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GLOBAL SPACE-TIME L^p-ESTIMATES FOR THE AIRY OPERATOR ON $L^2(\mathbb{R}^2)$ AND SOME APPLICATIONS

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Abstract. Let L be the Airy operator. The aim of this paper is to prove some a priori estimates for L defined as an unbounded operator on $L^2(\mathbb{R}^2)$. Some applications and counterexamples are also given.

1. Introduction

Consider the following initial value problem

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) u(x,t) = g(x,t), \\ u(x,\alpha) = f(x) \end{cases} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$.

The main question asked in this paper is to what space does u belong to whenever $u \in L^2(\mathbb{R}^2)$ such that $u_t + u_{x^3} \in L^2(\mathbb{R}^2)$? This is a natural question and it is important since it is about the L^2-domain of $\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}$. It has an analog for other operators (see [4–7]).

Throughout this paper L denotes the operator $\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}$, which is an unbounded linear operator with domain

$$D(L) = \{u \in L^2(\mathbb{R}^2) : Lu, \text{ as a distribution, is an } L^2(\mathbb{R}^2)\text{-function}\}.$$

Using the L^2-Fourier transform, we see that iL (where $i = \sqrt{-1}$) is unitarily equivalent to a multiplication operator by a *real-valued* function and hence iL is self-adjoint. So one of the questions asked in this paper is what "real potential" V can be added to iL without destroying the self-adjointness of $iL + V$.

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To this end, we first prove some *a priori* estimates of the type

$$\|u\| \leq a\|Lu\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)} \tag{1.2}$$

for some norm (on the left hand side). The proof is based on a Duhamel's principle, which we give without proof in the case of the Airy equation, and on Theorem 2.2 below. As an application, we use the Kato-Rellich perturbation theorem to deduce sufficient conditions on V making $iL + V$ self-adjoint.

We also give a counterexample showing that estimate (1.2) with L^p-norm ($8 < p \leq \infty$) on $\mathbb{R}^2$ (on the left hand-side) does not hold.

We note that throughout this paper, the value of the constant c may differ from line to line.

Finally, any result or definition used in this paper will be assumed to be known by the reader. The references needed are [1, 7].

2. Main Results

The following lemma is a Duhamel's principle for the Airy equation. To our best knowledge, this version of it does not exist for the Airy equation. However, we omit the proof since the latter very similar to that of the case of the heat equation (cf. [1]).

Lemma 2.1. *Let u_s be a solution of*

$$\begin{cases} Lu_s = 0, \ t > s, \\ u_s(x,s) = g(x,s). \end{cases}$$

Then

$$u(x,t) = \int_\alpha^t u_s(x,t)ds$$

is a solution of IVP 1.1 (with $f = 0$).

Before stating and proving the first main result in this article, we recall the following result which will be a key one for the proof of Theorem 2.3.

Theorem 2.2 (Kenig-Ponce-Vega [2]). *Let u be a solution of IVP 1.1 with $g = 0$. Then there exists a positive constant c such that*

$$\|u\|_{L^8(\mathbb{R}^2)} \leq c\|f\|_{L^2(\mathbb{R})}.$$

Here is the first main result.

Theorem 2.3. *For all $a > 0$, there exists a $b > 0$ such that*

$$\|u\|_{L^8(\mathbb{R}^2)} \leq a\|Lu\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)} \tag{2.1}$$

for all $u \in D(L)$.

PROOF. We first prove the theorem for $u \in C_0^\infty(\mathbb{R}^2)$ (the space of infinitely differentiable functions with compact support). In the end of this proof, we will extend this to $D(L)$-functions. We use the fact that any such u is, for any $\alpha \in \mathbb{R}$, the unique solution of IVP 1.1, where $f(x) = u(x,\alpha)$ and $g = Lu$.

Let $k \in \mathbb{Z}$ and let t and α be such that $k \le t \le k+1$ and $k \le \alpha \le k+1$ (hence $|t-\alpha| \le 1$).

Let u be a solution of IVP 1.1. Now we split u into two parts $u = u_1 + u_2$ where u_1, u_2 are the solutions of

$$\begin{cases} Lu_1(x,t) = g(x,t), \\ u_1(x,\alpha) = 0, \end{cases} \quad \text{and} \quad \begin{cases} Lu_2(x,t) = 0, \\ u_2(x,\alpha) = f(x) \end{cases}$$

respectively.

The following estimate is deduced from Theorem 2.2

$$\|u_2\|_{L^8(\mathbb{R}^2)} \le c\|f\|_{L^2(\mathbb{R})} = c\|u_2(\cdot,\alpha)\|_{L^2(\mathbb{R})} = \|u(\cdot,\alpha)\|_{L^2(\mathbb{R})}.$$

Hence

$$\|u_2\|_{L^8(\mathbb{R}\times[k,k+1])} \le c\|u(\cdot,\alpha)\|_{L^2(\mathbb{R})}.$$

Squaring both sides of the previous inequality, integrating with respect to α on $[k,k+1]$ and taking square roots yield

$$\|u_2\|_{L^8(\mathbb{R}\times[k,k+1])} \le c\|u\|_{L^2(\mathbb{R}\times[k,k+1])}.$$

For u_1, we cannot apply directly Theorem 2.2 and here is where Lemma 2.1 intervenes. Adopting the notations of Lemma 2.1, we see that Theorem 2.2 can be applied to u_s where

$$u_1(x,t) = \int_\alpha^t u_s(x,t)ds.$$

The Hölder inequality then gives

$$|u_1(x,t)| \le c|t-\alpha|^{\frac{7}{8}} \left(\int_\alpha^t |u_s(x,t)|^8 ds\right)^{\frac{1}{8}} \le c\left(\int_\alpha^t |u_s(x,t)|^8 ds\right)^{\frac{1}{8}}$$

and hence

$$\int_{\mathbb{R}} |u_1(x,t)|^8 dx \le c\int_{\mathbb{R}}\int_\alpha^t |u_s(x,t)|^8 dsdx$$

and since $[\alpha,t] \subset [k,k+1]$, one is led to

$$\int_{\mathbb{R}} |u_1(x,t)|^8 dx \le c\int_{\mathbb{R}}\int_k^{k+1} |u_s(x,t)|^8 dsdx.$$

Integrating against t over $[k,k+1]$ yields

$$\begin{aligned} \|u_1\|^8_{L^8(\mathbb{R}\times[k,k+1])} &= \int_k^{k+1}\int_{\mathbb{R}} |u_1(x,t)|^8 dxdt \\ &\le c\int_k^{k+1}\int_{\mathbb{R}}\int_k^{k+1} |u_s(x,t)|^8 dsdxdt \end{aligned}$$

or

$$\|u_1\|^8_{L^8(\mathbb{R}\times[k,k+1])} \le c\int_k^{k+1} \|u_s\|^8_{L^8(\mathbb{R}\times[k,k+1])}ds$$

and Theorem 2.2 implies

$$\|u_1\|^8_{L^8(\mathbb{R}\times[k,k+1])} \le c\int_k^{k+1} \|Lu\|^8_{L^2(\mathbb{R}\times[k,k+1])}ds = c\|Lu\|^8_{L^2(\mathbb{R}\times[k,k+1])}.$$

Therefore, one has in the end

$$\begin{aligned}\|u\|_{L^8(\mathbb{R}\times[k,k+1])} &\le \|u_1\|_{L^8(\mathbb{R}\times[k,k+1])} + \|u_2\|_{L^(\mathbb{R}\times[k,k+1])}\\ &\le c\|Lu\|_{L^2(\mathbb{R}\times[k,k+1])} + c\|u\|_{L^2(\mathbb{R}\times[k,k+1])}.\end{aligned}$$

Summing in k over $\mathbb{Z}$ gives us

$$\|u\|_{L^8(\mathbb{R}^2)} \le a\|Lu\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)},$$

establishing the result. Making the constant a in front of $\|Lu\|_{L^2(\mathbb{R}^2)}$ arbitrary follows easily from the change of variables

$$u_r(x,t) = u(rx, r^3t),\ r > 0.$$

To finish off the proof, we now show the validity of the theorem for functions u in $D(L)$. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $D(L)$ with respect to the graph norm of L (the proof is similar to the density of $C_0^\infty(\mathbb{R}^n)$ in Sobolev spaces, cf [3]), for each $u \in D(L)$, there is $u_n \in C_0^\infty(\mathbb{R}^2)$ such that

$$\|u_n - u\|_2 \longrightarrow 0 \text{ and } \|Lu_n - Lu\|_2 \longrightarrow 0$$

and hence

$$\|u_n\|_2 \longrightarrow \|u\|_2 \text{ and } \|Lu_n\|_2 \longrightarrow \|Lu\|_2.$$

We also know there is $u_{n(k)}$ such that $u_{n(k)}(x,t) \to u(x,t)$ a.e. Applying (2.1) to $u_{n(k)}$ and using Fatou's lemma yield

$$\begin{aligned}\int_{\mathbb{R}^2}|u(x,t)|^8dxdt &= \int_{\mathbb{R}^2}\liminf_{k\to\infty}|u_{n(k)}(x,t)|^8dxdt \le \liminf_{k\to\infty}\int_{\mathbb{R}^2}|u_{n(k)}(x,t)|^8dxdt\\ &\le \liminf_{k\to\infty}(a\|Lu_{n(k)}\|_2 + b\|u_{n(k)}\|_2)^8 = (a\|Lu\|_2 + b\|u\|_2)^8.\end{aligned}$$

Thus, for all $u \in D(L)$ one has

$$\|u\|_{L^8(\mathbb{R}^2)} \le a\|Lu\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)}$$

and the proof is over. □

Using a simple interpolation argument, we see that the inequality in the previous theorem holds for any $L^p(\mathbb{R}^2)$-norm on the left hand side where $2 \le p \le 8$ and hence we have

Corollary 2.4. *Let $2 \leq p \leq 8$. For all $a > 0$, there exists a $b > 0$ such that*

$$\|u\|_{L^p(\mathbb{R}^2)} \leq a\|Lu\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)} \tag{2.2}$$

for all $u \in D(L)$.

An application of the previous result is based on the following famous theorem of Kato and Rellich (for convenience of the reader, a proof may be found in [7]).

Theorem 2.5. *Let A and B be two densely defined operators and B is A-bounded with relative bound $a < 1$. If B is also symmetric and if A is self-adjoint, then $A + B$ is self-adjoint on $D(A)$.*

Corollary 2.6. *Let $\frac{8}{3} \leq q \leq \infty$. Let V be a real-valued function belonging to $L^q(\mathbb{R}^2)$. Then $iL + V$ is a self-adjoint operator on $D(L)$.*

Proof. Let $2 \leq p \leq 8$. Using the generalized Hölder inequality (and (2.2)) we can write

$$\|Vf\|_2 \leq \|V\|_q\|f\|_p \leq a\|V\|_q\|iLu\|_2 + b\|V\|_q\|u\|_2$$

where $q = \frac{2p}{p-2}$. Since a may be made arbitrary, we deduce that V is iL-bounded with relative bound $a\|V\|_q < 1$. Thus $iL+V$ is self-adjoint on $D(L)$ by the Kato-Rellich theorem. ∎

The next theorem settles the question of global space-time L^p estimates of the Airy operator on $\mathbb{R}^2$. We have

Theorem 2.7. *Let $p > 8$ (this includes the case $p = \infty$). There do not exist positive constants a and b such that*

$$\|u\|_{L^p(\mathbb{R}^2)} \leq a\|Lu\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)}$$

for all $u \in D(L)$.

Proof. We will show the existence of such a function u. Let $\delta > 0$. Consider

$$u_\delta(x,t) = \mathcal{F}^{-1}(g_\delta(\eta,\xi)) \text{ where } g_\delta(\eta,\xi) = \varphi(\delta\eta)V(\eta^3+\xi)$$

and where $\mathcal{F}^{-1}$ is the inverse L^2-Fourier transform, φ is a smooth function with compact support whereas V is a nonnegative smooth function of one variable with compact support (yet to be determined).

We want to show that $u_\delta \in D(L)$ which is equivalent (by means of the Fourier transform and the Plancherel theorem) to $\hat{u}_\delta$ and $|\xi + \eta^3|\hat{u}_\delta$ both belonging to $L^2(\mathbb{R}^2)$ which implies $(1 + |\xi + \eta^3|)\hat{u}_\delta \in L^2(\mathbb{R}^2)$. To get this condition we need an appropriate choice for support of V since $\hat{u}_\delta \in L^2(\mathbb{R}^2)$ (see below). We take the support of V to be $\{y : |y| \leq \frac{1}{2}\}$ so that

$$\operatorname{supp}\hat{u}_\delta \subset \{(\eta,\xi) \in \mathbb{R}^2 : |\xi + \eta^3| < 1\}$$

and hence $(1+|\xi+\eta^3|)\hat{u}_\delta \in L^2(\mathbb{R}^2)$. Thus we have $u_\delta \in D(L)$.

The next step is to show that u_δ does not belong to $L^p(\mathbb{R}^2)$ for any $p>8$. We are going to show that the ratio of the L^p-norm of u_δ and the L^2-norm of u_δ goes to infinity in a suitable limit.

We first compute the L^2-norm of u_δ. We have by the Plancherel theorem

$$\|u_\delta\|_2^2 = \|\hat{u}_\delta\|_2^2 = \|g_\delta\|_2^2 = \iint_{\mathbb{R}^2} |\varphi(\delta\eta)V(\eta^3+\xi)|^2 d\eta d\xi.$$

Then by the change of variables $\eta = s$ and $\xi = z - s^3$ we obtain

$$\|u_\delta\|_2 = \delta^{-\frac{1}{2}} \|\varphi\|_{L^2(\mathbb{R})} \|V\|_{L^2(\mathbb{R})}.$$

As for the L^p-norm we get

$$u_\delta(x,t) = \mathcal{F}^{-1}(g_\delta(\eta,\xi)) = \iint_{\mathbb{R}^2} \varphi(\delta\eta)V(\eta^3+\xi)e^{i\eta x+it\xi} d\eta d\xi.$$

By the same change of variables used for the 2-norm one gets

$$u_\delta(x,t) = \int_{\mathbb{R}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(\delta s)V(z)e^{isx-is^3t+izt} dz ds.$$

Setting $\delta s = r$ gives us

$$u_\delta(x,t) = \delta^{-1}\check{V}(t) \int_{\mathbb{R}} \varphi(r)e^{ir\frac{x}{\delta}-ir^3\frac{t}{\delta^3}} dr$$

which is equal to

$$u_\delta(x,t) = \delta^{-1}\check{V}(t)H\left(\frac{x}{\delta},\frac{t}{\delta^3}\right)$$

where H is some function of two variables. Thus

$$\|u_\delta\|_p^p = \delta^{4-p} \iint_{\mathbb{R}^2} |H(\mu,\tau)\check{V}(\delta^3\tau)|^p d\mu d\tau.$$

We need to investigate how the integral on the right hand side of the last equation behaves as $\delta \to 0$.

Since $\check{V}$ is continuous, $\lim_{\delta\to 0}\check{V}(\delta^3\tau) = \check{V}(0) = \|V\|_{L^1(\mathbb{R})}$ (as V is nonnegative). So

$$\lim_{\delta\to 0} |H(\mu,\tau)\check{V}(\delta^3\tau)|^p = |H(\mu,\tau)|^p \|V\|_{L^1(\mathbb{R})}^p.$$

Using the Fatou lemma gives us

$$\liminf_{\delta\to 0} \iint_{\mathbb{R}^2} |H(\mu,\tau)\check{V}(\delta^3\tau)|^p d\mu d\tau \geq \|H\|_{L^p(\mathbb{R}^2)}^p \|V\|_{L^1(\mathbb{R})}^p.$$

In the end, since

$$\frac{\|u_\delta\|_p}{\|u_\delta\|_2} = \delta^{\frac{4}{p}-\frac{1}{2}} \frac{\sqrt[p]{\iint_{\mathbb{R}^2} |H(\mu,\tau)\check{V}(\delta^3\tau)|^p d\mu d\tau}}{\|\varphi\|_{L^2(\mathbb{R})}\|V\|_{L^2(\mathbb{R})}},$$

using the argument above and sending $\delta \to 0$ yield

$$\frac{\|u_\delta\|_p}{\|u_\delta\|_2} \longrightarrow +\infty \text{ for } p > 8.$$

Finally, the last expression allows us to say that no inequality of the form

$$\|u\|_p \leq a\|Lu\|_2 + b\|u\|_2$$

can hold unless $p \leq 8$ and hence establishing the theorem. □

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