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EXCHANGE RINGS WITH MANY UNITS

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Abstract. A ring R satisfies Goodearl-Menal condition provided that for any $x, y \in R$, there exists a $u \in U(R)$ such that $x-u, y-u^{-1} \in U(R)$. If $R/J(R)$ is an exchange ring with primitive factors artinian, then R satisfies Goodearl-Menal condition if, and only if it has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2(\mathbb{Z}/2\mathbb{Z})$. Exchange rings satisfying the primitive criterion are also studied.

1. Introduction

A ring R is said to have unit 1-stable range if $aR + bR = R$ implies there exists a $u \in U(R)$ such that $a + bu \in U(R)$, where $U(R)$ denotes the group of all invertible elements in R. If R has unit 1-stable range, then $K_1(R) \cong U(R)/V(R)$, where $V(R) = \{(1+ab)(1+ba)^{-1} \mid 1+ab \in U(R)\}$ (cf. [9, Theorem 1.2]). Also we note that $K_2(R)$ is generated by $< a, b, c >_*$ if R is a commutative ring having unit 1-stable range (cf. [11]). In [6], Goodearl and Menal introduced a simple condition:

$$\text{For any } x, y \in R, \text{ there exists a } u \in U(R) \text{ such that } x - u, y - u^{-1} \in U(R).$$

They discovered that this condition supplied for many classes of rings having unit 1-stable range. As is well known, such condition coincides with unit 1-stable range for any unital complex C^*-algebra (see [6, Theorem 4.1]). This condition was also investigated in [3–6]. We say that a ring R satisfies Goodearl-Menal condition provided that such condition holds. In particular, Goodefroid observed that any topological ring R for which the group of units is open and dense in R satisfies Goodearl-Menal condition. If R satisfies Goodearl-Menal condition, by [9, Theorem 1.2 and Theorem

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1.3], the natural map $U(R)^{ab} \to K_1(R)$ is an isomorphism. Furthermore, $U(R)^{ab} \cong GL_n(R)/E_n(R)$ for any $n \geq 2$ (see [6, Theorem 1.4]).

A ring R is said to be an exchange ring provided that for any $a \in R$, there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - a)$. The class of exchange rings is very large. It includes regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings, and unit C^*-algebras of real rank zero. Such rings have been extensively studied by many authors (cf. [1–2], [7], [10] and [13–14]). For general theory of exchange rings, we refer the reader to [12]. In [13, Theorem 1], Yu proved that every exchange ring with artinian primitive factors has stable range one. If $R/J(R)$ is an exchange ring with primitive factors artinian, we prove that R satisfies Goodearl-Menal condition if, and only if it has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$. Exchange rings satisfying the primitive criterion are also studied.

Throughout, all rings are associative with an identity and all right R-modules are unital. $M_n(R)$ denotes the ring of all $n \times n$ matrices over R, $GL_n(R)$ denotes the n-dimensional general linear group of R. We use $|S|$ to stands for the cardinal number of the set S.

2. Division rings

In this section, we investigate Goodeal-Menal condition for the matrix rings over a division ring, which will be used in the sequel. A Morita context (A, B, M, N, ψ, ϕ) consists of two rings A, B, two bimodules ${}_AN_B, {}_BM_A$ and a pair of bimodule homomorphisms $\psi : N \bigotimes_B M \to A$ and $\phi : M \bigotimes_A N \to B$ which satisfy the following associativity: $\psi\big(n \bigotimes m\big)n' = n\phi\big(m \bigotimes n'\big)$ and $\phi\big(m \bigotimes n\big)m' = m\psi\big(n \bigotimes m'\big)$ for any $m, m' \in M, n, n' \in N$. These conditions insure that the set T of generalized matrices

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \qquad a \in A, b \in B, m \in M, n \in N$$

will form a ring, called the ring of the Morita context. The class of the rings of Morita contexts includes all 2×2 matrix rings and all triangular matrix rings. We start by the following elementary result.

LEMMA 2.1. *If A and B satisfy Goodearl-Menal condition, then so does T.*

PROOF. Let

$$\begin{pmatrix} a_1 & n_1 \\ m_1 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & n_2 \\ m_2 & b_2 \end{pmatrix} \in T.$$

Then there exist some $a \in U(A)$ and $b \in U(B)$ such that $a_1 - a = u_1 \in U(A), 1_A - a_2a = v_1 \in U(A), \big(b_1 - \phi(m_1u_1^{-1} \bigotimes n_1)\big) - b = u_2 \in U(B)$ and

$1_B - \big(\phi(m_2av_1^{-1} \bigotimes n_2) + b_2\big)b = v_2 \in U(B)$. One easily checks that

$$\begin{pmatrix} a_1 & n_1 \\ m_1 & b_1 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$
$$= \begin{pmatrix} u_1^{-1} + u_1^{-1}\psi(n_1u_2^{-1} \bigotimes m_1u_1^{-1}) & -u_1^{-1}n_1u_2^{-1} \\ -u_2^{-1}m_1u_1^{-1} & u_2^{-1} \end{pmatrix}^{-1}$$

and

$$1_T - \begin{pmatrix} a_2 & n_2 \\ m_2 & b_2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$
$$= \begin{pmatrix} v_1^{-1} + \psi(-v_1^{-1}n_2bv_2^{-1} \bigotimes -m_2av_1^{-1}) & -v_1^{-1}n_2bv_2^{-1} \\ -v_2^{-1}m_2av_1^{-1} & v_2^{-1} \end{pmatrix}^{-1},$$

and therefore we complete the proof. □

THEOREM 2.2. *Let A and B be right R-modules. If $End_R(A)$ and $End_R(B)$ satisfy Goodearl-Menal condition, then so does $End_R\big(A \oplus B\big)$.*

PROOF. Let $e : A \oplus B \to A \oplus B$ given by $e(a + b) = a$ for any $a \in A, b \in B$. Then $eEnd_R\big(A \oplus B\big)e \cong End_R\big(e(A \oplus B)\big) \cong End_R(A)$. Likewise, $\big(1_{A\oplus B} - e\big)End_R\big(A \oplus B\big)\big(1_{A\oplus B} - e\big) \cong End_R(B)$. As is well known, the endomorphisms of the direct sum are given by a suitable Morita context. Thus, we get

$$End_R\big(A \oplus B\big) \cong \begin{pmatrix} eEnd_R\big(A \oplus B\big)e & eEnd_R\big(A \oplus B\big)(1-e) \\ (1-e)End_R\big(A \oplus B\big)e & (1-e)End_R\big(A \oplus B\big)(1-e) \end{pmatrix}.$$

By hypothesis and Lemma 2.1, $End_R\big(A \oplus B\big)$ satisfies Goodearl-Menal condition. □

Let $e \in R$ be an idempotent. If eRe and $(1-e)R(1-e)$ satisfy Goodearl-Menal condition, it follows from Theorem 2.2 that R satisfies Goodearl-Menal condition. The converse is not true. For instance, choosing $R = M_3(\mathbb{Z}_2)$, and $e = diag(1, 0, 0)$. Then R satisfies Goodeal-Menal condition, but $eRe \cong \mathbb{Z}_2$ does not satisfy such condition.

COROLLARY 2.3. *A ring R satisfies Goodearl-Menal condition if, and only if so does the ring $TM_n(R)$ of all $n \times n$ upper triangular matrix over R.*

PROOF. $\Leftarrow$: This is obvious.
$\Rightarrow$: By Theorem 2.2 and induction, we complete the proof. □

A ring R is unit-regular provided that for any $x \in R$, there exists a $u \in U(R)$ such that $x = xux$, e.g., every division ring and the endomorphism ring of any finite-dimensional vector space over a division ring.

Lemma 2.4. *Let R be a unit-regular ring, and let $n \in \mathbb{N}$. Then $M_n(R)$ satisfies Goodearl-Menal condition if, and only if for any $X \in M_n(R)$ and diagonal matrix $Y \in M_n(R)$, there exists a $U \in GL_n(R)$ such that $X - U, Y - U^{-1} \in GL_n(R)$.*

Proof. $\Rightarrow$: This is an instance of the definition.

$\Leftarrow$: For any $X, Y \in M_n(R)$, there exist $U, V \in GL_n(R)$ such that $UXV = diag(x_1, \cdots, x_n)$ for some $x_1, \cdots, x_n \in R$. By hypothesis, we have some $W \in GL_n(R)$ such that $diag(x_1, \cdots, x_n) - W, VYU - W^{-1} \in GL_n(R)$. Thus, $A - U^{-1}WV^{-1}, Y - \big(U^{-1}WV^{-1}\big)^{-1} \in GL_n(R)$, as required. □

It is directly verified that $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ do not satisfy Goodearl-Menal condition. Choose

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2\big(\mathbb{Z}/2\mathbb{Z}\big).$$

For any $U \in GL_2\big(\mathbb{Z}/2\mathbb{Z}\big)$, we can check that $A - U \notin GL_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ or $B - U^{-1} \notin GL_2\big(\mathbb{Z}/2\mathbb{Z}\big)$. Thus, $M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ does not satisfy Goodearl-Menal condition. It is worth noting that the Goodearl-Menal condition is obviously preserved in homomorphic images.

Proposition 2.5. *Let D be a division ring. Then $M_n(D)$ satisfies Goodearl-Menal condition if $n = 1, D \ncong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$; or $n = 2, D \ncong \mathbb{Z}/2\mathbb{Z}$; or $n \geq 3$.*

Proof. It is proved by a computer in Microsoft Visual C++ that $M_3\big(\mathbb{Z}/2\mathbb{Z}\big), M_4\big(\mathbb{Z}/2\mathbb{Z}\big)$, $M_5\big(\mathbb{Z}/2\mathbb{Z}\big)$, $M_2\big(\mathbb{Z}/3\mathbb{Z}\big)$ and $M_3\big(\mathbb{Z}/3\mathbb{Z}\big)$ satisfy Goodearl-Menal condition.

Let $n \geq 2$. In view of Theorem 2.2, $M_{3n}\big(\mathbb{Z}/2\mathbb{Z}\big)$ and $M_{3(n-1)}\big(\mathbb{Z}/2\mathbb{Z}\big)$ satisfy Goodearl-Menal condition. Clearly, we see that $3n + 1 = 3(n - 1) + 4, 3n + 2 = 3(n - 1) + 5$. According to Theorem 2.2, $M_{3n+1}\big(\mathbb{Z}/2\mathbb{Z}\big)$ and $M_{3n+2}\big(\mathbb{Z}/2\mathbb{Z}\big)(n \in \mathbb{N})$ satisfy Goodearl-Menal condition. Consequently, $M_n\big(\mathbb{Z}/2\mathbb{Z}\big)(n \geq 3)$ satisfies Goodearl-Menal condition.

By virtue of Theorem 2.2, $M_{2n}\big(\mathbb{Z}/3\mathbb{Z}\big)$ satisfies Goodearl-Menal condition. Since $2n + 1 = 2(n - 1) + 3$, analogously, $M_{2n+1}\big(\mathbb{Z}/3\mathbb{Z}\big)$ satisfies Goodearl-Menal condition. Thus, $M_n\big(\mathbb{Z}/3\mathbb{Z}\big)(n \geq 2)$ satisfies Goodearl-Menal condition.

One easily checks that every division ring with at least 4 elements satisfies Goodearl-Menal condition. Therefore we complete the proof by Theorem 2.2. □

3. Exchange rings with primitive factors artinian

Lemma 3.1. *Let R be a ring. Then R satisfies Goodearl-Menal condition if, and only if for any $x, y \in R$, there exists a $u \in U(R)$ such that $(x - u)(yu - 1) \in U(R)$.*

PROOF. $\Rightarrow$: It is clear.

$\Leftarrow$: Assume that $ab = 1$. Then there exists a $u \in U(R)$ such that $(b - u)(au - 1) = 1$. Write $v = b - u$ and $w = a - u^{-1}$. Then $vwu = 1$ and $av = a(b - u) = 1 - au = -wu$, and so $a = -wuwu$. Thus,

$$ba = (-v^2)(-wuwu)ba = (-v^2)(ab)(-wuwu) = (-v^2)(-wuwu) = 1.$$

That is, R is directly finite, as required. ∎

THEOREM 3.2. *Let $R/J(R)$ be an exchange ring whose primitive factors are artinian. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, $M_2(\mathbb{Z}/2\mathbb{Z})$ as homomorphic images.*

PROOF. One direction is obvious by the observation on quotients of rings with Goodearl-Menal condition.

Conversely, letting $S = R/J(R)$, assume that there exist some $x, y \in S$ such that $(x-u)(yu-1) \notin U(S)$ for any $u \in U(S)$. Let Ω be the set of all ideals I of S such that $(x-u)(yu-1)$ is not a unit modulo I for any $u+I \in U(S/I)$. Clearly, $\Omega \neq \emptyset$. Choose an ascending chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ in Ω. Set $M = \bigcup\limits_{i=1}^{\infty} A_i$. Then M is an ideal of S. Assume that M is not in Ω. We have $u + M \in U(S/M)$ such that $(x - u)(yu - 1) + M \in U(S/M)$. So there are positive integers $n_i (1 \leq i \leq 4)$ such that

$$(x - u)(yu - 1)s - 1 \in A_{n_1}, \quad s(x - u)(yu - 1) - 1 \in A_{n_2},$$
$$ut - 1 \in A_{n_3} \text{ and } tu - 1 \in A_{n_4}$$

for some $s, t \in S$. Let $n = \max\{n_1, n_2, n_3, n_4\}$. Then $\overline{(x - u)(yu - 1)} \in U(S/A_n)$ for $u + A_n \in U(S/A_n)$, a contradiction. This implies that $M \in \Omega$. By using Zorn's Lemma, there exists an ideal Q of S such that it is maximal in Ω.

Set $T = S/Q$. If $J(T) \neq 0$, then $J(T) = K/Q$ for some $K \supsetneqq Q$. Clearly, $T/J(T) \cong S/K$. By the maximality of Q, there is some $(v + Q) + J(T) \in U\big(T/J(T)\big)$ such that

$$\big((x - v)(yv - 1) + Q\big) + J(T) \in U\big(T/J(T)\big).$$

Clearly, $v + Q \in U(S/Q)$. Further, we see that $(x - v)(yv - 1) + Q \in U(T)$. This gives a contradiction, and so $J(S/Q) = 0$.

Moreover, S/Q is an indecomposable ring. In view of [14, Lemma 3.7], $S/Q \cong M_n(D)$ for a division ring D. Since S has no isomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2(\mathbb{Z}/2\mathbb{Z})$, we have that $|D| = 2, n \geq 3$ or $|D| = 3, n \geq 2$ or $|D| \geq 4$. In view of Proposition 2.5, S/Q satisfies Goodearl-Menal condition. Thus, we have $w + Q \in U(S/Q)$ such that $\overline{(x - w)(yw - 1)} \in U(S/Q)$, a contradiction. According to Lemma 3.1, S satisfies Goodearl-Menal condition. For any $x, y \in R$, we can find some $\overline{w} \in R/J(R)$ such that $\overline{x} - \overline{u}, \overline{x} - \overline{u}^{-1} \in$

$U\big(R/J(R)\big)$. Clearly, $u \in U(R)$. Further, $x-u, y-u^{-1} \in U(R)$. Therefore R satisfies Goodearl-Menal condition. □

A ring R is said to be strongly π-regular provided that for any $x \in R$, there exists some $n \in \mathbb{N}$ such that $x^n \in x^{n+1}R$.

COROLLARY 3.3. *Let $R/J(R)$ be a strongly π-regular ring whose primitive factors are artinian. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, $M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ as homomorphic images.*

PROOF. Clearly, $R/J(R)$ is an exchange ring, and so the result follows by Theorem 3.2. □

Recall that a ring R is semilocal provided that $R/J(R)$ is artinian. Let $R = \{\frac{m}{n} \mid 2, 3 \nmid n, (m, n) = 1, m, n \in \mathbb{Z}\}$. Then R is semilocal with only two maximal ideals $2R$ and $3R$. In this case, $R/J(R)$ an exchange ring whose primitive factors are artinian. But R is not an exchange ring. In fact, R has only two idempotents, but $R/J(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ has four idempotents, and so idempotents can not be lifted modulo $J(R)$.

COROLLARY 3.4. *Let R be a semilocal ring. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ as homomorphic images.*

PROOF. Since R is semilocal, $R/J(R)$ is artinian. Thus, $R/J(R)$ is an exchange ring with all primitive factors artinian. Therefore we complete the proof by Theorem 3.2. □

COROLLARY 3.5. *Let A be an artinian right R-module. If $\frac{1}{2}, \frac{1}{3} \in R$, then $End_R(A)$ satisfies Goodearl-Menal condition.*

PROOF. Let $S = End_R(A)$. Then S is semilocal, by the Camps-Dicks theorem. Construct an R-morphism $\varphi : A \to A$ given by $\varphi(a) = a \cdot \frac{1}{2}$ for any $a \in A$. Then $\varphi \in Aut_R(A)$, and so $\frac{1}{2} \in S$. Likewise, $\frac{1}{3} \in S$. If there exists an ideal I of S such that $S/I \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ or $M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$, then $\frac{1}{2}, \frac{1}{3} \in S/I$. This gives a contradiction. In view of Corollary 3.4, $End_R(A)$ satisfies Goodearl-Menal condition. □

Recall that a ring R is of bounded index provided that there exists $n \in \mathbb{N}$ such that $x^n = 0$ for any nilpotent $x \in R$.

COROLLARY 3.6. *Let $R/J(R)$ be an exchange ring of bounded index. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ as homomorphic images.*

PROOF. By virtue of [13, Theorem 3], $R/J(R)$ is an exchange ring with primitive factors artinian. Thus, we obtain the result from Theorem 3.2. □

Example 3.7. Let $R = k[x]/(x^2) = \{a + bt \mid a, b \in k, t^2 = 0\}$ where k is a field of characteristic 5. Suppose $a + bt \in R$. If $a \neq 0$, then $(a + bt)^5 = (a + bt)^6(a - bt)a^{-2}$. If $a = 0$, then $(a + bt)^2 = (a + bt)^3$. Therefore R is a strongly π-regular ring. Assume that $(a+bt)^n = 0$ in R. Then $(a+bt)^{5n} = 0$, hence $a^{5n} = \big((a + bt)^5\big)^n = 0$. So $a = 0$, and then $(a + bt)^5 = a^5 = 0$. That is, R is a strongly π-regular ring of bounded index 5. Clearly, $\frac{1}{3!} \in R$. Hence, R is an exchange ring of bounded index. In addition, it has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$. In view of Corollary 3.6, R satisfies Goodearl-Menal condition.

A ring R is a right (left) quasi-duo if every maximal right (left) ideal is a two-sided ideal.

Corollary 3.8. *Let R be a right (left) quasi-duo exchange ring. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ as homomorphic images.*

Proof. $\Rightarrow$: In this case, $R/J(R)$ is abelian, and so it is clear as in the proof of Theorem 3.2.

$\Leftarrow$: Since R is a right (left) quasi-duo exchange ring, $R/J(R)$ is an exchange ring with all idempotents central. Similarly to [13, Theorem 6], $R/J(R)$ is an exchange ring of bounded index 1. By virtue of Corollary 3.6, R satisfies Goodearl-Menal condition. ∎

Let $R/J(R)$ be an exchange ring with all idempotents central. Analogously, we deduce that R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ as homomorphic images.

Proposition 3.9. *Let $R/J(R)$ be an exchange ring whose primitive factors are artinian. Then $M_n(R)$ satisfies Goodearl-Menal condition for all $n \geq 3$.*

Proof. Let $S = M_n\big(R/J(R)\big)(n \geq 3)$. Then S is an exchange ring with all primitive factors artinain. If there exists an ideal I of S such that $S/I \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$, then we have an ideal $K/J(R)$ of $R/J(R)$ such that $M_n\big(R/K\big) \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$. As $\big|M_n\big(R/K\big)\big| \geq 2^{n^2} \geq 512$, $M_n\big(R/K\big) \ncong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$. Hence, S has no homomorphic images $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$. According to Theorem 3.2, S satisfies Goodearl-Menal condition. Clearly, $S \cong M_n(R)/J\big(M_n(R)\big)$. From this, we deduce that $M_n(R)$ satisfies Goodearl-Menal condition, as asserted. ∎

Corollary 3.10. *Let R be a semilocal ring. Then $M_n(R)$ satisfies Goodearl-Menal condition for all $n \geq 3$.*

Proof. Since R is semilocal, $R/J(R)$ is an exchange ring with all primitive factors artinian. The result follows from Proposition 3.9. ∎

If G is a group and $[G,G]$ its commutator subgroup, then G^{ab} stands for $G/[G,G]$. If R satisfies Goodearl-Menal condition, then $K_1(R) \cong U(R)^{ab}$. Let $R = M_2(\mathbb{Z}/2\mathbb{Z})$. We note that $K_1(R) \ncong U(R)^{ab}$. Clearly, $K_1(R) \cong \mathbb{Z}/2\mathbb{Z} \cong \{1\}$. It is easy to verify that

$$U(R) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$\big[U(R), U(R)\big] = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Thus, we see that $\mid U(R)^{ab} \mid = 2$, and so $K_1(R) \ncong U(R)^{ab}$. But $K_1(R) \cong U(R)^{ab}$ if $R = M_n\big(\mathbb{Z}/2\mathbb{Z}\big)(n \geq 3)$. In general, $K_1(R) \cong GL_n(R)^{ab}(n \geq 3)$ if $R/J(R)$ is an exchange ring with primitive factors artinian, e.g., R is semilocal. This is an immediate consequence of Proposition 3.9.

Let $S(R)$ be the nonempty set of all ideals of a ring R generated by central idempotents. By Zorn's Lemma, $S(R)$ contains maximal elements. If P is a maximal element of the set $S(R)$, we say that R/P is a Pierce stalk of R.

Theorem 3.11. *Let R be an exchange ring whose Pierce stalks are of bounded index. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, $M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ as homomorphic images.*

Proof. One direction is clear. Conversely, letting $x, y \in R$. Let

$$\begin{aligned}
f_1(X_1,Y_1,X_2,Y_2,X_3,Y_3) &= 1-(X_1-X_2)X_3,\\
f_2(X_1,Y_1,X_2,Y_2,X_3,Y_3) &= 1-X_3(X_1-X_2),\\
f_3(X_1,Y_1,X_2,Y_2,X_3,Y_3) &= 1-(Y_1-Y_2)Y_3,\\
f_4(X_1,Y_1,X_2,Y_2,X_3,Y_3) &= 1-Y_3(Y_1-Y_2),\\
f_5(X_1,Y_1,X_2,Y_2,X_3,Y_3) &= 1-X_2Y_2,\\
f_6(X_1,Y_1,X_2,Y_2,X_3,Y_3) &= 1-Y_2X_2
\end{aligned}$$

be the polynomials in noncommutative indeterminate $X_1, Y_1, X_2, Y_2, X_3, Y_3$. Let R/P be an arbitrary Pierce stalk of R. Then R/P is an exchange ring of bounded index. This implies that R/P is an exchange ring with all primitive factors artinian. It is easy to check that $\big(R/P\big)/J\big(R/P\big)$ is an exchange ring whose primitive factors are artinian. By hypothesis, it is not easy to show that R/P does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ as homomorphic images. According to Theorem 3.2, R/P satisfies Goodearl-Menal condition. Thus, we can find a $u \in U(R/P)$ such that $\overline{x}-u, \overline{y}-u^{-1} \in U(R/P)$. Set $v = u^{-1}$. Then we have some $s, t, c, d \in R/P$ such that

$$1-(\overline{x}-u)s = 0, \quad 1-s(\overline{x}-u) = 0, \quad 1-(\overline{y}-v)t = 0,$$
$$1-t(\overline{y}-v) = 0, \quad 1-uv = 0, \quad 1-vu = 0.$$

This means that

$$\begin{aligned} f_1(\overline{x},\overline{y},u,v,s,t) &= 1-(x-u)s,\\ f_2(\overline{x},\overline{y},u,v,s,t) &= 1-s(x-u),\\ f_3(\overline{x},\overline{y},u,v,s,t) &= 1-(\overline{y}-v)t,\\ f_4(\overline{x},\overline{y},u,v,s,t) &= 1-t(\overline{y}-v),\\ f_5(\overline{x},\overline{y},u,v,s,t) &= 1-uv,\\ f_6(\overline{x},\overline{y},u,v,s,t) &= 1-vu. \end{aligned}$$

In view of [12, Lemma 11.4], there exist some $\alpha,\beta,\gamma,\delta \in R$ such that each $f_i(x,y,\alpha,\beta,\gamma,\delta)=0$. As a result, we deduce that $x-\gamma, y-\gamma^{-1} \in U(R)$. Therefore R satisfies Goodearl-Menal condition. □

Proposition 3.12. *Let R be a exchange ring ring whose Pierce stalks are right (left) quasi-duo. Then R satisfies Goodearl-Menal condition if, and only if it does not admit $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, M_2\big(\mathbb{Z}/2\mathbb{Z}\big)$ as homomorphic images.*

Proof. Let $x,y \in R$. Construct the polynomials $f_1,\cdots,f_6$ as in Theorem 3.11. Let R/P be an arbitrary Pierce stalk of R. Then R/P is a right (left) quasi-duo exchange ring. In view of Corollary 3.8, R/P satisfies Goodearl-Menal condition. As in the proof of Theorem 3.11, there exist some $\alpha,\beta,\gamma,\delta \in R$ such that each $f_i(x,y,\alpha,\beta,\gamma,\delta)=0$. Consequently, $x-\gamma, y-\gamma^{-1} \in U(R)$, as required □

A commutative ring R satisfies the primitive criterion if for each polynomial $f(x)=a_0+a_1x+\cdots+a_nx^n$ $(n\geq 0)$ with $a_0R+\cdots+a_nR=R$, i.e., $f(x)\in R[x]$ is primitive, then there exists an $\alpha \in R$ such that $f(\alpha)\in U(R)$ (cf. [8]). As is well known, every commutative ring satisfying the primitive criterion satisfies Goodearl-Menal condition. If $R/J(R)$ is a commutative exchange ring, it follows that R satisfies Goodearl-Menal condition if, and only if $|R/M|\geq 4$ for all maximal ideals M of R. Explicitly, we can derive the following.

Proposition 3.13. *Let $R/J(R)$ be a commutative exchange ring. Then the following are equivalent:*

(1) *R satisfies the primitive criterion.*
(2) *R/M is an infinite field for all maximal ideals M of R.*

Proof. (1) ⇒ (2) Suppose that R satisfies the primitive criterion and M is a maximal ideal of R. Then R/M is a field. Assume that $R/M = \{\overline{x_1},\cdots,\overline{x_n}\}$ is a finite field. Let $f(x)=(x-x_1)\cdots(x-x_n)\in R[x]$. Then $f(x)$ is primitive; hence, there exists some $\alpha \in R$ such that $f(\alpha)\in U(R)$. This implies that $\overline{f}(\overline{\alpha})=(\overline{\alpha}-\overline{x_1})\cdots(\overline{\alpha}-\overline{x_n})\in U(R/M)$, and so $\overline{\alpha}\notin R/M$. This gives a contradiction. Therefore R/M is an infinite field.

(2) ⇒ (1) If $R/J(R)$ satisfies the primitive criterion, then so does R. Thus, without loss of the generality, we may assume that R is a commutative

exchange ring. Assume that R doesn't satisfy the primitive criterion. Then there exists a primitive $f(x) = a_0 + a_1x + \cdots + a_nx^n$ such that $f(\alpha) \notin U(R)$ for all $\alpha \in R$. Let Ω be the set of all the ideals A of R such that $\overline{f}(\overline{\alpha}) = \overline{a_0} + \overline{a_1}\overline{\alpha} + \cdots + \overline{a_n}\overline{\alpha}^n \notin U(R/A)$ for all $\alpha \in R$. Clearly, $\Omega \neq \emptyset$.

Given any ascending chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq \cdots$ in Ω, we set $M = \bigcup_{i=1}^{\infty} A_i$. Then M is an ideal of R. If M is not in Ω, then there exists $\alpha \in R$ such that $\overline{f}(\overline{\alpha}) \in U(R/M)$. Hence, we have some $r \in R$ such that $f(\alpha)r-1 \in M$. Thus, we can find positive integers n such that $f(\alpha)r-1 \in A_n$; hence, $\overline{f}(\overline{\alpha}) \in U(R/A_n)$. This gives a contradiction. Thus, Ω is inductive. By using Zorn's Lemma, we have an ideal Q of R such that Q is maximal in Ω. Let $S = R/Q$. The maximality of $Q \in \Omega$ implies that S is indecomposable as a ring. If $J(S) \neq 0$, we may assume that $J(S) = N/Q$ with $Q \subsetneqq N$. By the maximality of Q, there exists some $\alpha \in R$ such that $\overline{f}(\overline{\alpha}) \in U(R/N)$. Since $S/J(S) \cong R/N$, we may assume that $\overline{f}(\overline{\alpha}) \in U\big(S/J(S)\big)$. As units lift modulo the Jacobson radical of S, we see that $\overline{f}(\overline{\alpha}) \in U(R/Q)$, and yields a contradiction. This implies that $J(S) = 0$, so S is an indecomposable ring with $J(S) = 0$. Since R is a commutative exchange ring, S is simple artinian. That is, S is a field. We infer that Q is a maximal ideal of R, and so R/Q is an infinite field. Thus, we can find some $\beta \in R$ such that $\overline{f}(\overline{\beta}) \in U(R/Q)$, a contradiction. Therefore R satisfies the primitive criterion. □

Corollary 3.14. *Let R be a commutative exchange ring. Then the following are equivalent:*

(1) *R satisfies the primitive criterion.*
(2) *R/M is an infinite field for all maximal ideals M of R.*

Proof. In view of [12, Theorem 29.2], $R/J(R)$ is an exchange ring. Therefore we complete the proof by Proposition 3.13. □

As an immediate consequence, we claim that R satisfies the primitive criterion if and only if R/M is an infinite field for all maximal ideals M of R if R is generalized n-like ($n \geq 2$), i.e., $(xy)^n - xy^n - x^ny + xy = 0$ for any $x, y \in R$. In this case, $R/J(R)$ is a commutative exchange ring, and we are done.

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