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2-MODULAR REPRESENTATIONS OF THE ALTERNATING GROUP A_8 AS BINARY CODES

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Abstract. Through a modular representation theoretical approach we enumerate all non-trivial codes from the 2-modular representations of A_8, using a chain of maximal submodules of a permutation module induced by the action of A_8 on objects such as points, Steiner $S(3,4,8)$ systems, duads, bisections and triads. Using the geometry of these objects we attempt to gain some insight into the nature of possible codewords, particularly those of minimum weight. Several sets of non-trivial codewords in the codes examined constitute single orbits of the automorphism groups that are stabilized by maximal subgroups. Many self-orthogonal codes invariant under A_8 are obtained, and moreover, 22 optimal codes all invariant under A_8 are constructed. Finally, we establish that there are no self-dual codes of lengths 28 and 56 invariant under A_8 and S_8 respectively, and in particular no self-dual doubly-even code of length 56.

1. Introduction

In [6] we described a method to investigate all non-trivial codes from the primitive 2-modular permutation representations of certain finite simple groups. In that paper, using a chain of maximal submodules of a permutation module induced by the action of the simple linear group $L_3(4)$ on objects like lines, hyperovals, Baer subplanes and unitals of $\mathrm{PG}(2,4)$ we obtained most of the non-trivial binary codes invariant under the group. However, it is well known, see for example [8,13], that there are two non-isomorphic simple groups of order 20160, respectively $L_3(4)$ and the alternating group A_8. It seems thus natural to ask for the codes invariant under A_8 and their weight

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distribution. Moreover, the isomorphism $A_8 \cong L_4(2) \cong \Omega^+(6,2)$ adds a rich geometrical structure that can be used to explore the connections with objects such as combinatorial designs, graphs, groups and irreducible modules.

In this paper, in a manner similar to that in [6] we consider the primitive representations of A_8, as described, for example, in [8]. Using Meat-Axe and Magma [5], we determine the irreducible constituents of the primitive 2-modular permutation representations and from these we determine the dimensions and constituents of all submodules of each of the subspaces. The incidences between the constituents are determined and used to describe the nature of the codewords of several weights. In addition, we used the Atlas of Brauer characters ([20]) to determine the irreducibility of the codes and the MacWilliams identities relating the weight enumerators of the dual codes. Similar to [6], in this paper we are able to determine and enumerate all submodules, and hence all non-trivial binary codes invariant under A_8. We thus prove the following main result:

Theorem 1.1. *Let G be the alternating group A_8 and Ω be a primitive G-set. Then up to equivalence there are exactly 52 non-trivial binary codes obtained from the 2-modular primitive representations of G as $\mathbb{F}_2G$-submodules of the permutation module $\mathbb{F}_2\Omega$ and admitting G as an automorphism group. The sets of non-trivial codewords of several of these codes constitute single orbits of the automorphism groups that are stabilized by maximal subgroups. Moreover, there is no A_8 and S_8-invariant self-orthogonal $[56,k,d]_2$ code C with $k = 10,19,20,20,21$ and $d = 16,16,10,16,10$, and no self-dual codes of lengths 28 and 56 invariant under A_8 and S_8.*

The proof of Theorem 1.1 follows from a series of propositions in Sections 7, 8, 10 and 11. The paper is organized as follows: after a brief description of our terminology and some background, Sections 3, 4, and 5 give respectively, a brief but complete description of A_8; the incidence relations among the geometric objects obtained through the primitive permutation representations and the 2-modular representations. In the remaining sections we describe the techniques used, and discuss our results.

2. Terminology and notation

Our notation for codes and groups will be standard, and it is as in [1] and [8]. The groups $G.H$, $G{:}H$, and $G^{\cdot}H$ denote a general extension, a split extension and a non-split extension respectively. For a prime p, the symbol p^m denotes an elementary abelian group of that order. The notation p_+^{1+2n} and p_-^{1+2n} are used for extraspecial groups of order p^{1+2n}. If p is an odd prime, the subscript is $+$ or $-$ according as the group has exponent p or p^2. For $p = 2$ it is $+$ or $-$ according as the central product has an even or odd number of quaternionic factors. For G a finite group acting on a finite set Ω, the set $\mathbb{F}_p\Omega$, that is, the vector space over $\mathbb{F}_p$ with basis Ω is called an $\mathbb{F}_pG$ *permutation module*, if the action of G is extended linearly on Ω.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The complementary design of $\mathcal{D}$ is obtained by replacing all blocks of $\mathcal{D}$ by their complements. The design $\mathcal{D}$ is *symmetric* if it has the same number of points and blocks. The numbers that occur as the size of the intersection of two distinct blocks are the *intersection numbers* of the design. A t-(v, k, λ) design is called *self-orthogonal* if the intersection numbers have the same parity as the block size. An automorphism of a design $\mathcal{D}$ is a permutation on $\mathcal{P}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by Aut$\mathcal{D}$.

The *code C of the design* $\mathcal{D}$ over the finite field $\mathbb{F}_p$ is the space spanned by the incidence vectors of the blocks over $\mathbb{F}_p$. The weight enumerator of C is defined as $\sum_{c \in C} x^{\mathrm{wt}(c)}$. The *dual* code $C^{\perp}$ is the orthogonal complement under the standard inner product $(,)$, i.e., $C^{\perp} = \{v \in F^n \mid (v, c) = 0 \text{ for all } c \in C\}$. A code C is *self-orthogonal* if $C \subseteq C^{\perp}$ and it is *self-dual* if $C = C^{\perp}$ and it is *self-complementary* if it contains the all-one vector. The all-one vector will be denoted by $\mathbf{1}$, and it is the constant vector of weight the length of the code. If C_1 is an $[n_1, k_1]$-code, and C_2 is an $[n_2, k_2]$-code, then we say that C is the *direct sum* of C_1 and C_2 if (up to reordering of coordinates) $C = \{(x, y) \mid x \in C_1, y \in C_2\}$. We denote this by $C = C_1 \oplus C_2$. If moreover C_1 and C_2 are nonzero, then we say that C decomposes into C_1 and C_2. A code C is said to be *decomposable* if there exist nonzero codes C_1 and C_2 such that C decomposes into C_1 and C_2. A binary code C is *doubly-even* if all codewords of C have weight divisible by four. Two linear codes are *isomorphic* if they can be obtained from one another by permuting the coordinate positions. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords and will be denoted Aut(C).

3. The primitive permutation representations of A_8

We consider G to be A_8, the alternating group on eight letters, i.e., the subgroup consisting of all even permutations of the symmetric group S_8, which is of order 20160, and its maximal subgroups and primitive permutation representations via the coset action on these subgroups are given in [8]. There are 6 primitive permutation representations of degrees 8, 15, 15, 28, 35 and 56 respectively (see [8]). We use the Atlas notation for the names of the geometric objects on which A_8 acts, namely points, Steiner $S(3, 4, 8)$ systems, duads, bisections and triads. These representations are depicted in Table 1: the first column gives the ordering of the primitive representations as given by Magma (or the Atlas) and as used in our computations; the second gives the maximal subgroups; the third gives the degree (the number of cosets of the point stabilizer).

No.	Max. sub.	Deg.
1	A_7	8
2	$2^3 : L_3(2)$	15
3	$2^3 : L_3(2)$	15
4	S_6	28
5	$2^4 : (S_3 \times S_3)$	35
6	$(A_5 \times 3) : 2$	56

Table 1. Maximal subgroups of A_8

We summarize the information obtained from the group and find notations for the objects which are permuted in each of its primitive permutation representations. The primitive representations may also be described (often is several ways, see for example the Atlas [8]) in terms of the action of G on various sets of geometrical objects: we shall use the notations $g(m)(m = 8, 15a, 15b, 28, 35, 56)$ to denote these sets. We will use names for all objects in terms of their alternating notation from [8], namely point, $S(3, 4, 8)$, duad, bisection and triad.

4. Incidence relations

The action of a group fixing an element of $g(m)$ may be transitive on the elements of $g(n)$ or may split these elements into several orbits or into two orbits if $m \neq n$, of which one has size one if $m = n$. The rows and columns of Table 2 represent the intersections of objects being permuted as named above. Denoting the entries as a_{mn}, the entry a_{42} corresponds to the transitive action of S_6 on $2^3{:}L_3(2)$. The entry a_{64} indicates that there are 3 orbits of an intransitive action of $(A_5 \times 3){:}2$ on S_6. The sizes of the orbits of the remaining actions are illustrated in Table 2.

	n					
m	8	$15a$	$15b$	28	35	56
8	1 - 7	15	15	7 - 21	35	21 - 35
$15a$	8	1 - 14	7 - 8	28	7 - 28	56
$15b$	8	7 - 8	1 - 14	28	7 - 28	56
28	2 - 6	15	15	1 - 12 - 15	15 - 20	6 - 20 - 30
35	8	3 - 12	3 - 12	12 - 16	1 - 16 - 18	8 - 48
56	3 - 5	15	15	3 - 10 - 15	5 - 30	1 - 10 - 15 - 30

Table 2. Orbits of $g(m)$ on $g(n)$.

5. The 2-Modular representations of A_8

Each conjugacy class of maximal subgroups of A_8 generates a permutation module over $\mathbb{F}_2$. We shall consider these $\mathbb{F}_2$-modules, and a chain of all their invariant maximal submodules under the action of A_8. Each maximal submodule constitutes in turn the binary code that is invariant under A_8. After eliminating isomorphic copies, we obtain a lattice of submodules. In this way, we classify and enumerate all submodules, hence codes invariant under A_8. Taking the submodules as the working modules, its corresponding maximal submodules are found recursively. The recursion terminates as soon as we reach an irreducible maximal submodule or a maximal submodule of dimension 1. In doing so we determine all codes associated with the permutation module of a given dimension and invariant under the group. Our construction is based on a method outlined in [6]. The sections that follow present the calculations on these modules. The vectors in each submodule form a code, over $\mathbb{F}_2$, whose length is the dimension of the permutation module and whose dimension is the dimension of the submodule. The weight enumerators of the submodules are therefore also the weight enumerators of these codes which are invariant under the action of A_8. Observe that the rank-2 representations of this group are pairwise equivalent under an outer automorphism (see Table 1 and [8]), and thus their submodules (resp. codes) are isomorphic (resp. equivalent). In this case we only consider the submodules (resp. codes) obtained from the first representation of that degree.

6. The 8-dimensional representation

In its natural representation on a set $\Omega = \{1, 2, \ldots, 8\}$ the group A_8 has for point stabilizer A_7 which has two orbits of lengths 1 and 7 respectively. Using the Atlas [8], we notice that the constituents being permuted by the group are the 8 symbols (points) of the set Ω. The permutation module splits into two absolutely irreducible constituents of dimensions 1 and 6 with multiplicities 2 and 1 respectively. There are only two irreducible maximal submodules of dimension 1 and 7. The permutation module has therefore just one composition series, and the lattice of submodules is as shown in Figure 1.

Figure 1. Submodule lattice for the 8-dimensional representation

It is evident that the codes of this representation are the trivial codes of length 8.

7. A 15-dimensional representation

Notice from Table 2 (see also Table 1) that there are two non-conjugate classes of maximal subgroups of A_8 of index 15 when G acts on the cosets of $2^3 : L_3(2)$. Under this action $2^3 : L_3(2)$ has two orbits, one of length 1 and another of length 14. Using this action and taking for m either $15a$ or $15b$ we form a 15-dimensional permutation module invariant under G. From the Atlas ([8]) we observe that the constituents being permuted by the group in these representations are Steiner $S(3, 4, 8)$ systems. The permutation module splits into four absolutely irreducible constituents of dimensions 1, 4, 4, and 6. There are only two irreducible submodules, one of dimension 1 and the other of dimension 4. These submodules are absolutely irreducible. By recursively determining a chain of maximal submodules of the permutation module (see [6]) we find that the permutation module has two maximal submodules of dimensions 11 and 14. From the 11-dimensional module we obtain two maximal submodules one of dimension 5 and the other of dimension 10. From the 14-dimensional module we get only one maximal submodule which is of dimension 10. This submodule is isomorphic to the submodule of dimension 10 obtained earlier from the 11-dimensional maximal submodule. The 10-dimensional submodule contains an irreducible maximal submodule of dimension 4, and the 5-dimensional submodule contains two irreducible maximal submodules, one of dimensions 1 and the other of dimension 4 respectively. We found that the 4-dimensional submodules are all isomorphic and irreducible. In all, from this permutation module we obtain four non-trivial submodules invariant under A_8 of dimensions 11, 10, 5 and 4. The lattice of the submodules is as given in Figure 2.

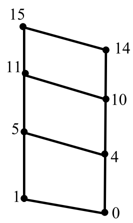

Figure 2. Submodule lattice for a 15-dimensional representation

Form the submodules described above we derive four non-trivial codes, namely $[15, 4, 8]_2$, $[15, 11, 3]_2$, $[15, 5, 7]_2$, and $[15, 10, 4]_2$. We denote the codes (resp. duals) as $C_{15,i}$ ($C_{15,i}{}^{\perp}$) where $i = 1, 2$ and list them in Table 3. From

Name	dim	0	3	4	5	6	7	8	9	10	11	12	15
$C_{15,1}$	4	1						15					
$C_{15,2}$	5	1					15	15					1
${C_{15,2}}^{\perp}$	10	1		105		280		435		168		35	
${C_{15,1}}^{\perp}$	11	1	35	105	168	280	435	435	280	168	105	35	1

TABLE 3. Weight distributions of the codes from a 15-dimensional representation.

Table 3 we deduce some obvious properties of the codes which we examine with certain detail in Proposition 7.1.

PROPOSITION 7.1. (i) *The code* $C_{15,1} = [15, 4, 8]_2$ *is self-orthogonal doubly-even, and its dual is a* $[15, 11, 3]_2$ *code.* $\mathrm{Aut}(C_{15,1}) \cong A_8$, *and* A_8 *acts irreducibly on* $C_{15,1}$ *as an* $\mathbf{F}_2$*-module.*

(ii) *The code* $C_{15,2} = [15, 5, 7]_2$ *is self-complementary, and its dual* $[15, 10, 4]_2$ *is singly-even. Moreover,* $\mathrm{Aut}(C_{15,2}) \cong A_8$, *and* $C_{15,1}$, $C_{15,2}$, *and their duals are optimal codes.*

PROOF. By construction we have that the code $C_{15,1}$ is a 4-dimensional code of length 15 and invariant under A_8, so $A_8 \subseteq \mathrm{Aut}(C_{15,1})$, this also follows since A_8 acts 2-transitively on the set of code coordinates. So $\mathrm{Aut}(C_{15,1})$ is a primitive permutation group of degree 15. Excluding the natural action, the primitive groups of degree 15, are $A_6, A_6{\cdot}2_1, A_7, S_7, A_8$ and S_8, see [14, p. 324]. From $A_8 \subseteq \mathrm{Aut}(C_{15,1})$, we eliminate all but A_8 and S_8 possibilities from the previous list. Moreover, direct calculations show that $C_{15,1}$ is not S_8-invariant, and since $|A_8| = |\mathrm{Aut}(C_{15,1})|$, the result follows. Now, from $C_{15,1} \subseteq {C_{15,1}}^{\perp}$ we deduce that $C_{15,1}$ is self-orthogonal. In addition, it can be observed from Table 3 that $C_{15,1}$ has precisely 15 non-zero vectors, and the zero vector. Also, note that the non-zero vectors (codewords) have weight divisible by four, hence $C_{15,1}$ is doubly-even. From [20] we have that 4 is the smallest dimension for any non-trivial irreducible $\mathbb{F}_2$-invariant module under A_8 and this gives yet another illustration of the isomorphism between A_8 and $L_4(2)$. Further, $C_{15,2}$ is the code obtained from $C_{15,1}$ by adjoining to it the all-ones vector $\mathbf{1}$, so if $\alpha \in \mathrm{Aut}(C_{15,1})$ then since $\alpha(\mathbf{1}) = \mathbf{1}$ and $C_{15,2} = \langle C_{15,1}, \mathbf{1}\rangle$, we have $\alpha \in \mathrm{Aut}(C_{15,2})$, and thus $\mathrm{Aut}(C_{15,1}) \subseteq \mathrm{Aut}(C_{15,2})$. Now $|\mathrm{Aut}(C_{15,1})| = |\mathrm{Aut}(C_{15,2})|$ gives $\mathrm{Aut}(C_{15,2}) = A_8$. The optimality of the codes can be verified from [16]. ∎

REMARK 7.2. The codes and groups found in Proposition 7.1 can be described geometrically: viewing A_8 as $L_4(2)$, notice that the action is that of $L_4(2)$ on the points of $\mathrm{PG}(3, 2)$. The codewords of weight 8 in $C_{15,1}$ form a single orbit under the automorphism group, with stabilizer isomorphic to $2^3{:}L_3(2)$, the affine subgroup of $GL_4(2)$. Taking the image of this orbit under the automorphism group we obtain a 2-$(15, 8, 4)$ design, which is the complement of the design of points and planes in $\mathrm{PG}(3, 2)$. By the fundamental theorem of projective geometry, the automorphism group of the design is $L_4(2)$. Since this design is self-orthogonal, it follows that its point-block incidence matrix spans a self-orthogonal code. In addition, the codes are spanned by their minimum weight codewords, so the assertion on the automorphism group follows as ${C_{15,1}}^{\perp}$ is the well-known Hamming code of length 15. A geometric significance of the codewords of non-zero weight could also be given for the remaining codes.

8. The 28-dimensional representation

The alternating group A_n where $n \geq 5$ acts as a rank-3 group of degree $\binom{n}{2}$ on the 2-subsets of $\Omega = \{1, 2, 3, \ldots, n\}$ known as duads. The stabilizer of a point (duad) $\mathcal{P} = \{a, b\} \in \Omega$ is a group isomorphic to the symmetric group S_{n-2}, and the orbits of the stabilizer consist of $\mathcal{P}$, one of length $2(n-2)$ and the other of length $\binom{n-2}{2}$, see for example [29]. In this action A_n defines a strongly regular graph on $\binom{n}{2}$ isomorphic to the triangular graph $T(n)$. In particular for $n = 8$, we have that S_6 has orbits of lengths 1, 12, and 15. The orbit of length 12 defines the graph $T(8)$ whose parameters are $(28, 12, 6, 4)$. The permutation module splits into three irreducible constituents of dimensions 1, 6, and 14 with multiplicities 2, 2, and 1 respectively. We found that the permutation module has two maximal submodules, one of dimension 22 and the other of dimension 27. From the 27-dimensional submodule we obtain one maximal submodule of dimension 21, while from the 22-dimension submodule we get four maximal submodules, one of which has dimension 8 and the remaining three have each dimension 21. These latter submodules are all non-isomorphic. From the 8-dimensional submodule we obtain three maximal submodules of dimension 7, no two of which are isomorphic. From each of the 21-dimensional submodules we obtain two maximal submodules, one being of dimension 7, and the other of dimension 20. The 20-dimensional submodules are all isomorphic. Each one of the three 7-dimensional submodules are isomorphic to one of those obtained from the earlier 8-dimensional submodule. The first 7-dimensional submodule contains two irreducible submodules, one of dimension 1 and the other of dimension 6. The remaining 7-dimensional submodules contain each an irreducible maximal submodule of dimension 6. We obtain in all four isomorphic and irreducible maximal submodules of dimension 6. Hence, we have a total of twelve maximal submodules invariant under A_8 from this permutation module, namely of dimensions 27, 22, 21, 21, 21, 20, 8, 7, 7, 7, 6 and 1. In Figure 3 we depict the lattice of submodules.

In all, we have obtained ten non-trivial codes invariant under A_8. The weight distributions of these codes are listed in Tables 4 and 5, and in Proposition 8.1 we summarize the 2-modular codes of this representation.

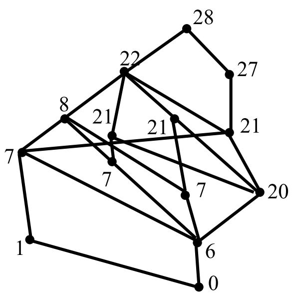

FIGURE 3. Submodule lattice for the 28-dimensional representation

Name	dim	0	3	4	5	6	7	8	9	10	11	12	13
$C_{28,1}$	6	1										28	
$C_{28,2}$	7	1										28	56
$C_{28,3}$	7	1										63	
$C_{28,4}$	7	1					8					28	
$C_{28,5}$	8	1					8					63	56
${C_{28,5}}^{\perp}$	20	1		210		2800		24087		103936		235228	
${C_{28,4}}^{\perp}$	21	1	56	210	672	2800	9320	24087	53760	103936	169008	235228	289856
${C_{28,3}}^{\perp}$	21	1		315		6048		47817		206976		472059	
${C_{28,2}}^{\perp}$	21	1		210	840	2800	9248	24087	54040	103936	166656	235228	295120
${C_{28,1}}^{\perp}$	22	1	56	315	1512	6048	18568	47817	107800	206976	335664	472059	584976

TABLE 4. Weight distributions of the codes from the 28-dimensional representation.

name	14	15	16	17	18	19	20	21	22	23	24	25	28
$C_{28,1}$			35										
$C_{28,2}$			35					8					
$C_{28,3}$			63										1
$C_{28,4}$		56	35										
$C_{28,5}$		56	63					8					1
${C_{28,5}}^{\perp}$	315360		236831		103040		23730		3248		105		
${C_{28,4}}^{\perp}$	315360	295120	236831	166656	103040	54040	23730	9248	3248	840	105		
${C_{28,3}}^{\perp}$	630720		472059		206976		47817		6048		315		1
${C_{28,2}}^{\perp}$	315360	289856	236831	169008	103040	53760	23730	9320	3248	672	105	56	
${C_{28,1}}^{\perp}$	630720	584976	472059	335664	206976	107800	47817	18568	6048	1512	315	56	1

TABLE 5. Table 4 continued.

PROPOSITION 8.1. (i) *The code $C_{28,1}$ is a self-orthogonal doubly-even, projective and optimal 2-weight $[28,6,12]_2$ code. Its dual ${C_{28,1}}^{\perp}$ is a $[28,22,3]_2$ uniformly packed code. $C_{28,1}$ is an irreducible A_8-invariant $\mathbb{F}_2$-module, and* $\mathrm{Aut}(C_{28,1}) \cong S_8$.

(ii) *$C_{28,2}$ is a $[28,7,12]_2$ code, and ${C_{28,2}}^{\perp} = [28,21,4]_2$. $C_{28,2}$ and ${C_{28,2}}^{\perp}$ are optimal, and* $\mathrm{Aut}(C_{28,2}) \cong S_8$.

(iii) *$C_{28,3}$ is a self-orthogonal, doubly-even $[28,7,12]_2$ code. Its dual ${C_{28,3}}^{\perp}$ is a $[28,21,4]_2$ code. $C_{28,3}$ is a decomposable code, $C_{28,3}$ and ${C_{28,3}}^{\perp}$ are optimal self-complementary codes, and* $\mathrm{Aut}(C_{28,3}) \cong S_6(2)$.

(iv) *$C_{28,4}$ is a projective $[28,7,7]_2$ code, and its dual ${C_{28,4}}^{\perp}$ is a $[28,21,3]_2$ code. Moreover,* $\mathrm{Aut}(C_{28,4}) \cong S_8$.

(v) *The code $C_{28,5}$ is a $[28,8,7]_2$ self-complementary code and its dual ${C_{28,5}}^{\perp}$ is a $[28,20,4]_2$ code. Moreover, $C_{28,5}$ is a decomposable code, $\mathrm{Aut}(C_{28,5}) \cong S_8$ and ${C_{28,5}}^{\perp}$ is an optimal code.*

Proof. (i) Notice first that $C_{28,1}$ forms part of an infinite family of codes with parameters $[\binom{n}{2}, n-2, 2n-4]_2$ obtained from the binary row span of the adjacency matrix of the triangular graph $T(n)$. The codes of the triangular graphs have been studied in [18], and with a view for permutation decoding these codes have been examined further in [25]. Our aim here is to give an illustration that explores the geometry of the graph and reveals the connections with modular representation theory. Thus, taking the images of the orbit of length 12 under A_8 on the duads of $\Omega = \{1, 2, \ldots, 8\}$ we obtain $C_{28,1}$ as the binary row span of the adjacency matrix of the triangular graph $T(8)$. Since $n = 28, k = 12, \lambda = 6$ and $\mu = 4$ are all even, it follows that $C_{28,1}$ is self-orthogonal. It is well-known and follows from [24] that $T(8)$ is the unique regular graph with spectrum $12^1, 4^7, -2^{20}$ for which the 2-rank is 6. Now, if we add two different vectors of the adjacency matrix M of $T(8)$ we obtain a vector of weight 12 if the corresponding vectors are adjacent and a vector of weight 16 if the vectors are not adjacent. Since there are 210 pairs of non-adjacent vertices, the binary row span of M has at least $\frac{210}{6}$ vectors of weight 16. Since the minimum weight codewords span the code and these have weight divisible by four, the code is doubly even. Further notice that $C_{28,1}$ has only two non-zero weights, i.e., it is a two-weight code. Let w_1 and w_2 (where $w_1 < w_2$) be the weights of a q-ary two-weight code C of length n and dimension k. To C we may associate a graph $\Gamma(C)$ on q^k vertices as follows. The vertices of the graph are identified with the codewords and two vertices corresponding to the codewords x and y are adjacent if and only if $d(x, y) = w_1$. From the above we obtain a strongly regular graph $\Gamma(C_{28,1})$ associated to $C_{28,1}$ with parameters $(64, 35, 18, 20)$ and its complement, a strongly regular $(64, 28, 12, 12)$ graph $\overline{\Gamma(C_{28,1})}$. Since $\lambda = \mu$ we have that $\overline{\Gamma(C_{28,1})}$ is in fact a 2-$(64, 28, 12)$ design with no absolute polarities. Designs with the parameters of $\overline{\Gamma(C_{28,1})}$ belong to a family of

$$v = 2^{2m},\ k = 2^{2m-1} - 2^{m-1},\ \lambda = 2^{2m-2} - 2^{m-1},\ n = 2^{2m-2}$$

symmetric designs (see [23]) termed symmetric difference property design following [22]. Recall that a code is called projective if its dual distance is at least 3. Moreover, a code with minimum distance $d = 3$ and covering radius 2 is called uniformly packed if every vector which is not a codeword is at distance 1 or 2 from a constant number of codewords ([34]). It then follows that ${C_{28,1}}^{\perp}$ is projective and $C_{28,1}$ is uniformly packed. Moreover ${C_{28,1}}^{\perp}$ has precisely $\binom{8}{3} = 56$ codewords of minimum weight 3 and the minimum weight codewords span the code. Now, from the 2-modular character table of A_8 (see [20, 36]), we have that the irreducible 6-dimensional $\mathbb{F}_2$-representation is unique. Since $\mathrm{Aut}(C_{28,1})$ contains A_8, by using the above weight enumerator we can easily see that $C_{28,1}$, under the action of A_8, does not contain an invariant subspace of dimension 1. Thus, if $C_{28,1}$ were reducible, it would contain an invariant irreducible subspace E of dimension d with $2 \leq d \leq 5$, which is not possible. Hence, C_Γ is irreducible and must be isomorphic to the 6-dimensional $\mathbb{F}_2$-module on which A_8 acts irreducibly. That the automorphism group of the code is the symmetric group S_8, follows by the classification of primitive groups ([31]), as this is the only primitive group of degree 28, and order 40320.

(ii) For the dimension of $C_{28,2}$, notice from Table 4 that $C_{28,1}$ is a subcode of codimension 1 in $C_{28,2}$ spanned by the words of weight divisible by four. Furthermore $C_{28,2}$ is not spanned by its minimum weight codewords; it is spanned by the words of weight 13. Moreover, and using Magma we verified that $\mathrm{Aut}(C_{28,2}) \cong S_8$. Under the action of S_8, the codewords of weight 13 form a single orbit, with the stabilizer of such a codeword a maximal subgroup isomorphic to $S_5 \times S_3$. Similarly, the codewords of weight 21 form a single orbit invariant under S_8 and these codewords are stabilized by a maximal subgroup isomorphic to S_7.

(iii) Notice first that $C_{28,3} = \langle C_{28,1}, \mathbf{1} \rangle = C_{28,1} \oplus \langle \mathbf{1} \rangle$. Hence $C_{28,3}$ is a decomposable module. Now, suppose that $\alpha \in \mathrm{Aut}(C_{28,1})$. Since $\alpha(\mathbf{1}) = \mathbf{1}$ and $C_{28,3} = \langle C_{28,1}, \mathbf{1} \rangle$, we have $\alpha \in \mathrm{Aut}(C_{28,3})$, so that $\mathrm{Aut}(C_{28,1}) \subseteq \mathrm{Aut}(C_{28,3})$. Hence we have by part (i) that $S_8 \leq \mathrm{Aut}(C_{28,3})$. Since $[\mathrm{Aut}(C_{28,3}){:}S_8] = 36$ and by Magma we have that S_8 is a maximal subgroup of $\mathrm{Aut}(C_{28,3})$. In order to describe the structure of $\mathrm{Aut}(C_{28,3})$ we need to determine a primitive group of degree 36 which contains S_8 maximally. Of the 20 primitive groups of degree 36 (see [31]) only one satisfies the above conditions, this being the symmpletic group $S_6(2)$. Hence $\mathrm{Aut}(C_{28,3}) \cong S_6(2)$. In addition, observe that $C_{28,2}$ and $C_{28,3}$ have the same parameters, however they are easily distinguished using their weight distributions.

In part (iv), for the dimension of $C_{28,4}$, notice from Table 4 that $C_{28,1}$ is a subcode of codimension 1 in $C_{28,4}$ spanned by the words of weight divisible by four. Furthermore, since $C_{28,4}$ is spanned by its minimum weight codewords, we can easily deduce its automorphism group. In fact we can show that $\mathrm{Aut}(C_{28,4}) \cong S_8$.

Finally, part (v) follows at once by noticing that $C_{28,5} = C_{28,3} \cup C_{28,4}$, and moreover $C_{28,4}$ is the subcode of $C_{28,5}$ spanned by any set of codewords of odd weight, while $C_{28,3}$ is the subcode generated by words of weight divisible by four. The codes $C_{28,3}$ and $C_{28,4}$ intersect in their doubly-even subcode of dimension 6, which is in fact $C_{28,1}$. Moreover, since $C_{28,5} = \langle C_{28,4}, \mathbf{1} \rangle = C_{28,4} \oplus \langle \mathbf{1} \rangle$ the assertions on the dimension and minimum weight follow. Considering the latter inclusion and the order of the groups, we obtain that $\mathrm{Aut}(C_{28,5}) \cong S_8$. Optimality of all codes can be verified in [16] and also using Magma ([2]). Finally, the codes $C_{28,2}, {C_{28,2}}^{\perp}, C_{28,4}, {C_{28,4}}^{\perp}, C_{28,5}$ and ${C_{28,5}}^{\perp}$ are all optimal. □

9. Designs held by the support of codewords in $C_{28,i}$

A careful examination of Tables 4 and 5 shows that the non-zero codewords of the codes tabulated are single orbits stabilized by maximal subgroups of the automorphism groups. Suppose that w_m is a codeword of non-zero weight m in $C = C_{28,i}$ where $i = 1, 2, \ldots, 5$. In this section we determine the structures of $(\mathrm{Aut}(C))_{w_m}$, that is the stabilizers of w_m in $\mathrm{Aut}(C)$. The structures of these stabilizers are listed in Table 6.

9.1. *Stabilizer in* $\mathrm{Aut}(C)$ *of a word* w_i *in* C. We now examine the action of $\mathrm{Aut}(C) = S_8$ or $\mathrm{Aut}(C) = S_6(2)$ on the set W_m of non-trivial codewords of C and describe their nature. In addition we look at the structure of the stabilizers $(\mathrm{Aut}(C))_{w_m}$ where $m \in M$ with M defined as follows.

Consider $M = \{7, 12, 13, 15, 16, 21\}$. For $m \in M$ we define $W_m = \{w_m \in C_{28,i} \mid \mathrm{wt}(w_m) = m\}$. We show in Lemma 9.1 that for all $m \in M$, $(\mathrm{Aut}(C))_{w_m} = H$ where $H \underset{\max}{<} \mathrm{Aut}(C)$ is a maximal subgroup of $\mathrm{Aut}(C)$. In addition for $w_m \in W_m$ we take image of the support of w_m under the action of $G = S_8$ or $G = S_6(2)$ to form the blocks of the t-$(28, m, k_m)$ $(1 \leq t \leq 2)$ designs $\mathfrak{D} = \mathcal{D}_{w_m}$, where $k_m = |(w_m)^G| \times \frac{m}{28}$ and show that $\mathrm{Aut}(C)$ acts primitively on $\mathcal{D}_{w_m}$. Information on these designs is given in Table 7.

Lemma 9.1. *Let C be a code of Proposition 8.1, and $0 \neq w \in C$. Then $\mathrm{Aut}(C)_w$ is a maximal subgroup of $\mathrm{Aut}(C)$. Moreover, the design $\mathfrak{D}$ obtained by orbiting the images of the support of any non-trivial codeword in C is primitive.*

Proof. Notice from Proposition 8.1 that $\mathrm{Aut}(C) = S_8$ or $\mathrm{Aut}(C) = S_6(2)$. We consider the following two cases.

Case I. $\mathrm{Aut}(C) = S_8$. Since W_m is invariant under the action of S_8 for all $m \in M$, Tables 4 and 5 show that ${w_m}^{S_8} = W_m$ except when $C = C_{28,5}$ and $m = 12$ or $m = 16$. Therefore each W_m is a single orbit under the action of S_8, so that S_8 is transitive on each W_m. From Tables 4 and 5 and the orbit stabilizer theorem we deduce that $[S_8{:}(S_8)_{w_m}] \in \{8, 28, 35, 56\}$. Looking at the list of maximal subgroups of S_8 (see Atlas [8]) and furthermore, using results of [28] we deduce that $(S_8)_{w_m} \in \{S_7, S_6 \times 2, (S_4 \times S_4){:}2, S_5 \times S_3\}$. Since S_8 is transitive on the code coordinates, the codewords of W_m form a 1-design $\mathcal{D}_{w_m}$ with the number of blocks being the indices of $(S_8)_{w_m}$ in S_8. This implies that S_8 is transitive on the blocks of $\mathcal{D}_{w_m}$ for each w_m and since $(S_8)_{w_m}$ is a maximal subgroup of S_8 for $m \in M$, we deduce that S_8 acts primitively on $\mathcal{D}_{w_m}$. See Table 6 and Table 7 for the parameters of these designs. Now, consider $C = C_{28,5}$ and $m = 12$. In this case $|W_{12}| = 63$ and W_{12} splits into two orbits of lengths 28 and 35, namely $W_{(12)_1}$ and $W_{(12)_2}$ under $\mathrm{Aut}(C)$. If $m = 16$ we have $|W_{16}| = 63$ and as earlier W_{16} also splits into two orbits of lengths 28 and 35, namely $W_{(16)_1}$ and $W_{(16)_2}$ respectively.

Case II. $\mathrm{Aut}(C) = S_6(2)$. In this case $C = C_{28,3}$ with $m = 12$ or $m = 16$. For either choices of m we have ${w_m}^{S_6(2)} = W_m$. Thus, W_m is a single orbit of $S_6(2)$, and arguing similarly as in Case I, we can show that $(S_6(2))_{w_m}$ is a maximal subgroup of $S_6(2)$ isomorphic to $2^5{:}S_6$. Lastly, $S_6(2)$ is primitive on the designs $D_{w_{12}} = 2\text{-}(28, 12, 11)$ and its complement $\overline{\mathcal{D}}_{w_{16}} = 2\text{-}(28, 16, 20)$ obtained by orbiting the images of the supports of the codewords of weights 12 and 16 respectively. ∎

In Table 6 the first column gives the codes $C_{28,i}$, the second column represents the codewords of weight m (the sub-indices of m represent the code from where the codeword is drawn), the third column gives the structure of the stabilizers in $\mathrm{Aut}(C)$ of a codeword w_m and the last column, tests the maximality $(\mathrm{Aut}(C))_{w_m}$.

C	m	$(\mathrm{Aut}(C))_{w_m}$	Maximal	C	m	$(\mathrm{Aut}(C))_{w_m}$	Maximal
$C_{28,4}$	7_4	S_7	Yes	$C_{28,4}$	15_4	$S_5 \times S_3$	Yes
$C_{28,5}$	7_5	S_7	Yes	$C_{28,5}$	15_5	$S_5 \times S_3$	Yes
$C_{28,1}$	12_1	$S_6 \times 2$	Yes	$C_{28,1}$	16_1	$(S_4 \times S_4) : 2$	Yes
$C_{28,2}$	12_2	$S_6 \times 2$	Yes	$C_{28,2}$	16_2	$(S_4 \times S_4) : 2$	Yes
$C_{28,3}$	12_3	$2^5 : S_6$	Yes	$C_{28,3}$	16_3	$2^5 : S_6$	Yes
$C_{28,4}$	12_4	$S_6 \times 2$	Yes	$C_{28,4}$	16_4	$(S_4 \times S_4) : 2$	Yes
$C_{28,5}$	$(12_5)_1$	$S_6 \times 2$	Yes	$C_{28,5}$	$(16_5)_1$	$S_6 \times 2$	Yes
	$(12_5)_2$	$(S_4 \times S_4) : 2$	Yes		$(16_5)_2$	$(S_4 \times S_4) : 2$	Yes
$C_{28,2}$	13_2	$S_5 \times S_3$	Yes	$C_{28,2}$	21_2	S_7	Yes
$C_{28,5}$	13_5	$S_5 \times S_3$	Yes	$C_{28,5}$	21_5	S_7	Yes

Table 6. Stabilizer in $\mathrm{Aut}(C)$ of a codeword w_m

In Table 7 the first column represents the codewords of weight m and the second column gives the parameters of the t-designs $\mathcal{D}_{w_m}$ as defined in Subsection 9. In the third column we list the number of blocks of $\mathcal{D}_{w_m}$. The final column shows whether or not a design $\mathcal{D}_{w_m}$ is primitive under the action of $\mathrm{Aut}(C)$.

m	$\mathcal{D}_{w_m}$	No. of blocks	Prim	m	$\mathcal{D}_{w_m}$	No. of blocks	Prim
7_4	1-$(28,7,2)$	8	Yes	15_4	1-$(28,15,30)$	56	Yes
7_5	1-$(28,7,2)$	8	Yes	15_5	1-$(28,15,30)$	56	Yes
12_1	1-$(28,12,12)$	28	Yes	16_1	1-$(28,16,16)$	35	Yes
12_2	1-$(28,12,12)$	28	Yes	16_2	1-$(28,16,20)$	35	Yes
12_3	2-$(28,12,11)$	63	Yes	16_3	2-$(28,16,20)$	63	Yes
12_4	1-$(28,12,12)$	28	Yes	16_4	1-$(28,16,20)$	35	Yes
$(12_5)_1$	1-$(28,12,12)$	28	Yes	$(16_5)_1$	1-$(28,16,16)$	28	Yes
$(12_5)_2$	1-$(28,12,15)$	35	Yes	$(16_5)_2$	1-$(28,16,20)$	35	Yes
13_2	1-$(28,13,26)$	56	Yes	21_2	1-$(28,21,6)$	8	Yes
13_5	1-$(28,13,26)$	56	Yes	21_5	1-$(28,21,6)$	8	Yes

TABLE 7. Primitive t-designs $\mathcal{D}_{w_m}$ invariant under $\mathrm{Aut}(C)$

In Remark 9.2, below we use Lemma 9.1 to give a geometric description of the nature of the codewords in each of the codes of Proposition 8.1. A geometric significance of the minimum weight codewords of the respective duals could also be given.

REMARK 9.2. (i) Notice that the minimum weight of of $C_{28,1}$ is precisely the valency of the graph, and that the minimum weight codewords are the rows of the adjacency matrix of $T(8)$ and these span the code. From a purely geometric perspective one can regard the minimum weight codewords as the duads $\mathcal{P} = \{a, b\}$ (2-element subsets) of the set $\Omega = \{1, 2, \dots, 8\}$ and those of weight 16 as the lines of the projective space $\mathrm{PG}(3, 2)$. Observe that all weights in $C_{28,2}$ are $\equiv 0, 1 \pmod 4$. Consequently, since $C_{28,1} \subseteq C_{28,2}$ we have that the words of weight $\equiv 0 \pmod 4$ in $C_{28,2}$ have the same geometrical significance as those of $C_{28,1}$, hence they are the duads of Ω, while the words of weight $\equiv 1 \pmod 4$ represent the set of triads $\mathcal{P} = \{a, b, c\}$ of Ω (see Section 10) and the points using the alternating notation (see Section 3 or the Atlas [8]).

For the code $C_{28,3}$ the codewords of minimum weight 12 represent the 63 isotropic points of the orthogonal space, and since the dimension of the code is 7, this provides an illustration of the isomorphism $S_6(2) \cong \Omega^+(7, 2)$. Moreover, the minimum weight codewords constitute the 63 rows of the incidence matrix of a quasi-symmetric 2-$(28, 12, 11)$ design $\mathcal{D}$ formed by taking the images of the support of the codewords of minimum weight under the action of the automorphism group. This is in fact a derived design with respect to a block of the 2-$(64, 28, 12)$ design given earlier. That the automorphism group of this design is isomorphic to $S_6(2)$ is well-known (see for example, [15], [23]).

Since $\mathrm{Aut}(\mathcal{D}) \subseteq \mathrm{Aut}(C_{28,3})$, and have the same order, it follows at once that $\mathrm{Aut}(C_{28,3}) \cong S_6(2)$. Since $\mathbf{1} \in C_{28,5}$ we have that the number of codewords of any weight w equals the number of words of weight $28 - w$. Moreover, all weights in $C_{28,5}$ are $\equiv 0, 1 \pmod 4$. Since $C_{28,5} = C_{28,3} \cup C_{28,4}$ and also $C_{28,5} = C_{28,4} \oplus \langle \mathbf{1} \rangle$ a geometric description of the codewords of $C_{28,5}$ can be given in terms of either $C_{28,3}$ and $C_{28,4}$ or simply using $C_{28,4}$.

(ii) The full set of minimum weight four vectors of ${C_{28,3}}^{\perp}$ defines a 2-$(28, 4, 5)$ design with 315 blocks, which we denote $\mathsf{H}(q)$. This design which was found by Hölz is in a class of well-known designs with parameters 2-$(q^3 + 1, q + 1, q + 2)$. It is known that $\mathsf{H}(q)$ contains the 2-rank 21 Hermitian and the 2-rank 19 Ree unitals i.e., 2-$(28, 4, 1)$ designs as subdesigns, see [1], and also [11] for a more recent account. The 315 vectors of minimum weight include the characteristic functions of the design forming these two unitals. Acting the subgroups $P\Gamma L_2(8)$ and $P\Gamma U_3(9)$ of $S_6(2)$ will isolate the weight four vectors that make up the blocks of copies of each of these unitals. When $q = 3$ the codes of the Hermitian and Ree unitals coincide with the code of $\mathsf{H}(q)$, which is in fact a code isomorphic to $C_{28,3}$ i.e., the dual code of the unital of order 3.

(iii) It was proved by Jungnickel and Tonchev in [21] that there exist four non-isomorphic symmetric 2-$(64, 28, 12)$ designs characterized by symmetric difference property and minimality of their 2-rank. Moreover, these designs have large full automorphism groups, and also large 2-subgroups. In particular, the orders of the full automorphism groups are divisible by 2^6, and their derived 2-$(28, 12, 11)$ quasi-symmetric designs give rise to four inequivalent $[28, 7, 12]_2$ codes whose weight distributions equals that of $C_{28,3}$. This shows that the code of the unital of order 3 is not unique. For more details, consult [23] and [15]. It has recently been proved (see [3, Theorem 5.1]) that there is exactly one primitive symmetric design with parameters 2-$(64, 28, 12)$ whose automorphism group is a maximal subgroup of index 694980 of the sporadic simple group Fi_{22} isomorphic to $2^6{:}S_6(2)$. Furthermore, using tactical decompositions, Crnković and Pavčević ([10]) constructed forty-six non-isomorphic symmetric 2-$(64, 28, 12)$ designs from orbit matrices having the Frobenius group of order 21 as an automorphism group. These designs do not satisfy the symmetric difference property, 2^6 is not a divisor of the order of their full automorphism groups, and their derived 2-$(28, 12, 11)$ designs are not quasi-symmetric. In a forthcoming paper [7] we plan to examine the binary codes of these designs.

10. The 35-dimensional representation

The group A_8, acts as a primitive rank-3 group of degree 35 on the set of lines of $V_4(2)$ (see [8]); with line stabilizer isomorphic to $2^4{:}(S_3 \times S_3)$. The orbits of the stabilizer of a line $\mathfrak{L}$ consist of $\{\mathfrak{L}\}, \Psi$ and Φ of lengths 1, 16 and 18 respectively. In addition, since A_8 acts as a rank-3, it is clear that the image of Ψ (or Φ) under A_8 defines a strongly regular graph. We denote this graph Γ and its complement $\overline{\Gamma}$. Note that Γ has parameters $(35, 16, 6, 8)$ and its complement is a $(35, 18, 9, 9)$ graph, which is in fact a 2-$(35, 18, 9)$ design whose polarity has no absolute points. According to the Atlas ([8]) the elements being permuted are the 35 bisections. The permutation module splits into five absolutely irreducible constituents of dimensions 1, 4, 4, 6 and 14 with multiplicities of 1, 1, 1, 2 and 1 respectively.

There are only two irreducible submodules in this representation, one of dimension 1 and the other of dimension 6. The permutation module has two maximal submodules of dimensions 29 and 34 respectively. From the 29 dimensional submodule we get four non-isomorphic maximal submodules of dimensions 15, 25, 25, and 28. The 34-dimensional submodule has one maximal submodule isomorphic to the 28 dimensional submodule given above. Chopping the modules recursively and filtering out the isomorphic copies, as it was for the representations of degrees 15 and 28 we get submodules of dimensions 34, 29, 28, 25, 25, 24, 24, 21, 20, 15, 14, 11, 11, 10, 10, 7 and 6. The two codes from submodules of dimension 25 are isomorphic and so are those from the submodules of dimensions 24, 11 and 10. Hence, we obtain twelve non-trivial binary codes. We denote the codes as $C_{35,i}$ and the duals ${C_{35,i}}^{\perp}$, where $i = 1, 2, \ldots, 6$. The lattice of submodules is given in Figure 4 and the weight distributions are given in Tables 8, 9 and 10 respectively.

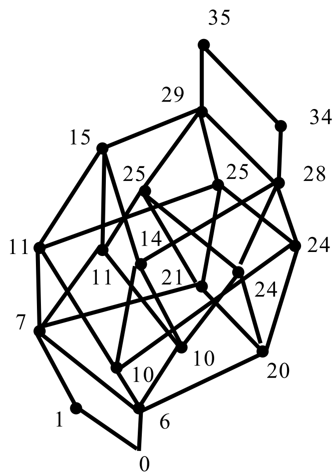

FIGURE 4. Submodule lattice for the 35-dimensional representation

Name	dim	0	3	4	5	6	7	8	9	10	11	12
$C_{35,1}$	6	1										
$C_{35,2}$	7	1										
$C_{35,3}$	10	1										105
$C_{35,4}$	11	1					15					105
$C_{35,5}$	14	1						105				105
$C_{35,6}$	15	1					30	105			315	630
${C_{35,6}}^{\perp}$	20	1				280		735		11648		52290
${C_{35,5}}^{\perp}$	21	1			56	280	210	735	4480	11648	26145	52290
${C_{35,4}}^{\perp}$	24	1		105		1960		21525		179648		813645
${C_{35,3}}^{\perp}$	25	1		105	56	1960	6735	21525	71680	179648	401415	813645
${C_{35,2}}^{\perp}$	28	1		840		25480		366660		2872688		13027560
${C_{35,1}}^{\perp}$	29	1	105	840	5096	25480	104760	366660	1104880	2872688	6513780	13027560

TABLE 8. Weight distributions of the codes from the 35-dimensional representation.

name	13	14	15	16	17	18	19	20	21	22
$C_{35,1}$				35				28		
$C_{35,2}$			28	35			35	28		
$C_{35,3}$				455				448		
$C_{35,4}$			448	455			455	448		
$C_{35,5}$		2640		4235		3360		3388		1680
$C_{35,6}$	1680	2640	3388	4235	3360	3360	4235	3388	2640	1680
${C_{35,6}}^{\perp}$		140360		244895		282240		195916		89320
${C_{35,5}}^{\perp}$	89320	140360	195916	244895	282240	282240	244895	195916	140360	89320
${C_{35,4}}^{\perp}$		2283560		3924515		4468800		3155131		1448440
${C_{35,3}}^{\perp}$	1448440	2283560	3155131	3924515	4468800	4468800	3924515	3155131	2283560	1448440
${C_{35,2}}^{\perp}$		36268760		63410270		70926240		50728216		23080120
${C_{35,1}}^{\perp}$	23080120	36268760	50728216	63410270	70926240	70926240	63410270	50728216	36268760	23080120

TABLE 9. Table 8 continued.

name	23	24	25	26	27	28	29	30	31	32	35
$C_{35,1}$											
$C_{35,2}$											1
$C_{35,3}$						15					
$C_{35,4}$	105					15					1
$C_{35,5}$		315				30					
$C_{35,6}$	630	315			105	30					1
${C_{35,6}}^{\perp}$		26145		4480		210		56			
${C_{35,5}}^{\perp}$	52290	26145	11648	4480	735	210	280	56			1
${C_{35,4}}^{\perp}$		401415		71680		6735		56			
${C_{35,3}}^{\perp}$	813645	401415	179648	71680	21525	6735	1960	56	105		1
${C_{35,2}}^{\perp}$		6513780		1104880		104760		5096		105	
${C_{35,1}}^{\perp}$	13027560	6513780	2872688	1104880	366660	104760	25480	5096	840	105	1

TABLE 10. Table 9 continued

In Proposition 10.1 which follows, we summarize the properties of the codes.

PROPOSITION 10.1. (i) *$C_{35,1} = [35, 6, 16]_2$ is a self-orthogonal, doubly-even and projective code. Its dual ${C_{35,1}}^{\perp} = [35, 29, 3]_2$ is a uniformly* *packed code. Moreover,* $C_{35,1}$ *and* ${C_{35,1}}^{\perp}$ *are optimal and* $\mathrm{Aut}(C_{35,1}) \cong S_8$ *acts irreducibly on* $C_{35,1}$ *as an* $\mathbb{F}_2$*-module.*

(ii) $C_{35,2} = [35, 7, 15]_2$ *and* ${C_{35,2}}^{\perp} = [35, 28, 4]_2$ *is an optimal and singly-even code. Furthermore* $\mathbf{1} \in C_{35,2}$ *and* $C_{35,2}$ *is a decomposable code, and* $\mathrm{Aut}(C_{35,2}) \cong S_8$.

(iii) $C_{35,3} = [35, 10, 12]_2$ *is a self-orthogonal and doubly-even code. Its dual* ${C_{35,3}}^{\perp}$ *is a* $[35, 25, 4]_2$ *code. Moreover,* $\mathrm{Aut}(C_{35,3}) \cong A_8$*, and* $C_{35,3}$ *and* ${C_{35,3}}^{\perp}$ *are optimal.*

(iv) $C_{35,4} = [35, 11, 7]_2$ *and* $\mathbf{1} \in C_{35,4}$*. The dual* ${C_{35,4}}^{\perp} = [35, 24, 4]_2$ *is singly-even. Moreover,* $C_{35,4}$ *is decomposable, and* $\mathrm{Aut}(C_{35,4}) \cong A_8$.

(v) $C_{35,5} = [35, 14, 8]_2$ *is a self-orthogonal and singly-even code. The dual* ${C_{35,5}}^{\perp} = [35, 21, 5]_2$ *and* $\mathrm{Aut}(C_{35,5}) \cong S_8$.

(vi) $C_{35,6} = [35, 15, 7]_2$ *and* $\mathbf{1} \in C_{35,6}$*, and the dual* ${C_{35,6}}^{\perp} = [35, 20, 6]_2$ *is singly-even. Moreover,* $C_{35,6}$ *is decomposable, and* $\mathrm{Aut}(C_{35,6}) \cong S_8$.

Proof. We start by observing that $C_{35,2} = C_{35,1} \oplus \langle \mathbf{1} \rangle$, $C_{35,4} = C_{35,3} \oplus \langle \mathbf{1} \rangle$, and $C_{35,6} = C_{35,5} \oplus \langle \mathbf{1} \rangle$. Consequently, it follows that $\mathrm{Aut}(C_{35,1}) \subseteq \mathrm{Aut}(C_{35,2})$, $\mathrm{Aut}(C_{35,3}) \subseteq \mathrm{Aut}(C_{35,4})$ and $\mathrm{Aut}(C_{35,5}) \subseteq \mathrm{Aut}(C_{35,6})$. Further, $C_{35,2}, C_{35,4}$ and $C_{35,6}$ are all decomposable codes (resp. decomposable $\mathbb{F}_2$-modules) invariant under A_8. For parts (i) and (ii), note that $C_{35,1}$ is the code defined by the row span over $\mathbb{F}_2$ of the adjacency matrix of the graph Γ (or equivalently of the row span of the adjacency matrix of a 1-design $\mathfrak{D}$ with parameters 1-$(35, 16, 16)$ formed by taking the vertices of Γ as the blocks of the design, and incidence in the design as adjacency in the graph). Thus, $C_{35,2}$ is the code of the complementary design $\overline{\mathfrak{D}}$ of $\mathfrak{D}$ and so the difference of any two codewords in $C_{35,1}$ is in $C_{35,2}$. As these differences span a subcode of dimension 6 in $C_{35,2}$, this subcode must be $C_{35,1}$. Moreover, from the weight distribution (see Table 8) we deduce that $C_{35,1}$ is the subcode of $C_{35,2}$ spanned by words of weight divisible by four. The above inclusion now follows, as $C_{35,2}$ is $C_{35,1}$ adjoined by the vector $\mathbf{1}$. Since Γ is a graph that appears in a partition of the symplectic graph $\mathcal{S}_6^+(2)$ it follows from [24, Theorem 5.3] that Γ possesses the triangle property and as such it is uniquely determined by its parameters and by the minimality of its 2-rank, which is 6. Thus the dimension of $C_{35,1}$ is 6. For completeness, we give some overview of the symmplectic graph. Let $\mathcal{A}$ be a $2n \times 2n$ nonsingular alternate matrix over $\mathbb{F}_q$, the symplectic graph relative to $\mathcal{A}$ over $\mathbb{F}_q$ is the graph with the set of one-dimensional subspaces of ${\mathbb{F}_q}^{(2n)}$ as its vertex set and with adjacency defined by $[u] \sim [v]$ if and only if $u\mathcal{A}^t v \neq 0$ for any $u \neq 0$ and $v \neq 0 \in {\mathbb{F}_q}^{(2n)}$, where $[u]$ and $[v]$ are one-dimensional subspaces of ${\mathbb{F}_q}^{(2n)}$, and $[u] \sim [v]$ means that $[u]$ and $[v]$ are adjacent. Furthermore, the minimum-weight 16 of $C_{35,1}$ can be deduced using results from [18, Section 4.4]. We note in addition that all codewords of $C_{35,1}$ are linear combinations of at most two rows of the adjacency matrix of Γ, and since there are exactly 35 codewords of minimum weight in $C_{35,1}$ and these are precisely all the rows of the adjacency matrix of Γ, these must span the code. Since the spanning vectors have weight 16, we have that $C_{35,1}$ is doubly-even and hence self-orthogonal. In addition $C_{35,1}$ is a two-weight code, and by an argument similar to that used in the proof of Proposition 8.1(i) we obtain a strongly regular $(64, 28, 12, 12)$ and its complement. Since ${C_{35,1}}^{\perp}$ has minimum weight 3 it follows from [4] that $C_{35,1}$ is a projective code, and moreover $C_{35,1}$ is uniformly packed ([34]). Optimality of $C_{35,1}$ and ${C_{35,1}}^{\perp}$ follows by Magma ([5]) and also from [16]. Further, using [16] we obtain that $C_{35,2}$ is a distance 1 less than the optimal. The assertion on irreducibility of $C_{35,1}$ follows an argument similar to that used in the proof of Proposition 8.1(i). Finally, $C_{35,1}$ is isomorphic as an $\mathbb{F}_2$-module to $C_{28,1}$ and $C_{35,2}$ is a decomposable 7-dimensional $\mathbb{F}_2$-module invariant under S_8. For the automorphism of the code we will use the knowledge on the design $\mathfrak{D}$ given above. Let $\overline{G} = \mathrm{Aut}(\mathfrak{D})$. By construction we have that $A_8 \subseteq \overline{G}$. However $|\overline{G}| = 2 \times |A_8|$ and $\overline{G}$ is a group generated by permutations such as $(1, 28)(3, 12)(4, 16)(6, 8)(10, 17)(13, 35)(15, 27)(18, 34)(23, 26)(30, 33)$ and $(1, 24, 33, 34, 26, 20, 5)(2, 4, 11, 18, 31, 8, 6)(3, 25, 14, 23, 16, 27, 19)(7, 21, 35, 12, 28, 13, 9)(10, 30, 32, 29, 15, 17, 22)$, which we denote x and y. Since these satisfy $x^2 = y^7 = (xy)^8 = 1$ and $\overline{G} = \langle x, y\rangle$ we have that $\overline{G} \cong S_8$. Moreover $\overline{G} \subseteq \mathrm{Aut}(C_{35,1})$ and since the minimum weight codewords are the blocks of $\mathfrak{D}$ and span $C_{35,1}$ we have that $\overline{G} \cong \mathrm{Aut}(C_{35,1})$.

For parts (iii) and (iv), we note that $C_{35,4}$ is the code spanned by the rows of the 35×15 triple symbol incidence matrix of the design $\mathcal{D}$ of points and lines of $\mathrm{PG}(3, 2)$. This is a quasi-symmetric 2-$(15, 3, 1)$ design with 35 blocks, and blocks meeting in 0 or 1 point respectively. It is well-known that the block graph of a quasi-symmetric design is a strongly regular graph, in this case we have a strongly regular $(35, 18, 9, 9)$ graph (see [33, Theorem 3.7]). The design $\mathcal{D}$ is the only 2-$(15, 3, 1)$ design for which the 2-rank is 11, as suggested by the well-known Hamada's conjecture on the minimality of the p-rank of geometric codes amongst those of the same parameters ([19]). Moreover, the minimum weight codewords span $C_{35,4}$ and since $r = 7$ is odd it follows that $C_{35,4}$ contains the all-one vector $\mathbf{1}$. In addition from $r = 7 \neq 2\lambda = 2$ and ${C_{35,4}}^{\perp} \neq 0$ we have that the minimum-weight of ${C_{35,4}}^{\perp}$ is at least 4. That the automorphism groups is as claimed follows using the fundamental theorem of projective geometry, and also considering the earlier inclusions and taking orders of the groups. Now, the words of weight divisible by four in $C_{35,4}$ form a linear subspace of codimension 1, so by the earlier containment we deduce that this code must be isomorphic to $C_{35,3}$(recall that $C_{35,4}$ is $C_{35,3}$ adjoined by the all-one vector). Further, since $C_{35,3}$ is spanned by its minimum weight codewords the assertion on the automorphism group follows.

Finally, for parts (v) and (vi) the dimension of the codes can be deduced immediately from the structures of the submodules, and the automorphism group follows from the earlier inclusions discussed at the beginning of the proof, and considering the orders. The codes $C_{35,5}, {C_{35,5}}^{\perp}, C_{35,6}$ and ${C_{35,6}}^{\perp}$ are all optimal. □

It can be deduced from Tables 8, 9 and 10 that the stabilizers of the codewords of the codes given in Proposition 10.1 are not always maximal subgroups of the automorphism group. However, in most cases a result along the lines of Lemma 9.1 could be derived. Such result although of interest would consist of many cases and its proof depend on a tedious case-by-case analysis. Thus, in Remark 10.2 below we concentrate only on examining the nature of the minimum weight codewords in each of the codes of Proposition 10.1. We give a geometric significance of those codewords and show in addition that they constitute single orbits of the corresponding automorphism groups.

Remark 10.2. (i) The words of weight 16 in $C_{35,1}$ have a geometrical significance: they are the rows of the adjacency matrix of Γ or equivalently the incidence vectors of the blocks of a 1-$(35, 16, 16)$ design formed by the images of the supports of the codewords of this weight. Since the symmetric group S_8 is the automorphism group of the code, we consider this action and provide a geometrical significance of some classes of codewords. From the Atlas [8] we have that the words of weight 16 in $C_{35,1}$ represent the bisections, while those of weight 20, represent the duads. The stabilizers in S_8 of a bisection, and of a duad are maximal subgroups isomorphic to $(S_4 \times S_4){:}2$ and to $S_6 \times 2$, respectively. We thus have shown that S_8 acts primitively on the set of duads, and on the set of bisections.

Furthermore, since A_8 acts on $C_{35,1}$ we can interpret the codewords of this code using A_8. Viewing A_8 as $L_4(2)$ we have from [8] that the objects permuted by the automorphism group are lines and copies of $S_4(2)$. The codewords of weight 16 represent the lines of $\mathrm{PG}(3, 2)$ while those of weight 20 are copies of $S_4(2)$, thereby explaining the connection with the symplectic graph $\mathcal{S}_6^+(2)$ found in the proof of Proposition 10.1. The stabilizers of a line, and of a copy of an $S_4(2)$ are maximal subgroups of A_8 isomorphic respectively to $2^4{:}(S_3 \times S_3)$ and S_6. Thus we have a primitive action of A_8 on the lines of $\mathrm{PG}(3, 2)$, and on the set of conjugates of $S_4(2)$ respectively.

The dimension 6 of $C_{35,1}$ provides a nice illustration of the isomorphism between A_8 and $\Omega^+(6, 2)$ (similar interpretation could be given for $C_{28,1}$). Therefore using $A_8 \cong \Omega^+(6, 2)$ we can regard the non-zero codewords of $C_{35,1}$ respectively as the sets of isotropic and the non-isotropic points of the orthogonal geometry. This in turn indicates that the objects being permuted are respectively the isotropic and non-isotropic points. The stabilizer of an isotropic point is a maximal subgroup of S_8 isomorphic to $(S_4 \times S_4){:}2$ (resp. maximal subgroup of A_8 isomorphic to $2^4{:}(S_3 \times S_3)$) and that of a non-isotropic point is a maximal subgroup isomorphic with $S_6 \times 2$ (resp. maximal subgroup of A_8 isomorphic to S_6). Now, since $C_{35,2} = C_{35,1} \oplus \langle \mathbf{1} \rangle$ the geometrical significance of the words of $C_{35,2}$ can be deduced in terms of the words of $C_{35,1}$.

(ii) The codewords of minimum weight in $C_{35,4}$ are precisely the 15 points of PG$(3,2)$, and the isotropic planes in the orthogonal geometry. The stabilizer of a codeword of minimum weight is a maximal subgroup of A_8 isomorphic to $2^3 : L_3(2)$. The codewords of minimum weight 12 in $C_{35,3}$ are invariant under A_8, and have for stabilizer a non-maximal subgroup isomorphic to $2^3 : S_4$. Using a result of Neumaier [27] (see also [9]), Haemers ([17]) gives an elegant geometric connection between PG$(3,2)$ and the Hoffman-Singleton graph, by taking the points and lines of PG$(3,2)$ to be the vertices of the graph. Points are mutually non-adjacent; lines are mutually adjacent if and only if the corresponding triples are disjoint. A point is adjacent to a line if and only if they are incident in PG$(3,2)$.

(iii) Finally, the 30 codewords of minimum weight 7 in $C_{35,6}$ are stabilized by a non-maximal subgroup of S_8 isomorphic to $2^3 : L_3(2)$, while the 105 codewords of minimum weight 8 in $C_{35,5}$ are stabilized by a maximal subgroup of S_8 isomorphic to $2^4 : S_4$.

Remark 10.3. Taking the point set $\mathcal{P}$ to be the 2-subsets of $V_4(2)$, Dempwolff in [12] constructed a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{B} = \{b(\mathcal{L}) \mid \mathcal{L} \in \mathcal{P}\}$, with blocks of the form $b(\mathcal{L}) = \{\mathcal{L}\} \cup \{\mathcal{Q} \in \mathcal{P} \mid \mathcal{L} \cap \mathcal{Q} = \varnothing\}$. This design has the group S_8 as its full automorphism group, and it is in fact isomorphic to the symmetric 2-$(35,17,8)$ design formed by orbiting the image of the union of the orbit of length 1 and 16, i.e., the set $\{\mathfrak{L}\} \cup \Psi$ under S_8. This is also the complementary design of the 2-$(35,18,9)$ design described earlier in this section. The design $\mathcal{D}$ is a Hadamard design, and it uniquely extends to a 3-$(36,18,8)$ design. The 3-ranks of these designs are 13 and 14 respectively, and their ternary codes were examined in [30].

The rows of the incidence matrix of the 2-$(15,3,1)$ design can be used as orthogonal parity checks that allow majority decoding of the code $C_{35,4}$ up to its full error-correcting capacity. We prove the following

Proposition 10.4. *The code $C_{35,4}$ can correct up to 3 errors by majority decoding.*

Proof. By applying the Rudolph's decoding algorithm [32] to the design 2-$(15,3,1)$ we obtain that $\lfloor \frac{r+\lambda-1}{2\lambda} \rfloor = \left\lfloor \frac{7+1-1}{2\times 1} \right\rfloor = 3$, and so the result follows. ∎

11. The 56-dimensional representation

As for the earlier representations, here we consider n to be a positive integer and Ω a set of size n, and $\Omega^{\{3\}}$ to be the set of all 3-element subsets of Ω. Adopting the terminology of the Atlas ([8]) we call these sets triads, although they are also called triples in [26]. The alternating group A_n where $n \geq 7$ acts primitively as a rank-4 group of degree $\binom{n}{3}$ on the points of $\Omega^{\{3\}}$. The stabilizer of a point $\mathcal{P} = \{a, b, c\}$ is a group isomorphic to $(A_{n-3} \times 3)$:2, with orbits of lengths 1, $\binom{n-3}{3}$, $3\binom{n-3}{2}$ and $3(n-3)$ respectively. In particular for $n = 8$, we see that A_8 has a unique primitive permutation representation of degree 56 on the cosets of $(A_5 \times 3) : 2$. The orbits of this action have lengths 1, 10, 15 and 30, respectively (see Table 1 and [8]). The elements being permuted in this action are the 56 triads. The permutation module splits into five absolutely irreducible constituents of dimensions 1, 4, 4, 6 and 14 with multiplicities of 2, 1, 1, 3 and 2, and there are only two irreducible submodules, namely of dimensions 1 and 14. Working similarly as in the representations of degrees 15 and 28 and 35 above, we get that the permutation module has two maximal submodules of dimensions 42 and 55. From the 42-dimensional submodule we get two non-isomorphic maximal submodules of dimensions 41 and 36 respectively. The 55-dimensional submodule has two maximal submodules, one of dimension 49 and the other of dimension 41, with the latter being isomorphic to the 41-dimensional submodule of the module of dimension 42 found above. Continuing recursively in this manner we get 31 non-isomorphic submodules of dimensions 55, 49, 48, 42, 41, 36, 35, 35, 35, 34, 32, 32, 31, 31, 29, 28, 28, 27, 25, 25, 24, 24, 22, 21, 21, 21, 20, 15, 14, 8 and 7, with the lattice of submodules as depicted in Figure 5.

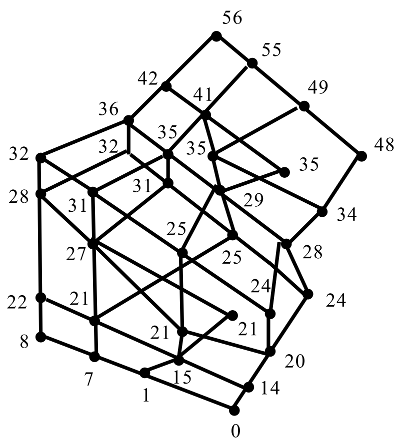

FIGURE 5. Submodule lattice for the 56-dimensional representation

The 32-dimensional submodules give two isomorphic codes, and so do the submodules of dimensions 31, 25, and 24. Thus, in total we obtain twenty-six non-trivial and non-isomorphic codes. In Tables 11, 12 and 13 we give the weight distribution for the codes that contain the all-one vector. In the tables the codes are denoted $C_{56,i}$ and their corresponding duals $C_{56,i}^{\perp}$, with $i = 1, 2, \ldots, 13$. In Tables 14, 15 and 16 we present the codes without the all-one vector.

name	dim	0 56	4 52	5	6 50	7 49	8 48	9 47	10 46	11 45	12 44	13 43
$C_{56,1}$	7	1										
$C_{56,2}$	8	1										
$C_{56,4}$	15	1										
$C_{56,6}$	21	1					35				280	
$C_{56,7}$	21	1			28				168		490	
$C_{56,8}$	21	1									210	
$C_{56,9}$	22	1								56	210	560
$C_{56,11}$	25	1					35				1120	
$C_{56,12}$	27	1			28		35		588		2240	
${C_{56,13}}^{\perp}$	28	1			28		35	280	588	1624	2240	7000
${C_{56,12}}^{\perp}$	29	1					455				15400	
${C_{56,11}}^{\perp}$	31	1			28		35		3108		37520	
${C_{56,10}}^{\perp}$	32	1			28	120	35	280	3108	4144	37520	117040
${C_{56,8}}^{\perp}$	35	1					4235		16800		564480	
${C_{56,7}}^{\perp}$	35	1	70		560		7315		76272		735980	
${C_{56,6}}^{\perp}$	35	1			448		1715		49728		538160	
${C_{56,5}}^{\perp}$	36	1			448	240	1715	2800	49728	109984	538160	1808800
${C_{56,4}}^{\perp}$	41	1	70		2688		77035		2208640		34062140	
${C_{56,3}}^{\perp}$	42	1	70	560	2688	12560	77035	457520	2208640	9146256	34062140	115171280
${C_{56,1}}^{\perp}$	49	1	6020		505232		22206275		556315760		8724879800	

TABLE 11. Codes from the 56-dimensional representation containing the all-ones vector.

name	14 42	15 41	16 40	17 39	18 38	19 37	20 36	21 35
$C_{56,1}$								
$C_{56,2}$								8
$C_{56,4}$			266				672	
$C_{56,6}$			5201				83104	
$C_{56,7}$	840		5306		12460		46872	
$C_{56,8}$	840		266		7000		64792	
$C_{56,9}$	840	728	266	840	7000	27720	64792	95960
$C_{56,11}$			79121				1450624	
$C_{56,12}$	21980		159761		668780		3381504	
${C_{56,13}}^{\perp}$	21980	51352	159761	357280	668780	1616160	3381504	5051680
${C_{56,12}}^{\perp}$	84480		649061		2956800		12347104	
${C_{56,11}}^{\perp}$	300020		2780561		11067980		52753344	
${C_{56,10}}^{\perp}$	300020	995512	2780561	5819800	11067980	25247040	52753344	80315200
${C_{56,8}}^{\perp}$	6061360		39938241		198847600		744783578	
${C_{56,7}}^{\perp}$	5540000		39662553		202366080		750029056	
${C_{56,6}}^{\perp}$	5476160		39789281		200742080		756458304	
${C_{56,5}}^{\perp}$	5476160	15745072	39789281	93578800	200742080	404154240	756458304	1283739520
${C_{56,4}}^{\perp}$	354062720		2542912057		12957463680		47955156634	
${C_{56,3}}^{\perp}$	354062720	992213488	2542912057	5981538640	12957463680	25917985520	47955156634	82200927200
${C_{56,1}}^{\perp}$	90698577200		650765629745		3317623509200		12275213890940	

TABLE 12. Table 11 continued.

name	22 34	23 33	24 32	25 31	26 30	27 29	28
$C_{56,1}$					28		70
$C_{56,2}$				56	28		70
$C_{56,4}$			9205				12480
$C_{56,6}$			532875				854160
$C_{56,7}$	98840		272965		346416		528380
$C_{56,8}$	117320		251125		333592		546860
$C_{56,9}$	117320	170520	251125	293832	333592	458360	546860
$C_{56,11}$			8146635				14199360
$C_{56,12}$	7030520		18029515		21638232		32351360
${C_{56,13}}^{\perp}$	7030520	11727520	18029515	20693008	21638232	27602960	32351360
${C_{56,12}}^{\perp}$	29944320		69292755		92843520		120603120
${C_{56,11}}^{\perp}$	111765080		291842635		346625832		513131360
${C_{56,10}}^{\perp}$	111765080	188861680	291842635	332235568	346625832	440145440	513131360
${C_{56,8}}^{\perp}$	2048051040		4172993195		6328509152		7279729640
${C_{56,7}}^{\perp}$	2041226880		4157300000		6332396224		73011493760
${C_{56,6}}^{\perp}$	2027795840		4182322795		6288761472		7355866400
${C_{56,5}}^{\perp}$	2027795840	3022186720	4182322795	5317674208	6288761472	7040868800	7355866400
${C_{56,4}}^{\perp}$	130758826240		265841299675		405616141056		466898830280
${C_{56,3}}^{\perp}$	82200927200	193317079200	265841299675	340234799072	405616141056	450742296480	466898830280
${C_{56,1}}^{\perp}$	33477848112800		68047373268875		103850752946528		119510793724560

TABLE 13. Table 12 continued.

name	dim	0	4	6	8	10	12	14	16	18
$C_{56,3}$	14	1							210	
$C_{56,5}$	20	1					280		2065	
$C_{56,10}$	24	1					1120		36505	
$C_{56,13}$	28	1			420		6580	41280	327145	1485120
${C_{56,9}}^{\perp}$	34	1			3570		296800	2775200	19785031	101230080
${C_{56,2}}^{\perp}$	48	1	2940	253400	11097730	278187784	4362301300	45349812360	325381240751	1658815478040

TABLE 14. Codes from the 56-dimensional representation without the all-ones vector.

name	20	22	24	26	28	30	32
$C_{56,3}$			5040		6240		4165
$C_{56,5}$	42616		265440		427080		267435
$C_{56,10}$	732256		4066440		7099680		4080195
$C_{56,13}$	6170416	14952000	34648320	46455360	60301560	46388160	34644435
${C_{56,9}}^{\perp}$	375024832	1020557440	2078710620	3166170112	3650746680	3166226112	2078605375
${C_{56,2}}^{\perp}$	6137600058464	16738933866480	34023676352540	51925383202000	59755396862280	51925369744528	34023696916335

TABLE 15. Table 14 continued.

name	34	36	38	40	42	44	46	48	50	52
$C_{56,3}$		672		56						
$C_{56,5}$		40488		3136				35		
$C_{56,10}$		718368		42616				35		
$C_{56,13}$	14992320	6176688	1471680	321916	43200	8820		35		
${C_{56,9}}^{\perp}$	1020669440	375004224	101136000	19877522	2764800	267680	16800	665		
${C_{56,2}}^{\perp}$	325384388994	45348764840	4362578500	278127976	11108545	251832	3080	16738914246320	6137613832476	1658808031160

TABLE 16. Table 15 continued.

REMARK 11.1. (i) The code $C_{56,7}$ is part of the family of codes with parameters $[\binom{n}{3}, n, \binom{n-1}{2}]_2$ studied in [26, Theorem 1], and obtained from non-trivial undirected graphs with vertex set $\Omega^{\{3\}}$. The edges of the graphs are defined by the rules that two vertices are adjacent in a graph if and only if they have exactly zero, one or two elements of Ω in common. With the exception of $C_{56,7}$ and its dual all the remaining codes found in this representation are new.

(ii) In this representation we found the following decomposable codes: $C_{56,2} \oplus C_{56,2}{}^{\perp} = \mathbb{F}_2^{56}$, $C_{56,4} = C_{56,3} \oplus \langle \mathbf{1} \rangle$, $C_{56,6} = C_{56,5} \oplus \langle \mathbf{1} \rangle$, $C_{56,7} \oplus C_{56,7}{}^{\perp} = \mathbb{F}_2^{56}$, $C_{56,11} = C_{56,10} \oplus \langle \mathbf{1} \rangle$, $C_{56,12}{}^{\perp} = C_{56,13}{}^{\perp} \oplus \langle \mathbf{1} \rangle$, $C_{56,8}{}^{\perp} = C_{56,9}{}^{\perp} \oplus \langle \mathbf{1} \rangle$ and $C_{56,1}{}^{\perp} = C_{56,2}{}^{\perp} \oplus \langle \mathbf{1} \rangle$.

(iii) The properties of the codes whose weight distributions are listed in Tables 11, 12, 13, 14, 15, and 16 are summarized in Table 17. In Table 17, the first column gives the parameters of the code, the second, third and fourth columns are true ("yes") if the code is self-orthogonal (s.o.), singly-even (s.e.) or doubly even (d.e.), and false ("no") otherwise. The fifth column gives the structure of the automorphism group. The sixth, seventh and eighth columns are true ("yes") if the code contains the all-ones vector, is optimal or is generated by minimum weight vectors, and false ("no"), otherwise. If the code is generated by minimum weight vectors, or other codewords, then the ninth column gives the 1-$(56, m, \lambda)$ design held by the support of those codewords, and last column gives the number of blocks of the design.

By determining all G-invariant submodules, the number of distinct codes of lengths 28 and 56, obtainable from the 2-modular primitive representations of G is determined. Consequently, the number of self-dual G-invariant codes is also determined. Hence, a combination of the results of Proposition 8.1 and Table 17 give a result concerning non-existence of A_8 and S_8-invariant self-dual codes which follows

PROPOSITION 11.2. *There are no self-dual codes of lengths 28 and 56 obtained from the* 2*-modular primitive representations of* A_8 *and* S_8*. Moreover, there is no self-dual doubly-even code of length 56 invariant under* A_8 *and* S_8*.*

12. CONCLUDING REMARKS

We determine in total 52 non-trivial and non-isomorphic codes invariant under A_8. We describe the nature of the non-trivial codewords of some codes by providing their geometrical significance, and show that in many cases, these codewords are single orbits of the automorphism groups stabilized by maximal subgroups. Several of these codes are decomposable, optimal or near optimal for the given lengths and dimensions.

code	s.o.	s.e.	d.e.	aut	**1**	optimal	min words	design	no. of blocks
$[56, 7, 26]$	no	yes	no	S_8	yes	yes	yes	1-$(56, 26, 13)$	28
$[56, 8, 21]$	yes	no	no	S_8	yes	no	yes	1-$(56, 21, 3)$	8
$[56, 14, 16]$	yes	yes	yes	S_8	no	no	yes	1-$(56, 16, 60)$	280
$[56, 15, 16]$	yes	yes	yes	S_8	yes	no	yes	1-$(56, 16, 76)$	266
$[56, 20, 12]$	no	yes	yes	S_8	no	no	yes	1-$(56, 12, 60)$	280
$[56, 21, 8]$	no	yes	no	S_8	yes	no	yes	1-$(56, 8, 5)$	35
$[56, 21, 6]$	no	yes	no	S_8	yes	no	yes	1-$(56, 6, 3)$	28
$[56, 21, 12]$	yes	yes	yes	S_8	yes	no	yes	1-$(56, 12, 45)$	210
$[56, 22, 11]$	no	no	no	S_8	yes	no	yes	1-$(56, 11, 11)$	56
$[56, 24, 12]$	yes	yes	yes	A_8	no	yes	yes	1-$(56, 12, 240)$	1120
$[56, 25, 8]$	yes	yes	yes	A_8	yes	no	no	1-$(56, 16, 22606)$	79121
$[56, 27, 6]$	no	yes	no	S_8	yes	no	no	1-$(56, 10, 105)$	588
$[56, 28, 6]$	no	no	no	S_8	yes	no	no	1-$(56, 9, 45)$	280
$[56, 28, 8]$	no	yes	no	S_8	no	no	yes	1-$(56, 8, 60)$	420
$[56, 29, 8]$	no	yes	no	S_8	yes	no	yes	1-$(56, 8, 65)$	455
$[56, 31, 6]$	no	yes	no	A_8	yes	no	no	no	
$[56, 32, 6]$	no	yes	no	A_8	yes	no	no	no	
$[56, 34, 8]$	no	yes	no	S_8	no	yes	yes	1-$(56, 8, 510)$	3570
$[56, 35, 8]$	no	yes	no	S_8	yes	no	yes	1-$(56, 8, 605)$	4235
$[56, 35, 4]$	no	yes	no	S_8	yes	no	yes	1-$(56, 4, 5)$	70
$[56, 35, 6]$	no	yes	no	S_8	yes	yes	yes	1-$(56, 6, 48)$	448
$[56, 36, 6]$	no	no	no	S_8	yes	no	no	no	
$[56, 41, 4]$	no	yes	no	S_8	yes	no	no	no	
$[56, 42, 4]$	no	no	no	S_8	yes	no	no	no	
$[56, 48, 4]$	no	yes	no	S_8	no	yes	yes	1-$(56, 4, 210)$	2940
$[56, 49, 4]$	no	yes	no	S_8	yes	yes	yes	1-$(56, 4, 430)$	6020

TABLE 17. Summary of the properties of the codes

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