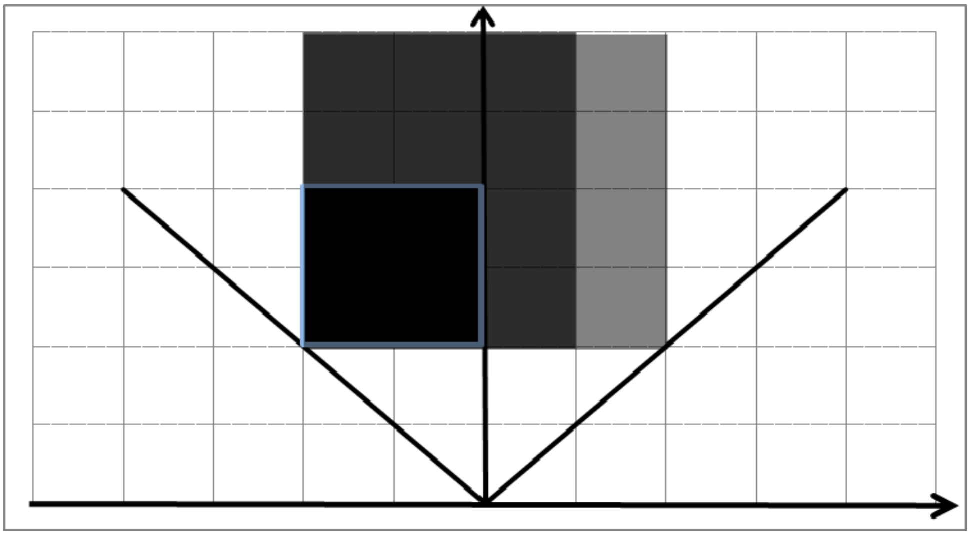

Figure 1: Tolerance Regions: the dark shading denotes B($\bar{u}$), the medium shading denotes points added for B(u'), and the light shading denotes points added for B(u*)

A somewhat different approach to obtain a tolerance region was proposed by Wang and Huang [1993]. Specifically, they assume a partial symmetry in that $u_i = u_{s+i}$ for each i and maximize volume by letting ρ(u) = $\prod_{i=1}^{s} u_i$. Furthermore, their approach does not include the possibility of having a prior information. Critiques of this approach include: a lack of an intuitive rationale for the maximizing volume objective (see Wendell [1997]); and the need to solve a nonlinear optimizition problem to perform sensitivity analysis on an optimal basis of a linear program (see Filippi [2005]).

Rather than letting P(x*) denote the set where the same basis remains optimal, as noted in Ward and Wendell [1990], one could choose P(x*) as an optimal coefficient set corresponding to those γ for which x^* remains an optimal solution to (1). Ward and Wendell [1990] note that the optimal coefficient set is polyhedral when (1) corresponds to a linear programming problem and that it is conceptually easy to extend the tolerance approach to consider the optimal coefficient set (presuming that the set could be characterized by inequality constraints) . While considering tolerances with respect to the optimal coefficient set is appealing, as noted by Hladik [2008d] characterizing the polyhedral set may be time consuming and computing even a symmetric tolerance with respect to an optimal coefficient set is an NP-hard problem. Also, as noted by

Filippi [2005], it is "a heavy computational task" in which "the computational effort is not justified by the obtained results". Other characterizations of $P(x^*)$ have been considered. For objective function perturbations in linear programming, Hladik [2008d, 2010b] shows that $P(x^*)$ under support set invariance is exactly the optimal coefficient set, and he considers extensions to optimal partition invariance.

An approach for dealing indirectly with the optimal coefficient set and more generally with almost optimal solutions was recently proposed by Wendell [2004]. Here, using the original symmetric characterization of tolerance τ as $\{\gamma : \|\gamma\|_\infty \le \tau\}$, the approach considers a maximum regret function $\alpha^*(\tau)$ defined as Max $\{\varphi(\gamma) - cx^* : \|\gamma\|_\infty \le \tau, \gamma \in \Gamma\}$, where $c = \hat{c} + \gamma G$, where x^* is an optimal basic feasible solution when $\gamma = 0$, and where $\varphi(\gamma)$ is the optimal value function of (1) in the linear case. Observe that the function $\alpha^*(\tau)$ corresponds to a potential loss of optimality of x^* for $\gamma \in \Gamma$ and for $\|\gamma\|_\infty \le \tau$ in that $\alpha^*(\tau) \ge \varphi(\gamma) - cx^*$.

Of course, for $\tau \le \tau^*$ we have that $\alpha^*(\tau) = 0$. For computing $\alpha^*(\tau)$ beyond τ^*, Wendell [2004] gives a ranking procedure that enables this to be done easily for small value of τ, but which approaches completes enumeration of all basic feasible solutions for large value of τ. As Wendell's paper notes "because only the first part of the maximum regret curve (corresponding to small value of τ) would usually be required in practice, the computational effect to compute it should be modest. Graphs of $\alpha^*(\tau)$ are given in Figure 2 for the three cases in the example and illustrate how a priori information about variability in the parameters can be explored. As noted by Wendell, $\alpha^*(\tau)$ is convex when $\Gamma = R^s$ (as in Figure 2), but in general may be neither convex nor concave. Interestingly, it can be shown that the inverse function of $\alpha^*(\tau)$ corresponds to the optimal objective value of (2) when $B(u)$ is the symmetric tolerance hyperbox for a tolerance τ and when $P(x^*) = \{\gamma: \varphi(\gamma) - cx^* \le \alpha\}$. Note that this corresponds to x^* being approximately optimal.

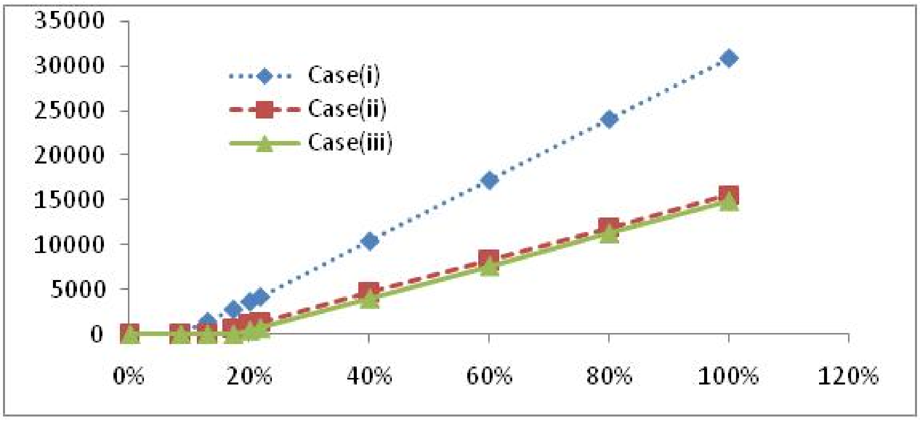

Figure 2: Maximum Regret Curves for the Example

5. OTHER RESULTS AND OBSERVATIONS

One attractive application of the tolerance approach is in multiple objective problems. An early application of it was given by Wendell [1984b] to sensitivity of the weights in goal programming. Hansen, Labbé, and

Wendell [1989] considered a weighted-sum formulation of a linear multiple objective problem in that an overall objective, corresponding to a weighted sum of the multiple objectives, is optimized. The original LP tolerance results were extended to consider perturbations in the weights, including general LP characterizations for finding the maximum tolerance and an easy procedures for calculating it when a priori information on the weights corresponds to a hyperbox (namely, when specific lower and upper bounds could be specified for each weight). These latter results on the hyperbox were subsequently extended by Marmol and Puerto [1997] by expanding the concept of a hyperbox to include other meaningful relations (e.g., one weight is at least as large as another and/or lower bounds on the differences between weights). Subsequently, Borges and Antunes [2002] presented a visual interactive approach to tolerance analysis in multiple objective linear programming that can be used "to visualize dynamically the changes of the tolerance region and study distinct sets of estimated weights as well as additional information". More recently, Hladik [2008a, 2008c] directly considers the multiple objective linear programming problem and extends the tolerance approach for considering changes in the coefficients in the objective functions while preserving the efficiency of a given solution.

The tolerance approach has been extended to deal with right-hand side terms (e.g., see Wendell [1984a, 1985]) and matrix coefficients (e.g., see Ravi and Wendell [1985, 1989], Wang and Huang [1992]) and Hladik [2008b]). It has also been applied to special applications of linear programming by exploiting the special structure of the problem. This includes, in particular, work on the transportation problem (e.g., see Arsham [1992], Doustdargholi, Asl, and Abasgholipour [2009], Ravi and Wendell [1988]). Also, Labbé, Thisse and Wendell [1991] consider an application to facility location. In Data Envelopment Analysis (DEA) one approach (e.g., see Charnes, Haag, Jaska and Semple [1992]) using the Tchebycheff norm to characterizing variations of an inefficient DMU while remaining inefficient is in effect tolerance sensitivity analysis. More recently, Singh [2010] discussed the application of the maximum volume characterization of the tolerance approach to DEA.

Finally, we note that some extensions and related work for nonlinear problems have been considered. In contrast to the characterization in Wendell [2004] of regret in a linear problem as a difference in optimality, Oguz [2000] gives a simple bound for problems with a linear objective over any closed, bounded, nonempty set. Since the feasible set need not be convex, the problem class for this bound includes a wide range of problems like the TSP, knapsack problems, etc. Other related work in nonlinear problems includes fractional programming (see Gupta and Singh [2006] and Hladik [2010a]) and sum-of-ratios programming (see Singh and Gupta [2010]).

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