Taxes, subsidies and unemployment - a unified optimization approach

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Abstract

Like a linear programming (LP) problem, linear-fractional programming (LFP) problem can be usefully applied in a wide range of real-world applications. In the last few decades a lot of research papers and monographs were published throughout the world where authors (mainly mathematicians) investigated different theoretical and algorithmic aspects of LFP problems in various forms. In this paper we consider these two approaches to optimization (based on linear and linear-fractional objective functions on the same feasible set), compare results they lead to and give interpretation in terms of taxes, subsidies and manpower requirement. We show that in certain cases both approaches are closely connected with one another and may be fruitfully utilized simultaneously.

Key words: *Linear programming, Linear-fractional programming, conflicting economic interests, redirection, unemployment*

1. INTRODUCTION

Problems of linear programming arise in a wide range of real-world applications, for example in the case when there appears a necessity to optimize results of some activity: profit gained by a company, cost of production, cost of transportation, etc. Problems of linear-fractional programming appear in the same cases as LP problems (Bajalinov, 2003), (Martos, 1964), (Stancu-Minasian, 1997), but in contrast to LP, in LFP objective functions are of fractional type: profit gained by company per unit of expenditure, cost of production per unit of produced goods, cost of transportation per unit of transported goods, etc. Often such objective functions (linear and linear-fractional) appear on the same feasible set (i.e. subject to the same constraints) and express two different economic interests.

It is well known that two or more objective functions defined on the same feasible set in general case lead to different (non-coincident) optimal solutions. It means that the economic interests expressed by linear objective function and linear-fractional objective function on the same feasible set in general case result different optimal solutions (and, hence, different decisions) which may conflict with one another. However sometimes it may occur that these objective functions lead to the same (coincident) optimal solutions (Bajalinov, 1999).

Below we investigate all such possible cases and try to give suitable economic interpretations.

Consider the following linear programming and linear fractional programming problems

$$P(x) \to max \ , \ x \in S \tag{1}$$

$$Q(x) \to max \ , \ x \in S \tag{2}$$

where Q(x)=P(x)/C(x), P(x)= $\sum_{j=1}^{n} p_j x_j + p_0$, C(x)= $\sum_{j=1}^{n} c_j x_j + c_0$ are affine functions and C(x) > 0 for all

$x \in S=\{x \in R^n: Ax \le b, x \ge 0\}$, A is (m x n) matrix, i.e. $A = |a_{ij}|_{mxn}$, $x = (x_1, x_2, \cdots, x_n)^T$, $b = (b_1, b_2, \cdots, b_n)^T$,

a_{ij}, b_i, p_j, c_j are scalar constants and T denotes the transpose of a vector. Note that constants p_0 and c_0 do not play a principal role in LP case but may affect optimality in LFP case. Here and in what follows we suppose that these problems are solvable, i.e. feasible set S is not empty and objective functions P(x) and Q(x) on set S are bounded from above.

2. NON-COINCIDENT SOLUTIONS

Let us consider the case when problems (1) and (2) have different optimal solutions. Let vector x* be an optimal solution of LP problem (1) and x' denote an optimal solution of LFP problem (2). Obviously, in this case we have the following inequalities:

$$P(x^*) \ge P(x') \tag{3}$$

$$Q(x^*) \le Q(x') \tag{4}$$

The theoretical sense of these two inequalities (in terms of mathematical programming theory) is obvious:

1. since vector x* maximizes linear objective function P(x) on feasible set S, it means that $P(x^*) \ge P(x)$, for all $x \in S$, including vector $x' \in S$;
2. analogously, since vector x' maximizes linear-fractional objective function Q(x) on feasible set S, it means that $Q(x') \ge Q(x)$, for all $x \in S$, including vector $x^* \in S$.

But what do they mean these two inequalities in a practical aspect? To give an answer to this question let us suppose that there is a company which would like to optimize its activity using LP model (1) and LFP model (2). Let linear function C(x) express the cost of the company and function P(x) be a profit function. It is clear that in this case linear-fractional function Q(x)=P(x)/C(x) may be interpreted as efficiency expressed as *profit/cost*. So, if the company prefers to maximize its profit it has to organize its activity in accordance with optimal plan x* which provides P(x*) units of profit, but may not provide maximal efficiency Q(x'), since $Q(x^*) \le Q(x')$. On the other hand, if company prefers to maximize its efficiency (calculated as profit/cost) it has to organize its activity in accordance with optimal plan x' which provides Q(x') units of profit per one unit of cost, but may not provide maximal profit P(x*), since $P(x^*) \ge P(x')$.

Which optimal plan the company should prefer? To give an answer to this question let us return to inequality (4) and rewrite it in the following form: $P(x^*)/C(x^*) \le P(x')/C(x')$ or (here we suppose that P(x')>0 and C(x*)>0)

$$P(x^*)/P(x') \leq C(x^*)/C(x') \qquad (5)$$

From (3) we have that $P(x^*)/P(x') \geq 1$. Using the latter in (5) we obtain that $C(x^*)/C(x') \geq 1$ or (assuming that $C(x')>0$)

$$C(x^*) \geq C(x'). \qquad (6)$$

Inequality (6) shows that the cost of optimal plan x* cannot be less than the cost of optimal plan x'. In other words, in general case we can say that **optimal plan x' is cheaper than x***. Thus, the answer to the question "**Which optimal plan the company should prefer?"** may be formulated as follows: it depends on the costs C(x*) and C(x'), and the amount of money the company can spend. So there are following three possible scenarios:

1. If the company **cannot** spend C(x*) units of money, there is only one possibility - to accept cheaper optimal plan x', implement it and obtain P(x') units of profit;
2. If the company **can** spend C(x*) units, then there are following two cases:
   a. to accept optimal plan x*, implement it and obtain P(x*) units of profit, or
   b. to accept optimal plan x', implement it $k = C(x^*)/C(x') \geq 1$ times and obtain kP(x') units of profit.

Observe, that

$$k\,P(x') = P(x')\,C(x^*)/C(x') = C(x^*)\,Q(x') \geq C(x^*)\,Q(x^*) = P(x^*). \qquad (7)$$

(7) shows, that profit in case (2.b) cannot be less then profit in the case (2.a). To illustrate all these possible scenarios let us consider the following numerical example. Let be the following two linear functions:

$$P(x) = 3.5x_1+2x_2+1.5x_3+1x_4, \quad C(x) = 2x_1+3x_2+0.5x_3+4x_4, \qquad (8)$$

and feasible set S defined by the following constraints:

$$\begin{aligned} 2x_1+2x_2+1x_3+4x_4 &\leq 1000 \qquad (9)\\ 1x_1+0x_2+1x_3+2x_4 &\leq 1500\\ 1x_1+1x_2+1x_3+1x_4 &\geq 250\\ x_j \geq 0,\ j&=1,2,3,4. \end{aligned}$$

Solving LP problem

$$P(x) = 3.5x_1+2x_2+1.5x_3+1x_4 \rightarrow \max$$

$$x = (x_1, x_2, x_3, x_4) \in S,$$

we obtain optimal solution x*=(500, 0, 0, 0) and the following values: P(x*)=1750, C(x*)=1000, Q(x*)=1.75. Then, solving LFP problem

$$Q(x)= \frac{3.5x_1 + 2x_2 + 1.5x_3 + 1x_4}{2x_1 + 3x_2 + 0.5x_3 + 4x_4} \rightarrow max$$

$$x=(x_1, x_2, x_3, x_4) \in S,$$

we obtain optimal solution x' = (0, 0, 1000, 0) and the following values: P(x') = 1500, C(x') = 500, Q(x') = 3. So, we have the following three possible cases:

1. If the company **cannot** spend C(x*) = 1000 units, there is only one possibility - to accept cheaper optimal plan x', implement it and obtain P(x') = 1500 units of profit;
2. If the company **can** spend C(x*) =1000 units, then there are two following cases:
   a. to accept optimal plan x*, spend C(x*) = 1000 units to implement plan x* and then obtain P(x*) = 1750 units of profit, or
   b. to accept optimal plan x', implement it C(x*)/C(x')=1000/500=2 times and obtain 2*P(x') = 2*1500 = 3000 units of profit.

Obviously, the most attractive case is (2.b) since it leads to the greatest possible profit. Note this case requires usage both LP and LFP approaches for modelling and optimization.

3. COINCIDENT SOLUTIONS

In this section we consider such situations when the problems do not have any common (coincident) optimal solutions and we show how it is possible to redirect objective functions in such a way that all optimization problems considered lead to the same optimal solution.

3.1. Case 1

Let be given LP problem (1). Consider the following new LP problem

$$C(x) \to max \ , \ x \in S \tag{10}$$

Let vector $x'' = (x_1'', x_2'', \cdots, x_n'')$ be an optimal solution for problem (10). Consider some hypothetic economic system consisting of n industries. Let us suppose that linear function $C(x)$ describes the manpower requirement of the system in point $x \in R^n$, here x_j denotes the output of j-th industry. Further, let us suppose that the main economic interest of the society is minimization of unemployment. So, from the point of view of the society the activity of the economic system must be organized in accordance with output vector x'' which solves problem (10). At the same time let us suppose that the main aim of the owners of the system is maximization of profit function P(x). In other words, from the point of view of the owners the economic system must operate in accordance with output vector $x*$ which maximizes profit function P(x), i.e. solves problem (1). Note in general case $x'' \neq x*$ It is obvious that

$$P(x^*) \geq P(x'') \quad \text{and} \quad C(x^*) \leq C(x'') \tag{11}$$

i.e. economic interests mentioned above are conflicting in the following sense:

1. if the economic system operates according to plan x'' manpower requirement is maximal but profit may be not maximal. In this case we have $P(x*) - P(x'') \geq 0$ units of lost profit;
2. if the system operates according to plan $x*$ profit is maximal but manpower requirement may be not maximal. In this case we have $C(x'') - C(x*) \geq 0$ units of lost manpower requirement;

Our aim now is to show that there is a very simple way to redirect these conflicting economic interest in such a way that profit function $P(x)$ on feasible set S will lead to the same output plan x'' as objective function $C(x)$. Let $J''=\{s_1,s_2,\cdots,s_m\}$ denote the index-set of basis variables for vector x'' and $B=(A_{s_1},A_{s_2},\cdots,A_{s_m})$ be the appropriate basis, where $A_j=(a_{1j},a_{2j},\cdots,a_{mj})^T$ is j-th column-vector of matrix $A=|a_{ij}|_{m\times n}$. In accordance with theory of simplex method (Dantzig, 1963), (Dantzig, Thapa, 2003), (Gass, 1985), (Vanderbei, 2007) we have:

$$\Delta_j=\sum_{i=1}^{m}c_{s_i}x_{ij}-c_j\geq 0\,,\ j=1,2,\cdots,n \tag{12}$$

where coefficients $x_{ij}(i=1,2,\cdots,m,\ \ j=1,2,\cdots,n)$ are defined from the following systems of linear equations:

$$\sum_{i=1}^{m}A_{s_i}x_{ij}=A_j\,,\ \ j=1,2,\cdots,n$$

Let us consider new vector $t=(t_0,t_1,t_2,\ldots,t_n)$ and replace vector $p=(p_0,p_1,p_2,\ldots,p_n)$ in LP problem (1) with new vector

$$p'=(p'_0,p'_1,p'_2,\ldots,p'_n)\ \text{ where }\ p'_j=p_j+t_j\ ,\ j=0,1,2,\cdots,n\,. \tag{13}$$

Thus, we have a new LP problem

$$P'(x)\to max\,,\ x\in S \tag{14}$$

where $P'(x)=\sum_{j=1}^{n}p'_j\,x_j+p_0$. Now using system (12) we construct the following system of conditions

$$\sum_{i=1}^{m}p'_{s_i}\,x_{ij}-p'_j\geq 0,\ \ \ j=1,2,\cdots,n \tag{15}$$

Obviously, if coefficients $p'_j\,,\ j=0,1,2,\cdots,n$ satisfy system (15), it means that vector x'' solves problem (14). Let us rewrite (15) in the following form

$$\sum_{i=1}^{m}t_{s_i}x_{ij}-t_j\geq-\sum_{i=1}^{m}p_{s_i}x_{ij}+p_j\,,\ \ \ j=1,2,\cdots,n \tag{16}$$

Observe, in system (16) elements p_j and x_{ij} are known constants and coefficients, so only n unknown elements $t_j,\ \ j=1,2,\cdots,n$ have place in the left-hand side of the system. Further, it is easy to show that system (16) defines non-empty set of points $t\in R^{n+1}$ since constraints (16) can be satisfied with any vector t determined from the following constraints

$$p_j+t_j=\mu c_j,\ \ j=1,2,\cdots,n,\ \ \forall\mu>0 \tag{17}$$

Indeed, if we replace elements t_j in the left-hand side of (16) with $(\mu c_j-p_j),\ \ j=1,2,\cdots,n$ we obtain:

$\mu\Delta_j-\sum_{i=1}^{m}p_{s_i}x_{ij}+p_j\geq-\sum_{i=1}^{m}p_{s_i}x_{ij}+p_j\,,\ \ j=1,2,\cdots,n$. So we have that $\mu\Delta_j\geq 0,\ \ j=1,2,\cdots,n$.

Thus, we have shown that there exist such vectors $t \in R^{n+1}$ which (being used as a correction vector for the original profit vector *p*) can redirect the objective function $P(x)$ of problem (1) in such a way that vector *x*" provides not only the maximal level of manpower requirement but provides maximal profit too. Obviously, values t_j may be interpreted as taxes or subsidies (depending on their signs) for *j*-th industry per unit of output.

Furthermore, let us consider the following function: $T(x) = \sum_{j=1}^{n} t_j x_j + t_0$. It can be shown that there are such vectors $t \in R^{n+1}$ that $T(x'') = \sum_{j=1}^{n} t_j x''_j + t_0 = 0$ i.e. there is (at least one) vector *t* of such taxes and subsidies that their total sum in output point *x*" is equal to zero. Indeed, let $t_j = \mu c_j - p_j$, $j = 0,1,2,\cdots,n$, $\mu \geq 0$, then we have

$$T(x'') = \sum_{j=1}^{n} t_j x''_j + t_0 = \sum_{j=1}^{n} (\mu c_j - p_j) x''_j + (\mu c_0 - p_0) =$$

$$= \mu \left(\sum_{j=1}^{n} c_j x''_j + c_0 \right) - \left(\sum_{j=1}^{n} p_j x''_j + p_0 \right) = \mu C(x'') - P(x'') \tag{18}$$

Choosing

$$\mu = \frac{P(x'')}{C(x'')} \tag{19}$$

from (18) we obtain that $T(x'') = 0$.

To illustrate these results we consider problems (1) and (10) using functions $P(x)$ and $C(x)$ as defined in (8). First, solving problems (1) and (10) we have

$$x^* = (500,\ 0,\ 0,\ 0,\ 0),\ \ P(x^*) = 1750,\ \ C(x^*) = 1000,$$

$$x'' = (0,\ 500,\ 0,\ 0,\ 0),\ \ P(x'') = 1000,\ \ C(x'') = 1500.$$

Hence (see (13) and (19))

$$\mu = \frac{P(x'')}{C(x'')} = \frac{2}{3}\ ,\ \ t = (t_0, t_1, t_2, t_3, t_4) = \left(0, -2\frac{1}{6}, 0, -1\frac{1}{6}, 1\frac{2}{3}\right)$$

and

$$p' = (p'_0, p'_1, p'_2, p'_3, p'_4) = \left(0,\ 1\frac{1}{3},\ 2,\ \frac{1}{3},\ 2\frac{2}{3}\right),\ \ P'(x'') = 1000,\ \ T(x'') = 0\ .$$

3.2. Case 2

Let be given LFP problem (2) and LP problem (10). Consider the economic system described in the previous subsection. Following notations and assumptions introduced above we suppose that linear function $C(x)$ describes the manpower requirement of the economic system in point $x \in R^n$, linear function $P(x)$ expresses profit and linear-fractional function $Q(x)$ is efficiency calculated as *profit/cost*. Furthermore, as before, we assume that the main economic aim of the decision maker is maximization of manpower requirement in the society. But the owners, in contrast to previous case, prefer to maximize the efficiency of the economic system calculated as *profit/manpower requirement*.

Let vector x' denote an optimal solution of LFP problem (2) and vector x'' be an optimal solution of LP problem (10) with associated basis B (see previous subsection). Obviously,

$$Q(x') \geq Q(x'') \tag{20}$$

and

$$C(x') \leq C(x'') \tag{21}$$

i.e. economic interests mentioned above are conflicting in the following sense:

1. if the economic system operates according to plan x'' the level of manpower requirement is maximal but efficiency may be not maximal; lost efficiency is $Q(x') - Q(x'') \geq 0$;
2. if the given system operates according to plan x' the level of efficiency is maximal but manpower requirement may be not maximal; lost manpower requirement in this case is $C(x'') - C(x') \geq 0$.

Our aim now is to show that there is a simple way to redirect these conflicting economic interest in such a way that LFP function $Q(x)$ on feasible set S will lead to output plan x''. Since vector x'' is optimal solution of LP problem (10) it means that has place system (12). Consider new vector $t = (t_0, t_1, \cdots, t_n)$ and replace vector $p = (p_0, p_1, \cdots, p_n)$ in LFP problem (2) with new vector

$$p' = (p'_0, p'_1, \cdots, p'_n), \quad \text{where} \quad p'_j = p_j + t_j, \; j = 0,1,2,\cdots,n \tag{22}$$

Thus, we have a new LFP problem

$$Q'(x) = \frac{P'(x)}{C(x)} \to max, \quad x \in S \tag{23}$$

where $P'(x) = \sum_{j=1}^{n} p'_j x_j + p_0$.

Consider the following LFP *reduced costs* (Bajalinov, 2003), (Martos, 1964) of problem (23) in point x'' :

$$\Delta_j(x'') = \Delta'_j \, C(x'') - \Delta''_j \, P(x''), \quad j = 1,2,\cdots,n \tag{24}$$

where $\Delta'_j = \sum_{i=1}^{m} p'_{s_i} x_{ij} - p'_j$, $\Delta''_j = \sum_{i=1}^{m} c_{s_i} x_{ij} - c_j$, $j = 1,2,\cdots,n$.

In accordance with theory of simplex method for LFP, if $\Delta_j(x'') \geq 0$, $j = 1,2,\cdots,n$ then vector x'' is optimal solution for LFP problem (23). In other words, this optimality criteria defines such coefficients p'_j, $j = 0,1,2,\cdots,n$ that vector x'' solves LFP problem (23). Thus we have the following constraints for p'_j:

$$\Delta_j(x'') = \left(\sum_{i=1}^{m} p'_{s_i} x_{ij} - p'_j \right) C(x'') - \Delta''_j \left(\sum_{i=1}^{m} p'_{s_i} x''_{s_i} + p'_0 \right) \geq 0, \quad j = 1,2,\cdots,n$$

or

$$\sum_{i=1}^{m} p'_{s_i} \left(C(x'') x_{ij} - \Delta''_j x''_{s_i} \right) - p'_j C(x'') - \Delta''_j p'_0 \geq 0, \quad j = 1,2,\cdots,n \tag{25}$$

Using (22) in (25) we rewrite the latter in the following form

$$\sum_{i=1}^{m} t_{s_i} F_{ij} - t_j C(x'') - \Delta''_j t_0 \geq \hat{\Delta}_j(x''), \quad j = 1,2,\cdots,n \tag{26}$$

where $F_{ij} = C(x'') x_{ij} - \Delta''_j x''_{s_i}$, and $\hat{\Delta}_j(x'') = \sum_{i=1}^{m} p_{s_i} F_{ij} - p_j C(x'') - \Delta''_j p_0$.

Constraints (26) define set of such vectors $t \in R^{n+1}$ which redirect original fractional objective function $Q(x)$ in such a way that it leads to optimal solution x''. Moreover, it is easy to show that this set is not empty. Indeed, let us choose components t_j in the following way:

$$t_j = \mu c_j - p_j, \; j = 0,1,2,\cdots,n, \; \mu \geq 0 \tag{27}$$

then we have

$$\Delta'_j = \sum_{i=1}^{m} \left(p_{s_i} + \mu c_{s_i} - p_{s_i} \right) x_{ij} - \left(p_j + \mu c_j - p_j \right) = \mu \Delta''_j, \quad j = 0,1,2,\cdots,n,$$

and

$$P'(x'') = \sum_{i=1}^{m} \left(p_{s_i} + \mu c_{s_i} - p_{s_i} \right) x_{s_i} + \left(p_0 + \mu c_0 - p_0 \right) = \mu C'(x'').$$

Hence, $\Delta_j(x'') = (\mu \Delta''_j) C(x'') - \Delta''_j (\mu C(x'')) = 0$, $j = 1,2,\cdots,n$. The latter equalities mean that when choosing vector t by formulas (27) the new linear-fractional objective function $Q'(x)$ on feasible set S leads to optimal solution x''. Thus, we have shown that there exist such vectors $t \in R^{n+1}$ which (being used as a correction vector for the original profit vector p) can redirect original fractional objective function $Q(x)$ of problem (2) in such a way that vector x'' provides not only maximal level of manpower requirement but it provides maximal efficiency too. Obviously, values t_j may be interpreted as taxes or subsidies (depending on their signs) for *j*-th industry per unit of output. Moreover, in the same manner as it was done in previous subsection, it may be shown that there are such vectors $t \in R^{n+1}$ that their total sum in output point x'' is equal to zero.

To illustrate these results we consider problems (2) and (10) using functions, $P(x), C(x)$ and $Q(x)$ as defined in (8)-(9). First, solving problems (2) and (10) we have

$$x' = (0,\, 0,\, 1000,\, 0),\;\; P(x') = 1500,\;\; C(x') = 500,\;\; Q(x') = \frac{1500}{500} = 3$$

$$x'' = (0,\, 500,\, 0,\, 0),\;\; P(x'') = 1000,\;\; C(x'') = 1500,\;\; Q(x'') = \frac{1000}{1500} = \frac{2}{3}$$

Hence (see (19), (22) and (27)), we have

$$\mu = \frac{P(x'')}{C(x'')} = \frac{2}{3}\;,\;\; t = \left(0,\, -2\frac{1}{6},\, 0,\, -1\frac{1}{6},\, 1\frac{2}{3}\right),\;\; p' = \left(0,\, 1\frac{1}{3},\, 2,\, \frac{1}{3},\, 2\frac{2}{3}\right)$$

$$P'(x'') = 1000,\;\; Q'(x'') = \frac{1000}{1500} = \frac{2}{3},\;\; T(x'') = 0\,.$$

CONCLUSIONS

If a real-world optimization problem can be reduced to a linear programming model, often it automatically means that the problem may be re-formulated as an LFP problem too. In this case the following question may appear: what type of objective function do we have to apply - linear or linear-fractional? The investigation of such situations has led us to the results presented above. These results in economic terms may be briefly summarized as follows.

We considered the following three possible situations when different economic interests lead to different (in some sense conflicting) optimal solutions. In the first case profit maximization and maximization of efficiency lead to different optimal solutions and we do not try to redirect these objective functions but we show how it is possible to utilize optimal solution obtained for maximal efficiency in order to obtain some more profit. The second and the third cases deal with the situations when conflicting economic interest should be redirected. We show that such redirection often (at least mathematically) can be implemented by using suitable taxes and subsidies. Such taxes and subsidies may be found from the system of constraints presented above. Moreover, as it was shown above such set of taxes and subsidies contains at least one vector of such taxes and subsidies which may be referred to as *equitable* since their total sum in the point of optimal solution is equal to zero, so such “reconciliation” may be free of charge for the both sides of such types of conflicting situations. In other words, minimization of unemployment may be done (at least theoretically) free of charge.

The results presented in this paper are based on the theoretical investigations of interconnections between LP and LFP models. There are plenty of practical applications for the results of this study. Future work in this domain includes development and implementation of special module/procedure (in the frame of programming package WinGULF) for determining proper correction vector *t* for given LP and LFP problems. In the next step it could be highly useful to perform real-life data based numerical experiments for some regions and/or economics.

Finally, we have just to note that all optimization problems in the numerical examples were solved by package WinGULF 4.2 (Linear and linear-fractional programming package based on special "fractional" extension of primal simplex method with built in branch-and-bound engine for integer problems. For more information see author's Web-site *zeus.nyf.hu/~bajalinov*) developed by the author for educational purposes, and then checked in Microsoft Excel (Spreadsheet software trademark of Microsoft Corporation) using add-in Solver (Trademark of Frontline Systems, Inc.).

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