GLASNIK MATEMATIČKI
Vol. 47(67)(2012), 307 – 324

COMPOSITION OF GENERALIZED DERIVATIONS AS A LIE DERIVATION

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Abstract. Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R. If the composition (FG) acts as a Lie derivation on R, then (FG) is a derivation of R and one of the following holds:

1. there exist $\alpha \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \alpha x$, for all $x \in R$;
2. G is an usual derivation of R and there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;
3. there exist $\alpha, \beta \in C$ and a derivation h of R such that $F(x) = \alpha x + h(x)$, $G(x) = \beta x$, for all $x \in R$, and $\alpha\beta + h(\beta) = 0$. Moreover in this case h is not an inner derivation of R;
4. there exist $a', c' \in U$ such that $F(x) = a'x$, $G(x) = c'x$, for all $x \in R$, with $a'c' = 0$;
5. there exist $b', q' \in U$ such that $F(x) = xb'$, $G(x) = xq'$, for all $x \in R$, with $q'b' = 0$;
6. there exist $c', q' \in U$, $\eta, \gamma \in C$ such that $F(x) = \eta(xq' - c'x) + \gamma x$, $G(x) = c'x + xq'$, for all $x \in R$, with $\gamma c' - \eta c'^2 = -\gamma q' - \eta q'^2$.

1. Introduction

Throughout this paper, R always denotes a prime ring with center $Z(R)$, U the Utumi quotient ring of R and $C = Z(U)$ the center of U. We refer the reader to [1] for the definitions and the related properties of these objects.

Let $F : R \longrightarrow R$ be an additive mapping of R into itself. It is said to be a derivation of R if $F(xy) = F(x)y + xF(y)$, for all $x, y \in R$. If $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$ and d a derivation of R, then the

2010 *Mathematics Subject Classification.* 16N60, 16W25.
Key words and phrases. Prime rings, differential identities, generalized derivations.

mapping F is called a generalized derivation on R. Obviously any derivation of R is a generalized derivation of R.

A typical example of a generalized derivation is a map of the form $x \mapsto ax + xb$, where a, b are fixed elements in R; such generalized derivations are called inner.

The well known Posner's first theorem states that if δ and d are two non-zero derivations of R, then the composition $(d\delta)$ cannot be a non-zero derivation of R if $char(R) \neq 2$ ([13], Theorem 1). An analogue of Posner's result for Lie derivations was proved by Lanski in [10]. More precisely Lanski showed that if δ and d are two non-zero derivations of R and L is a Lie ideal of R, then $(d\delta)$ cannot be a Lie derivation of L into R unless $char(R) = 2$ and either R satisfies $s_4(x_1, ..., x_4)$, the standard identity of degree 4, or $d = \alpha\delta$, for $\alpha \in C$.

In [9] Hvala initiated the algebraic study of generalized derivations. In particular, generalized derivations whose product is again a generalized derivation was characterized. More precisely Hvala (Theorem 1 in [9]) proved that:

THEOREM 1.1. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R. If the composition FG acts as a generalized derivation on R, then one of the following holds:*

1. *there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
2. *there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;*
3. *there exist $a, b \in U$ such that $F(x) = ax$, $G(x) = bx$, for all $x \in R$;*
4. *there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = xb$, for all $x \in R$;*
5. *there exist $a, b \in U$, $\alpha, \beta \in C$ such that $F(x) = ax + xb$, $G(x) = \alpha x + \beta(ax - xb)$, for all $x \in R$.*

Results concerning generalized derivations can also be found in [3], [7], [15] and [16]. Moreover the results in [12] and [6] evidence the relationship between the behaviour of generalized derivations in a prime (or semiprime) ring and the structure of the ring. In light of the above cited Lanski's result, one might wonder if it is possible that the composition of two generalized derivations with special forms may act like a Lie derivation on R. Under this assumption, we give a description of the forms of the involved generalized derivations F and G. The statement of our result is the following:

THEOREM 1.2. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R. If the composition (FG) acts as a Lie derivation on R, then (FG) is a derivation of R and one of the following holds:*

1. *there exist $\alpha \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \alpha x$, for all $x \in R$;*

2. *G is an usual derivation of R and there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
3. *there exist $\alpha, \beta \in C$ and a derivation h of R such that $F(x) = \alpha x + h(x)$, $G(x) = \beta x$, for all $x \in R$, and $\alpha\beta + h(\beta) = 0$. Moreover in this case h is not an inner derivation of R;*
4. *there exist $a', c' \in U$ such that $F(x) = a'x$, $G(x) = c'x$, for all $x \in R$, with $a'c' = 0$;*
5. *there exist $b', q' \in U$ such that $F(x) = xb'$, $G(x) = xq'$, for all $x \in R$, with $q'b' = 0$;*
6. *there exist $c', q' \in U$, $\eta, \gamma \in C$ such that $F(x) = \eta(xq' - c'x) + \gamma x$, $G(x) = c'x + xq'$, for all $x \in R$, with $\gamma c' - \eta c'^2 = -\gamma q' - \eta q'^2$.*

2. Preliminaries

In all the paper we will make implicit use of some well known results. We would like to dedicate this first Section to state and prove them. We begin with:

Remark 2.1. We would like to point out that in [11] Lee proves that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U. In particular Lee proves the following result:

Theorem 3 in [11]. *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U.*

Remark 2.2. Let R be a non-commutative prime ring and $F : R \to R$ a generalized derivation of R. If F acts as a Lie derivation of R, then F is an usual derivation of R. In particular if F is an inner generalized derivation of R acting as a Lie derivation on R, then F is an inner derivation of R.

Proof. Since F acts as a Lie derivation, we have that R satisfies $F([x_1, x_2]) - [F(x_1), x_2] - [x_1, F(x_2)]$, and using Remark 2.1, by easy calculations it follows that R satisfies the generalized identity

$$x_1 a x_2 - x_2 a x_1. \tag{2.1}$$

In particular, if replace x_1 with $x_2 t$, for any $t \in R$, we have $x_2 t a x_2 - x_2 a x_2 t = 0$, that is $x_2[t, ax_2] = 0$. Again pick $t = rz$, for any $r, z \in R$. Thus $0 = x_2[rz, ax_2] = x_2 r[z, ax_2]$ and by the primeness of R it follows $[z, ax_2] = 0$. This implies $ax_2 \in Z(R)$ (and this holds for all $x_2 \in R$). Of course we may suppose there exists at least one $y_0 \in R$ such that $0 \neq ay_0 \in Z(R)$ (if not $a = 0$ and we are done). Hence for all $t \in R$, $[a(ay_0), t] = 0$ which implies $[a, t]ay_0 = 0$. Since ay_0 is a central element in R, we get $[a, t] = 0$ that is $a \in Z(R)$. Finally by (2.1), R satisfies $a[x_1, x_2]$ and since R is not commutative we conclude that $a = 0$. ∎

Remark 2.3. Let R be a non-commutative prime ring of characteristic different from 2, D_1 and D_2 be derivations of R such that $D_1(x)D_2(x) = 0$ for all $X \in R$. Then either $D_1 = 0$ or $D_2 = 0$.

Proof. It is a reduced version of Theorem 3 in [14]. □

Remark 2.4. Let R be a non-commutative prime ring and $a, b \in R$. If $a[r_1, r_2] + [r_1, r_2]b = 0$ for all $r_1, r_2 \in R$ then $a = -b \in Z(R)$.

Proof. Since If $a[r_1, r_2] + [r_1, r_2]b = 0$ for all $r_1, r_2 \in R$, we have in particular $a[x, yx] + [x, yx]b = 0$ for all $x, y \in R$ and expanding this we get $[x, y][x, b] = 0$. Thus $b \in Z(R)$, by Remark 2.3. Hence $(a + b)[r_1, r_2] = 0$ for all $r_1, r_2 \in R$, and since R is prime it follows $a + b = 0$. □

We also premit the following (Proposition 2.5 in [5]):

Theorem 2.5. *Let R be a prime ring with $char(R) \neq 2$. Assume that R does not embed in $M_2(L)$, the algebra of 2×2 matrices over a field L. If there exist $a, b, c, q, v, w \in R$ such that $a(cx + xq) + (cx + xq)b = vx + xw$ for all $x \in [R, R]$, then one of the following holds:*

1. *c and q are central elements of R;*
2. *a and b are central elements of R;*
3. *b, q and w are central elements of R;*
4. *a, c and v are central elements of R;*
5. *there exists $\alpha \in C$ such that $a + \alpha c$ and $b - \alpha q$ are central elements of R.*

As a reduction we also have

Proposition 2.6. *Let R be a prime ring with $char(R) \neq 2$. Assume that R does not embed in $M_2(L)$, the algebra of 2×2 matrices over a field L. Let F, G be non-zero additive mapping on R defined as $F(x) = ax + xb$ and $G(x) = cx + xq$, for all $x \in R$ and fixed suitable a, b, c, q elements of U. If there exists $p \in U$ such that $F(G(x)) = px - xp$ for all $x \in [R, R]$, then one of the following holds:*

1. *c and q are central elements of R, and $(c + q)a = -(c + q)b = p$; that is $F(x) = [a, x]$ and $G(x) = (c + q)x$;*
2. *a and b are central elements of R, and $(a + b)c = -(a + b)q = p$; that is $F(x) = (a + b)x$ and $G(x) = [c, x]$;*
3. *b, q and p are central elements of R, and $(a + b)(c + q) = 0$; that is $F(x) = (a + b)x$ and $G(x) = (c + q)x$;*
4. *a, c and p are central elements of R, and $(c + q)(a + b) = 0$; that is $F(x) = x(a + b)$ and $G(x) = x(c + q)$;*
5. *there exist $\alpha, \eta \in C$ such that $F(x) = \alpha(xq - cx) + \eta x$, for all $x \in R$, with $\eta c - \alpha c^2 = -\eta q - \alpha q^2$.*

PROOF. Conclusions 1-4 follows directly from 1-4 in Theorem 2.5. Here we would like just to show how conclusion 5 follows by an easy computation. In fact, from conclusion (5) in Theorem 2.5 we have that there exists $\lambda, \mu \in C$ such that $a = \lambda - \alpha c$ and $b = \mu + \alpha q$. Since $a(cx + xq) + (cx + xq)b = px - xp$ for all $x \in [R, R]$, we have that R satisfies

$$(\lambda c - \alpha c^2 + \mu c)[x_1, x_2] + [x_1, x_2](\lambda q + \mu q + \alpha q^2) = p[x_1, x_2] - [x_1, x_2]p$$

which implies $(\lambda + \mu)c - \alpha c^2 = -(\lambda + \mu)q - \alpha q^2$. If one denotes $\eta = \lambda + \mu$, it follows $\eta c - \alpha c^2 = -\eta q - \alpha q^2$ and $F(x) = ax + xb = \alpha(xq - cx) + \eta x$, for all $x \in R$. ∎

3. The Results.

In light of Remark 2.1, we recall that every generalized derivation H of R can be uniquely extended to U and assumes the form $H(x) = ax + d(x)$, for some $a \in U$ and a derivation d on U Thus we can write $F(x) = ux + h(x)$, $G(x) = vx + g(x)$, for suitable $u, v \in U$ and h, g derivations of U. Usually h and g are called derivation associated respectively with F and G. Firstly we analyse the case when h and g are not simultaneously inner derivations of U:

THEOREM 3.1. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero generalized derivations of R. Assume that the derivations h and g associated respectively with F and G are not simultaneously inner. If the composition (FG) acts as a Lie derivation on R, then (FG) is a derivation of R and one of the following holds:*

1. *there exist $\alpha \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \alpha x$, for all $x \in R$;*
2. *G is an usual derivation of R and there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
3. *there exist $\alpha, \beta \in C$ and a derivation h of R such that $F(x) = \alpha x + h(x)$, $G(x) = \beta x$, for all $x \in R$, and $\alpha\beta + h(\beta) = 0$. Moreover in this case h is not an inner derivation of R;*
4. *there exist $a', c' \in U$ such that $F(x) = a'x$, $G(x) = c'x$, for all $x \in R$, with $a'c' = 0$;*
5. *there exist $b', q' \in U$ such that $F(x) = xb'$, $G(x) = xq'$, for all $x \in R$, with $q'b' = 0$;*
6. *there exist $c', q' \in U$, $\eta, \gamma \in C$ such that $F(x) = \eta(xq' - c'x) + \gamma x$, $G(x) = c'x + xq'$, for all $x \in R$, with $\gamma c' - \eta c'^2 = -\gamma q' - \eta q'^2$.*

PROOF. Since FG acts as a Lie derivation on R, then R satisfies

$$(FG)[x_1, x_2] = [(FG)(x_1), x_2] + [x_1, (FG)(x_2)]. \tag{3.1}$$

As we said above $F(x) = ux + h(x)$, $G(x) = vx + g(x)$, for suitable $u, v \in U$ and h, g derivations of U. Therefore R satisfies the differential identity

$$\begin{aligned}
&u\left(v[x_1,x_2] + [g(x_1),x_2] + [x_1,g(x_2)]\right) + h(v)[x_1,x_2]\\
&\quad + v[h(x_1),x_2] + v[x_1,h(x_2)] + [(hg)(x_1),x_2]\\
&\quad + [g(x_1),h(x_2)] + [h(x_1),g(x_2)] + [x_1,(hg)(x_2)]\\
&\quad - [uvx_1 + ug(x_1) + h(v)x_1 + vh(x_1) + (hg)(x_1), x_2]\\
&\quad - [x_1, uvx_2 + ug(x_2) + h(v)x_2 + vh(x_2) + (hg)(x_2)].
\end{aligned} \tag{3.2}$$

First consider the case when $\{h, g\}$ is a set of linearly C-independent derivations modulo X-inner derivations (i.e., modulo the space of inner derivations of R). In light of Kharchenko's theory (see [8]) and starting from (3.2), R satisfies:

$$\begin{aligned}
&u\left(v[x_1,x_2] + [z_1,x_2] + [x_1,z_2]\right) + h(v)[x_1,x_2] + v[t_1,x_2]\\
&\quad + v[x_1,t_2] + [u_1,x_2] + [z_1,t_2] + [z_1,t_2] + [x_1,u_2]\\
&\quad - [uvx_1 + uz_1 + h(v)x_1 + vt_1 + u_1, x_2]\\
&\quad - [x_1, uvx_2 + uz_2 + h(v)x_2 + vt_2 + u_2]
\end{aligned}$$

and in particular R satisfies the blended component $[z_1, t_2] + [t_1, z_2]$ which implies the contradiction that R is commutative.

Hence we suppose that $\{h, g\}$ is linearly C-dependent modulo X-inner derivations. Then here we may assume that there exist $\alpha, \beta \in C$ such that

$$\alpha h + \beta g = ad(q)$$

the inner derivation induced by some element $q \in U$, moreover at least one of $\{h, g\}$ is not an inner derivation.

We divide the proof into three cases:

The case $\alpha = 0$. For $\alpha = 0$, we have $\beta \neq 0$ and $g = ad(c)$, the inner derivation induced by $c = \beta^{-1}q$. It follows that h is not an inner derivation of U. By (3.2) and Kharchenko's result in [8], U satisfies

$$\begin{aligned}
&u\left(v[x_1,x_2] + [[c,x_1],x_2] + [x_1,[c,x_2]]\right)\\
&\quad + h(v)[x_1,x_2] + v[z_1,x_2] + v[x_1,z_2] + \Big[[h(c),x_1] + [c,z_1],x_2\Big]\\
&\quad + \Big[[c,x_1],z_2\Big] + \Big[z_1,[c,x_2]\Big] + \Big[x_1,[h(c),x_2] + [c,z_2]\Big]\\
&\quad - \Big[abx_1 + u[c,x_1] + h(v)x_1 + vz_1 + [h(c),x_1] + [c,z_1],x_2\Big]\\
&\quad - \Big[x_1, abx_2 + u[c,x_2] + h(v)x_2 + vz_2 + [h(c),x_2] + [c,z_2]\Big]
\end{aligned} \tag{3.3}$$

and in particular U satisfies the component

$$\Big[z_1,[c,x_2]\Big] - [v,x_2]z_1 \tag{3.4}$$

and for $z_1 = [c, x_2]$ we have $[v, x_2][c, x_2] = 0$ in U. Therefore by Remark 2.3 either $v \in C$ or $c \in C$. If $v \in C$, by (3.4) U satisfies $[z_1, [c, x_2]]$ and this implies $c \in C$ (for example see the well known result of Posner in [13]). On the other hand if we assume $c \in C$, again by (3.4), $[v, x_2]z_1 = 0$ in U and we get $v \in C$, since U is prime.

Therefore in any case both $v \in C$ and $c \in C$, which implies $G(x) = vx$ for all $x \in R$. Hence $(FG)(x) = u(vx) + h(vx) = (uv + h(v))x + (vh)(x)$, where $vh : U \to U$ is the derivation of U defined as $(vh)(x) = v \cdot h(x)$. Therefore (FG) is a generalized derivation. Since by hypothesis (FG) acts as a Lie derivation, then by Remark 2.2, we have $uv + h(v) = 0$ and $(FG) = (vh)$, with $v \in C$. Moreover notice that in this case also $u \in C$.

THE CASE $\beta = 0$. In this case we have $\alpha \neq 0$ and $h(x) = [p, x]$ for all $x \in U$, where $p = \alpha^{-1}q$. Moreover g is not an inner derivation. By (3.2) and Kharchenko's result in [8], U satisfies

(3.5)
$$\begin{aligned} &u\left(v[x_1, x_2] + [t_1, x_2] + [x_1, t_2]\right) \\ &\quad + h(v)[x_1, x_2] + v[[p, x_1], x_2] + v[x_1, [p, x_2]] + [[p, t_1], x_2] + [t_1, [p, x_2]] \\ &\quad - \Big[[p, x_1], t_2] + [x_1, [p, t_2]] - [uvx_1 + ut_1 + [p, v]x_1 + v[p, x_1] + [p, t_1], x_2\Big] \\ &\quad - \Big[x_1, uvx_2 + ut_2 + [p, v]x_2 + v[p, x_2] + [p, t_2]\Big] \end{aligned}$$

and in particular U satisfies

$$\Big[t_1, [p, x_2]\Big] - [u, x_2]t_1 \tag{3.6}$$

and for $t_1 = [p, x_2]$ we have $[u, x_2][p, x_2] = 0$ in U. As above this implies both $u \in C$ and $p \in C$. Therefore $F(x) = \alpha x$ with $\alpha = u \in C$ and $(FG)(x) = (\alpha G)(x) = \alpha vx + (\alpha g)(x)$, for all $x \in R$. Since (FG) acts as a Lie derivation, by Remark 2.2 it follows $\alpha v = 0$, that is $v = 0$, because we may assume $\alpha \neq 0$ (if not $F = 0$). Hence G is an usual derivation of U.

THE CASE $\alpha \neq 0$ AND $\beta \neq 0$. In this case we may write $g(x) = [c, x] + \gamma h(x)$, with $c = \beta^{-1}q$ and $\gamma = -\alpha\beta^{-1} \neq 0$. Notice that if h is inner then also g is inner, and analogously in case g is inner then also h is inner. Therefore we may assume both h and g are not inner derivation of U. This means in particular that $c \in C$ and $g = \gamma h$. Again by (3.2) and since g and h are both outer derivations of U, we have that U satisfies

$$\begin{aligned} &u\left(v[x_1, x_2] + \gamma[t_1, x_2] + \gamma[x_1, t_2]\right) \\ &\qquad + h(v)[x_1, x_2] + v[t_1, x_2] + b[x_1, t_2] + \gamma[z_1, x_2] + 2\gamma[t_1, t_2] \\ &\qquad + \gamma[x_1, z_2] - \Big[uvx_1 + \gamma ut_1 + h(v)x_1 + bt_1 + \gamma z_1, x_2\Big] \\ &\qquad - \Big[x_1, uvx_2 + \gamma ut_2 + h(v)x_2 + vt_2 + \gamma z_2\Big] \end{aligned} \tag{3.7}$$

and in particular U satisfies $2\gamma[t_1, t_2]$, which gives the contradiction that U is commutative. $\square$

REMARK 3.2. In light of previous Theorem, in all that follows we'll always assume that h and g are inner derivations of U, that is there exist p_1, p_2 elements of U such that $h(x) = [p_1, x]$ and $g(x) = [p_2, x]$ for all $x \in R$. Thus $F(x) = (u + p_1)x - xp_1$ and $G(x) = (v + p_2)x - xp_2$, for all $x \in R$. For sake of clearness we denote $(u + p_1) = a$, $-p_1 = b$, $(v + p_2) = c$ and $-p_2 = q$, so that $F(x) = ax + xb$ and $G(x) = cx + xq$.

Now we consider the case when R does not satisfy the standard identity of degree 4, that is $\dim_C(RC) > 4$:

THEOREM 3.3. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, C the extended centroid of R, F and G non-zero inner generalized derivations of R defined as $F(x) = ax + xb$ and $G(x) = cx + xq$, for suitable $a, b, c, q, \in U$. Assume that $\dim_C(RC) > 4$. If the composition (FG) acts as a Lie derivation on R, then (FG) is a derivation of R and one of the following holds:*

1. *there exist $\alpha \in C$ and $a \in U$ such that $F(x) = [a, x]$ and $G(x) = \alpha x$, for all $x \in R$;*
2. *G is an inner usual derivation of R and there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
3. *there exist $a', c' \in U$ such that $F(x) = a'x$, $G(x) = c'x$, for all $x \in R$, with $a'c' = 0$;*
4. *there exist $b', q' \in U$ such that $F(x) = xb'$, $G(x) = xq'$, for all $x \in R$, with $q'b' = 0$;*
5. *there exist $c', q' \in U$, $\eta, \gamma \in C$ such that $F(x) = \eta(xq' - c'x) + \gamma x$, $G(x) = c'x + xq'$, for all $x \in R$, with $\gamma c' - \eta c'^2 = -\gamma q' - \eta q'^2$.*

PROOF. Here we denote $H(x) = FG(x)$, for all $x \in R$. By the main hypothesis we have that H acts as a Lie derivation of R. Since $char(R) \neq 2$ and $\dim_C(RC) > 4$, by Theorem 3 in [2], there exist a derivation D from R into its central closure RC and f an additive mapping of R into C sending commutators in zero, such that $H = D + f$. We also recall that derivations, Lie derivations and generalized derivations of R can be extended to RC in a natural way, for instance as pointed out in [1]. Therefore $H([x, y]) = D([x, y])$ for all $x, y \in R$. In this situation the result follows from Proposition 2.6. $\square$

Thanks to the results contained in Theorem 3.1 and Theorem 3.3, we have to prove our main result just in the case $F(x) = ax + xb$ and $G(x) = cx + xq$ and $\dim_C(RC) \leq 4$. In particular this last means that $U \cong M_2(C)$, the ring of all 2×2 matrices over C. In other words R satisfies the generalized polynomial identity

$$
\begin{aligned}
&a(c[x_1,x_2]+[x_1,x_2]q)+(c[x_1,x_2]+[x_1,x_2]q)b\\
&\quad-[a(cx_1+x_1q)+(cx_1+x_1q)b,x_2]\\
&\quad-[x_1,a(cx_2+x_2q)+(cx_2+x_2q)b]
\end{aligned}
\tag{3.8}
$$

for $a,b,c,q \in U$. Since U and R satisfy the same generalized polynomial identities with coefficients in U (see [4]), then all $r_1, r_2 \in R$. If $q \in Z(R)$, then (3.8) is an identity also for U. Thus, without loss of generality we may replace R by U and consider the case $R \cong M_2(C)$.

We dedicate the last Section of the paper to analyse this case.

4. The 2×2 Matrix Case.

Although we need to prove our result only in the case $R \cong M_2(C)$, the ring of 2×2 matrices over C, we would like to point out that the same techniques and results also hold in the case $R \cong M_m(C)$, for $m \geq 3$.

Firstly we fix the following:

Remark 4.1. Since both F and G are inner generalized derivations, namely $F(x) = ax + xb$ and $G(x) = cx + xq$ for all $x \in R$, then we have that

$$
\begin{aligned}
&a(c[r_1,r_2]+[r_1,r_2]q)+(c[r_1,r_2]+[r_1,r_2]q)b\\
&\quad-[a(cr_1+r_1q)+(cr_1+r_1q)b,r_2]-[r_1,a(cr_2+r_2q)+(cr_2+r_2q)b]=0
\end{aligned}
$$

for all $r_1, r_2 \in R$. Moreover, for any inner automorphism φ of $M_m(K)$, we have that

$$
\begin{aligned}
&\varphi(a)(\varphi(c)[r_1,r_2]+[r_1,r_2]\varphi(q))+(\varphi(c)[r_1,r_2]+[r_1,r_2]\varphi(q))\varphi(b)\\
&\quad-[\varphi(a)(\varphi(c)r_1+r_1\varphi(q))+(\varphi(c)r_1+r_1\varphi(q))\varphi(b),r_2]\\
&\quad-[r_1,\varphi(a)(\varphi(c)r_2+r_2\varphi(q))+(\varphi(c)r_2+r_2\varphi(q))\varphi(b)]=0
\end{aligned}
$$

for all $r_1, r_2 \in R$. Notice that, for any authomorphism φ of R, a matrix X is central iff $\varphi(X)$ is central. Hence, to prove our result, we may replace a, b, c, q respectively with $\varphi(a)$, $\varphi(b)$, $\varphi(c)$, $\varphi(q)$.

In order to prove the main Theorem, we also need the following lemma:

Lemma 4.2. *Let F be a infinite field and $n \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_n(F)$ then there exists some invertible matrix $Q \in M_n(F)$ such that each matrix $QA_1Q^{-1}, \dots, QA_kQ^{-1}$ has all non-zero entries (for the proof see [5, Lemma 1.5])*

We begin with:

Lemma 4.3. *Let K be an infinite field, let $R = M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist*

$a, b, c, q \in R$ such that

$$a(c[r_1, r_2] + [r_1, r_2]q) + (c[r_1, r_2] + [r_1, r_2]q)b - [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] - [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] = 0$$

for all $r_1, r_2 \in R$. If $q \in Z(R)$, then one of the following holds:

1. *c is a central matrix and $c + q = 0$;*
2. *c is a central matrix and $a = -b$;*
3. *b is a central matrix and $(a + b)(c + q) = 0$.*

Proof. Since $q \in Z(R)$, by the assumption we have that

$$a(c + q)[r_1, r_2] + (c + q)[r_1, r_2]b - [a(c + q)r_1 + (c + q)r_1b, r_2] - [r_1, a(c + q)r_2 + (c + q)r_2b] = 0 \tag{4.1}$$

for all $r_1, r_2 \in R$. Since in case $c + q = 0$ we are done, we assume $0 \neq c + q$. Notice that if $0 \neq c + q \in Z(R)$, then $c \in Z(R)$ and by (4.1) it follows that

$$a[r_1, r_2] + [r_1, r_2]b - [ar_1 + r_1b, r_2] - [r_1, ar_2 + r_2b] = 0$$

for all $r_1, r_2 \in R$. Thus by Remark 2.2 we get $a = -b$ and the proof is finished. Analogously, in case $b \in Z(R)$, again by (4.1)

$$(a + b)(c + q)[r_1, r_2] - [(a + b)(c + q)r_1, r_2] - [r_1, (a + b)(c + q)r_2] = 0$$

for all $r_1, r_2 \in R$. In particular for $r_1 = e_{ii}$, $r_2 = e_{ij}$ with $i \neq j$, we have $e_{ij}(a+b)(c+q)e_{ii} - e_{ii}(a+b)(c+q)e_{ij} = 0$, which implies easily $(a+b)(c+q) = 0$, and we are done again. Therefore we may assume $c+q$ and b both non-scalar matrices. We will prove that in this case we get a contradiction.

By Remark 4.1 and Lemma 4.2, we can assume that $c + q$ and b have all non-zero entries, say $c + q = \sum_{kl} t_{kl}e_{kl}$ and $b = \sum_{kl} b_{kl}e_{kl}$, for $0 \neq t_{kl}, 0 \neq b_{kl} \in K$.

Starting again from (4.1), for any $i \neq j$ and $r_1 = e_{ii}$, $r_2 = e_{ij}$, we get

$$(c + q)e_{ij}b - (c + q)e_{ii}be_{ij} + e_{ij}a(c + q)e_{ii} + e_{ij}(c + q)e_{ii}b - e_{ii}a(c + q)e_{ij} - e_{ii}(c + q)e_{ij}b + (c + q)e_{ij}be_{ii} = 0$$

moreover left multiplying by e_{jj} and right multiplying by e_{ii} it follows $2t_{ji}b_{ji} = 0$, a contradiction. □

Analogously one may prove the following (we omit the proof for brevity):

Lemma 4.4. *Let K be an infinite field, let $R = M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a, b, c, q \in R$ such that*

$$a(c[r_1, r_2] + [r_1, r_2]q) + (c[r_1, r_2] + [r_1, r_2]q)b - [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] - [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] = 0$$

for all $r_1, r_2 \in R$. If $c \in Z(R)$, then one of the following holds:

1. *q is a central matrix and $c + q = 0$;*

2. *q is a central matrix and $a = -b$;*
3. *a is a central matrix and $(c+q)(a+b) = 0$.*

LEMMA 4.5. *Let K be an infinite field, let $R = M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a, b, c, q \in R$ such that*

$$\begin{aligned} &a(c[r_1,r_2] + [r_1,r_2]q) + (c[r_1,r_2] + [r_1,r_2]q)b \\ &\quad - [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] - [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] = 0 \end{aligned}$$

for all $r_1, r_2 \in R$. If $b \in Z(R)$, then one of the following holds:

1. *a is a central matrix and $a + b = 0$;*
2. *a is a central matrix and $c = -q$;*
3. *q is a central matrix and $(a+b)(c+q) = 0$.*

PROOF. By Lemma 4.3 we may suppose $q \notin Z(R)$, moreover here we assume $a + b$ is a non-scalar matrix and prove that a contradiction follows. Denote $q = \sum_{kl} q_{kl}e_{kl}$ and $a + b = \sum_{kl} t_{kl}e_{kl}$ for $q_{kl}, t_{kl} \in K$.

By Remark 4.1 and Lemma 4.2, we may assume that q and a have all non-zero entries. Since $b \in Z(R)$, we have that

$$\begin{aligned} &(a+b)(c[r_1,r_2] + [r_1,r_2]q) - [(a+b)(cr_1 + r_1q), r_2] \\ &\quad - [r_1, (a+b)(cr_2 + r_2q)] = 0 \end{aligned} \tag{4.2}$$

for all $r_1, r_2 \in R$. Once again for any $i \neq j$ and $r_1 = e_{ii}$, $r_2 = e_{ij}$, we get

$$\begin{aligned} &(a+b)e_{ij}q - (a+b)e_{ii}qe_{ij} + e_{ij}(a+b)ce_{ii} + e_{ij}(a+b)e_{ii}q \\ &\quad - e_{ii}(a+b)ce_{ij} - e_{ii}(a+b)e_{ij}q + (a+b)e_{ij}qe_{ii} = 0. \end{aligned}$$

As in previous Lemma, left multiplying by e_{jj} and right multiplying by e_{ii} it follows $2t_{ji}q_{ji} = 0$, a contradiction.

Therefore $a + b$ must be a central matrix, that is $a, b \in Z(R)$. In case $a + b = 0$ we are done, in the other case by (4.2) we have

$$(c[r_1,r_2] + [r_1,r_2]q) - [(cr_1 + r_1q), r_2] - [r_1, (cr_2 + r_2q)] = 0$$

for all $r_1, r_2 \in R$ and by Remark 2.2 it follows $c = -q$ □

Analogously:

LEMMA 4.6. *Let K be an infinite field, let $R = M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a, b, c, q \in R$ such that*

$$\begin{aligned} &a(c[r_1,r_2] + [r_1,r_2]q) + (c[r_1,r_2] + [r_1,r_2]q)b \\ &\quad - [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] - [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] = 0 \end{aligned}$$

for all $r_1, r_2 \in R$. If $a \in Z(R)$, then one of the following holds:

1. *b is a central matrix and $a + b = 0$;*
2. *b is a central matrix and $c = -q$;*

3. *c is a central matrix and* $(c+q)(a+b)=0$.

LEMMA 4.7. *Let K be an infinite field, let $R=M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a,b,c,q\in R$ such that*

$$a(c[r_1,r_2]+[r_1,r_2]q)+(c[r_1,r_2]+[r_1,r_2]q)b \\ -[a(cr_1+r_1q)+(cr_1+r_1q)b,r_2]-[r_1,a(cr_2+r_2q)+(cr_2+r_2q)b]=0$$

for all $r_1,r_2\in R$. Denote $a=\sum_{kl}a_{kl}e_{kl}$, $b=\sum_{kl}b_{kl}e_{kl}$, $c=\sum_{kl}c_{kl}e_{kl}$, $q=\sum_{kl}q_{kl}e_{kl}$, for suitable a_{kl}, b_{kl}, c_{kl} and q_{kl} elements of K. If there are $i\neq j$ such that $q_{ji}\neq 0$, $c_{ji}\neq 0$ and $b_{ji}=0$, then $a_{ri}=0$ and $b_{rk}=0$ for all $r\neq i$ and $k\neq r$ (that is the only non-zero off-diagonal elements of b fall in the i-th row).

PROOF. By our hypothesis, for $r_1=e_{ii}$ and $r_2=e_{ij}$ we have:

$$\begin{aligned} X=&a(ce_{ij}+e_{ij}q)+(ce_{ij}+e_{ij}q)b \\ &-[a(ce_{ii}+e_{ii}q)+(ce_{ii}+e_{ii}q)b,e_{ij}] \\ &-[e_{ii},a(ce_{ij}+e_{ij}q)+(ce_{ij}+e_{ij}q)b]=0 \end{aligned}$$

and in particular, for all $r\neq i$, the (r,i)-entry of the matrix X is $2a_{ri}q_{ji}=0$, that is $a_{ri}=0$. Notice that since $m=2$, the proof of Lemma is complete. Thus in the following we assume $m\geq 3$. Moreover, for all $s\neq i,j$, the (j,s)-entry of X is $a_{ji}q_{js}+c_{ji}b_{js}=0$, and in light of previous argument, it follows $b_{js}=0$. Analogously for for $r_1=e_{ii}$ and $r_2=e_{it}$, with $t\neq i,j$, we also have

$$\begin{aligned} Y=&a(ce_{it}+e_{it}q)+(ce_{it}+e_{it}q)b \\ &-[a(ce_{ii}+e_{ii}q)+(ce_{ii}+e_{ii}q)b,e_{it}] \\ &-[e_{ii},a(ce_{it}+e_{it}q)+(ce_{it}+e_{it}q)b]=0. \end{aligned}$$

Notice that the (j,i)-entry of Y is $2c_{ji}b_{ti}=0$ and the (j,k)-one is $c_{ji}b_{tk}=0$, for all $k\neq i,t$. These imply that $b_{ti}=0$ and $b_{tk}=0$. From all the previous equalities it follows that $b_{tr}=0$, for all $t\neq i$ and for all $r\neq t$. □

LEMMA 4.8. *Let K be an infinite field, let $R=M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a,b,c,q\in R$ such that*

$$(*)\qquad \begin{aligned} &a(c[r_1,r_2]+[r_1,r_2]q)+(c[r_1,r_2]+[r_1,r_2]q)b \\ &\quad-[a(cr_1+r_1q)+(cr_1+r_1q)b,r_2] \\ &\quad-[r_1,a(cr_2+r_2q)+(cr_2+r_2q)b]=0 \end{aligned}$$

for all $r_1,r_2\in R$. Denote

$$a=\sum_{kl}a_{kl}e_{kl},\quad b=\sum_{kl}b_{kl}e_{kl},\quad c=\sum_{kl}c_{kl}e_{kl},\quad q=\sum_{kl}q_{kl}e_{kl},$$

for suitable a_{kl}, b_{kl}, c_{kl} and q_{kl} elements of K. If there are $i \neq j$ such that $b_{ji} = 0$ and $q_{rs} \neq 0$, $c_{rs} \neq 0$ for all $r \neq s$, then b is central in R.

PROOF. Without loss of generality we may assume $b_{21} = 0$. In case a is a central matrix and since c is not central, by Lemma 4.6 we have the required conclusion $b \in Z(R)$. Therefore we assume that $a \notin Z(R)$ and prove that a contradiction follows. Firstly we notice that by Lemma 4.7 we have $a_{21} = 0$.

In (*) consider $r_1 = e_{22}$ and $r_2 = e_{12}$, thus we have

$$\begin{aligned} X = & -ae_{12}q - ce_{12}b - ae_{22}qe_{12} - ce_{22}be_{12} - e_{22}qbe_{12} + e_{12}ace_{22} \\ & + e_{12}ae_{22}q + e_{12}ce_{22}b - e_{22}ace_{12} - e_{22}ae_{12}q - e_{22}ce_{12}b \\ & + ae_{12}qe_{22} + ce_{12}be_{22} + e_{12}qbe_{22} = 0 \end{aligned}$$

in particular the $(1,1)$-entry of X is $q_{21}(a_{11} - a_{22}) = 0$, that is

$$a_{11} = a_{22} \tag{4.3}$$

and in light of this the $(1,2)$-entry of X is

$$(a_{11} + b_{22})(c_{22} + q_{22}) + q_{21}(b_{12} - a_{12}) = 0. \tag{4.4}$$

Choose now $r_1 = e_{11}$ and $r_2 = e_{12}$, thus by (*) we have

$$\begin{aligned} Y = & \, ae_{12}q + ce_{12}b - ae_{11}qe_{12} - ce_{11}be_{12} - e_{11}qbe_{12} + e_{12}ace_{11} \\ & + e_{12}ae_{11}q + e_{12}ce_{11}b - e_{11}ace_{12} - e_{11}ae_{12}q - e_{11}ce_{12}b \\ & + ae_{12}qe_{11} + ce_{12}be_{11} + e_{12}qbe_{11} = 0. \end{aligned}$$

The $(2,2)$-entry of the matrix Y is $c_{21}(b_{22} - b_{11}) = 0$, that is

$$b_{11} = b_{22} \tag{4.5}$$

and by (4.4) we have

$$(a_{11} + b_{11})(c_{22} + q_{22}) + q_{21}(b_{12} - a_{12}) = 0. \tag{4.6}$$

Finally in (*) let $r_1 = e_{12}$ and $r_2 = e_{21}$, then by calculations we have that

$$\begin{aligned} T = & \, ae_{11}q - ae_{22}q + ce_{11}b - ce_{22}b - ae_{12}qe_{21} - ce_{12}be_{21} - e_{12}qbe_{21} \\ & - e_{21}ace_{12} + e_{21}ae_{12}q + e_{21}ce_{12}b - e_{12}ace_{21} - e_{12}ae_{21}q - e_{12}ce_{21}b \\ & + ae_{21}qe_{12} + ce_{21}be_{12} + e_{21}qbe_{12} = 0. \end{aligned}$$

In particular the $(1,1)$-entry of T is

$$(a_{11} + b_{11})(c_{22} + q_{22}) + q_{21}(b_{12} + a_{12}) = 0. \tag{4.7}$$

Hence, by comparing (4.6) with (4.7), we get $q_{21}a_{12} = 0$, that is $a_{12} = 0$ and the contradiction $a \in Z(R)$ follows. $\square$

For sake of clearness, if we replace the element ac with u and the element qb with p, we may write the previous Lemma as follows.

Remark 4.9. Let K be an infinite field, let $R = M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a, b, c, q, p, u \in R$ such that

$$u[r_1, r_2] + a[r_1, r_2]q + c[r_1, r_2]b + [r_1, r_2]p \\ - [ur_1 + ar_1q + cr_1b + r_1p, r_2] - [r_1, ur_2 + ar_2q + cr_2b + r_2p] = 0$$

for all $r_1, r_2 \in R$. Denote

$$b = \sum_{kl} b_{kl}e_{kl}, \quad c = \sum_{kl} c_{kl}e_{kl}, \quad q = \sum_{kl} q_{kl}e_{kl},$$

for suitable b_{kl}, c_{kl} and q_{kl} elements of K. If there are $i \neq j$ such that $b_{ji} = 0$ and $q_{rs} \neq 0$, $c_{rs} \neq 0$ for all $r \neq s$, then b is central in R.

Proposition 4.10. *Let K be an infinite field, let $R = M_2(K)$ be the algebra of 2×2 matrices over K, $Z(R)$ the center of R. Assume that there exist $a, b, c, q \in R$ such that*

$$a(c[r_1, r_2] + [r_1, r_2]q) + (c[r_1, r_2] + [r_1, r_2]q)b \\ - [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] - [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] = 0$$

for all $r_1, r_2 \in R$. Then one of the following holds:

1. *$b, q \in Z(R)$ and $(a + b)(c + q) = 0$;*
2. *$a, c \in Z(R)$ and $(c + q)(a + b) = 0$;*
3. *$c, q \in Z(R)$ and $a + b = 0$;*
4. *$a, b \in Z(R)$ and $c + q = 0$;*
5. *there exist $\lambda, \mu \in C$ such that $b + \mu q = \beta \in C$, $a - \mu c = \alpha \in C$ and $\mu c^2 + \lambda c = \mu q^2 - \lambda q$.*

Proof. Let

$$a = \sum_{kl} a_{kl}e_{kl}, \quad b = \sum_{kl} b_{kl}e_{kl}, \quad c = \sum_{kl} c_{kl}e_{kl}, \quad q = \sum_{kl} q_{kl}e_{kl},$$

for suitable a_{kl}, b_{kl}, c_{kl} and q_{kl} elements of K.

Clearly if one of q, c, b or a is a scalar matrix we are done by Lemmas 4.3, 4.4, 4.5 or 4.6 respectively. In order to prove the Proposition, we may assume that q, c, b and a are non-central matrices.

By Remark 4.1 and Lemma 4.2, there exists some invertible matrix $Q \in M_2(K)$ such that $QqQ^{-1} = q'$, $QcQ^{-1} = c'$, $QbQ^{-1} = b'$ and $QaQ^{-1} = a'$ have all non-zero entries. By this conjugation we denote

$$a' = \sum_{kl} a'_{kl}e_{kl}, \quad b' = \sum_{kl} b'_{kl}e_{kl}, \quad c' = \sum_{kl} c'_{kl}e_{kl}, \quad q' = \sum_{kl} q'_{kl}e_{kl},$$

for suitable a'_{kl}, b'_{kl}, c'_{kl} and q'_{kl} elements of K, the conjugates of elements a, b, c, q. Of course

$$\begin{aligned}a'(c'[r_1,r_2]+[r_1,r_2]q')+(c'[r_1,r_2]+[r_1,r_2]q')b'\\-[a'(c'r_1+r_1q')+(c'r_1+r_1q')b',r_2]\\-[r_1,a'(c'r_2+r_2q')+(c'r_2+r_2q')b']=0\end{aligned}$$

for all $r_1, r_2 \in R$.

Since $q'_{rs} \neq 0$ and $c'_{rs} \neq 0$ for all $r \neq s$, then the following holds: if for some $i \neq j$ there is some $b'_{ji} = 0$ then by Lemma 4.8 b' is a central matrix, that is also b is a central matrix, a contradiction.

Hence assume that $b'_{rs} \neq 0$ for all $r \neq s$. Let $\eta = \frac{b'_{ji}}{q'_{ji}} \neq 0$ and $a'' = a' + \eta c'$. By replacing a' with $a'' - \eta c'$ in the main equation we get

$$\begin{aligned}(a''-\eta c')(c'[r_1,r_2]+[r_1,r_2]q')+(c'[r_1,r_2]+[r_1,r_2]q')b'\\-[(a''-\eta c')(c'r_1+r_1q')+(c'r_1+r_1q')b',r_2]\\-[r_1,(a''-\eta c')(c'r_2+r_2q')+(c'r_2+r_2q')b']=0\end{aligned}$$

for all $r_1, r_2 \in R$, that is

$$\begin{aligned}(a''-\eta c')c'[r_1,r_2]+a''[r_1,r_2]q'+c'[r_1,r_2](b'-\eta q')+[r_1,r_2]q'b'\\-[(a''-\eta c')c'r_1+a''r_1q'+c'r_1(b'-\eta q')+r_1q'b',r_2]\\-[r_1,(a''-\eta c')c'r_2+a''r_2q'+c'r_2(b'-\eta q')+r_2q'b']=0\end{aligned}$$

for all $r_1, r_2 \in R$, where the (j, i) entry of the matrix $b' - \eta q'$ is zero and $q'_{rs} \neq 0$ and $c'_{rs} \neq 0$ for all $r \neq s$. Thus by Remark 4.9 it follows that $b' - \eta q'$ is a central matrix, that is $b - \eta q = \beta \in Z(R)$.

Thus, by replacing b with $\eta q + \beta$ in the main assumption, we get

$$\begin{aligned}a(c[r_1,r_2]+[r_1,r_2]q)+(c[r_1,r_2]+[r_1,r_2]q)(\eta q+\beta)\\-[a(cr_1+r_1q)+(cr_1+r_1q)(\eta q+\beta),r_2]\\-[r_1,a(cr_2+r_2q)+(cr_2+r_2q)(\eta q+\beta)]=0\end{aligned}$$

for all $r_1, r_2 \in R$.

Suppose here that $a + \eta c$ is not a scalar matrix. Since q and c are not a scalar matrices, then there exists some invertible matrix $P \in M_m(K)$ such that $PqP^{-1} = q'''$, $PcP^{-1} = c'''$ and $P(a + \eta c)P^{-1} = p'''$ have all non-zero entries. As above, by this conjugation we denote

$$a''' = \sum_{kl} a'''_{kl}e_{kl}, \quad c''' = \sum_{kl} c'''_{kl}e_{kl}, \quad q''' = \sum_{kl} q'''_{kl}e_{kl}, \quad p''' = \sum_{kl} p'''_{kl}e_{kl}$$

for suitable a'''_{kl}, c'''_{kl}, q'''_{kl} and p'''_{kl} elements of K, the conjugates of elements $a, c, q, (a+\eta c)$. Then

$$\begin{aligned} a'''(c'''[r_1,r_2] + [r_1,r_2]q''') + (c'''[r_1,r_2] + [r_1,r_2]q''')(\eta q''' + \beta) \\ - [a'''(c'''r_1 + r_1q''') + (c'''r_1 + r_1q''')(\eta q''' + \beta), r_2] \\ - [r_1, a'''(c'''r_2 + r_2q''') + (c'''r_2 + r_2q''')(\eta q''' + \beta)] = 0 \end{aligned}$$

for all $r_1, r_2 \in R$. Choosing $r_1 = e_{ii}$ and $r_2 = e_{ij}$, with $i \neq j$, right multiplying by e_{ii} and left multiplying by e_{jj} it follows $(a'''_{ji} + \eta c'''_{ji})q'''_{ji} = 0$, a contradiction again.

Therefore $a + \eta c$ must be a central matrix, say $a + \eta c = \alpha \in Z(R)$. Thus by calculations we notice that

$$\begin{aligned} a(c[r_1,r_2] + [r_1,r_2]q) + (c[r_1,r_2] + [r_1,r_2]q)b \\ = \Big((\alpha+\beta)c - \eta c^2\Big)[r_1,r_2] + [r_1,r_2]\Big((\alpha+\beta)q + \eta q^2\Big) \end{aligned}$$

and by the main assumption

$$\begin{aligned} \Big((\alpha+\beta)c - \eta c^2\Big)[r_1,r_2] + [r_1,r_2]\Big((\alpha+\beta)q + \eta q^2\Big) \\ = [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] + [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] \end{aligned}$$

for all $r_1, r_2 \in R$. Hence by Remark 2.2 we conclude $(\alpha+\beta)c - \eta c^2 = -(\alpha+\beta)q - \eta q^2$. $\square$

Finally we may prove:

Theorem 4.11. *Let $R = M_2(C)$ be the algebra of 2×2 matrices over the field C. Assume that there exist $a, b, c, q \in R$ such that*

$$\begin{aligned} a(c[r_1,r_2] + [r_1,r_2]q) + (c[r_1,r_2] + [r_1,r_2]q)b \\ - [a(cr_1 + r_1q) + (cr_1 + r_1q)b, r_2] - [r_1, a(cr_2 + r_2q) + (cr_2 + r_2q)b] = 0 \end{aligned}$$

for all $r_1, r_2 \in R$. Then one of the following holds:

1. *$b, q \in C$ and $(a+b)(c+q) = 0$;*
2. *$a, c \in C$ and $(c+q)(a+b) = 0$;*
3. *$c, q \in C$ and $a + b = 0$;*
4. *$a, b \in C$ and $c + q = 0$;*
5. *there exist $\lambda, \mu \in C$ such that $b + \mu q = \beta \in C$, $a - \mu c = \alpha \in C$ and $\mu c^2 + \lambda c = \mu q^2 - \lambda q$.*

Proof. If one assumes that C is infinite, the conclusion follows from Proposition 4.10.

Now let K be an infinite field which is an extension of the field C and let $\overline{R} = M_m(K) \cong R \otimes_C K$. Consider the generalized polynomial

$$\begin{aligned} P(x_1, x_2) = a(c[x_1, x_2] + [x_1, x_2]q) + (c[x_1, x_2] + [x_1, x_2]q)b \\ - [a(cx_1 + x_1q) + (cx_1 + x_1q)b, x_2] \\ - [x_1, a(cx_2 + x_2q) + (cx_2 + x_2q)b] \end{aligned}$$

which is a generalized polynomial identity for R. Since $P(x_1, x_2)$ is a multilinear generalized polynomial in the indeterminates x_1, x_2, then it is a generalized polynomial identity for $\overline{R}$ and the conclusion follows from Proposition 4.10. □

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Received: 17.5.2011.
Revised: 24.8.2011.