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LINEAR INDEPENDENCE AND SETS OF UNIQUENESS

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Abstract. Consider the Bessel system of integer translates $\{\psi_k\}$ of a square integrable function ψ. We show that ℓ^p-linear independence of $\{\psi_k\}$ is equivalent to periodization function $p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + k)|^2$ vanishing almost everywhere on a set which is an ℓ^p-set of uniqueness, where $1 \leqslant p \leqslant 2$. General result, concerning no restriction on Bessel systems is then proved for the case $p = 1$.

Integer translates of a single function $\psi \in L^2(\mathbb{R})$ exhibit several interesting connections with deep results and notions of Fourier analysis. For example, consider the Schauder basis property vs. the Muckenhoupt A_p-condition (see [11]) or the use of the Carleson theorem in the recent proof of the ℓ^2-linear independence characterization by S. Saliani ([12]). For the systematic treatment of such properties see [5] and for the extension to the general, LCA groups, case see [4]. In this paper we would like to establish yet another connection, as far as we know not yet observed, between the ℓ^p-linear independence of the system of translates and the ℓ^p-sets of uniqueness, where $1 \leqslant p \leqslant 2$.

Integer translates $\psi_k(x) = \psi(x - k)$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$, of a given function $\psi \in L^2(\mathbb{R})$, form a generating set $\mathcal{B}_\psi := \{\psi_k : k \in \mathbb{Z}\}$ for the principal shift-invariant space

$$\langle\psi\rangle := \overline{\operatorname{span}}\,\mathcal{B}_\psi, \tag{1}$$

generated by ψ. The study of the properties of $\mathcal{B}_\psi$ within $\langle\psi\rangle$ is interesting by itself. However, we would like to add that $\langle\psi\rangle$ forms also the main resolution level for various reproducing function systems generated by

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ψ, like wavelets and Gabor systems (see, for example, [6] for details). In particular, the systematic study of Parseval frame wavelets relies heavily on the properties of $\mathcal{B}_\psi$ within $\langle\psi\rangle$ (see [14] for details and further references). An important feature of frames is that in principle they allow for redundancy (see the discussion in I. Daubechies [3]). In the case of Parseval frame wavelets, however, the redundancy of the system is not possible via finite sums, since the system is linearly independent; as proved recently in M. Bownik and D. Speegle ([1]). If we add to this that $\mathcal{B}_\psi$ is obviously (see [13] or [1]) linearly independent whenever $\psi \neq 0$, then it is natural to study various levels of linear independence for infinite sums (see [15] for numerous such notions in the Banach space basis theory).

The first attempt was to study ℓ^2-linear independence of $\mathcal{B}_\psi$. The characterization (in the case of Parseval frame wavelets or, essentially, Bessel systems) was established in [13] (see also the discussion in [5]), but the general case remained open for several years. Most recently, S. Saliani ([12]) solved the problem using a very interesting idea. She established the connection with the Menšov type results (see [17] for the introduction to the subject), using theorems of S. V. Kislyakov (see [10]) and S. A. Vinogradov (see [16]).

We show in this article that the connection with the Menšov result is not "by accident". We characterize the ℓ^p-linear independence, for $1 \leqslant p \leqslant 2$, in the case of Bessel systems, in terms of the sets of uniqueness. Among others, this also provides an additional perspective to the ℓ^2 result. The proof for the general case still eludes us (except in the cases $p = 1$ and $p = 2$) and we leave it for the future research.

Let us introduce the notation. The following notion can be introduced in a much more abstract setting, but such a level of abstraction is not needed in this paper.

Definition 1. *Let $1 \leqslant p \leqslant 2$. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space $(\mathbb{H}, \|\ \|)$ is ℓ^p-linearly independent if*

$$\Big((c_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}),\ \lim_{n\to\infty} \|\sum_{k=1}^{n} c_k x_k\| = 0 \Rightarrow c_n = 0,\ \textit{for every } n \in \mathbb{N}\Big).$$

Observe that in this definition the order of vectors (x_n) matters. Our set $\mathcal{B}_\psi$ is indexed by $\mathbb{Z}$. Having Fourier analysis background in mind we order $\mathbb{Z}$ as $\{0, 1, -1, 2, -2, \ldots\}$. Since all vectors in $\mathcal{B}_\psi$ are of the same norm, the convergence of partial sums of ψ_k's (in this, just introduced ordering of $\mathbb{Z}$) is equivalent to the convergence of symmetric partial sums (see also [11])

$$\sum_{|k|\leq n} c_k \psi_k. \tag{2}$$

In the study of $\langle\psi\rangle$, the fundamental role is played by *the periodization function* (see [4, 5, 14] and the numerous references within) $p_\psi : \mathbb{R} \to [0, +\infty\rangle$

defined by

$$p_\psi(\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+k)|^2, \xi \in \mathbb{R}, \tag{3}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. In particular, of interest for us here is the set of zeroes of p_ψ

$$Z_\psi := \{\xi \in \mathbb{R} : p_\psi(\xi) = 0\}. \tag{4}$$

Observe that p_ψ is 1-periodic (hence, so is Z_ψ), so p_ψ can be considered as a function on the torus $\mathbb{T} := {}^{\mathbb{R}}/_{\mathbb{Z}}$ and Z_ψ could be considered also in the sense $Z_\psi \subseteq \mathbb{T}$; and we shall do so here most of the time. It is known, see [12], that

$$\mathcal{B}_\psi \text{ is } \ell^2 - \text{ linearly independent} \Leftrightarrow |Z_\psi| = 0, \tag{5}$$

where $|A|$ denotes the Lebesgue measure of A. Since, for $1 \leqslant p \leqslant 2$, $\ell^p \subseteq \ell^2$, it is obvious that $|Z_\psi| = 0$ implies the ℓ^p-linear independence of $\mathcal{B}_\psi$. However, since $\ell^p \neq \ell^2$ for $p < 2$, one could expect that ℓ^p-linear independence allows also $|Z_\psi| > 0$. As we shall see, this is indeed so. Let us also remind our reader that (see [5])

$$\mathcal{B}_\psi \text{ is a Bessel system for } \langle\psi\rangle \Leftrightarrow (\exists B > 0) \quad p_\psi \leq B \quad a.e. \tag{6}$$

Before stating the following definition, let us remind our reader that for every $1 \leqslant p \leqslant 2$ and for every $(c_k)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ there exists a function $f \in L^2(\mathbb{T})$ such that $\widehat{f}(k) = c_k$, for every $k \in \mathbb{Z}$; where $\widehat{f}(k)$ denotes the k-th Fourier coefficient of f.

Definition 2. *Let $1 \leqslant p \leqslant 2$. A measurable subset $E \subseteq \mathbb{T}$ is an ℓ^p-set of uniqueness if there is no $0 \neq f \in L^2(\mathbb{T})$ such that $f|_{\mathbb{T}\setminus E} = 0$ a.e. and $(\widehat{f}(k))_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$.*

The early period of the theory of uniqueness sets is closely connected with the standard result of Menšov (see, for example, [17] for details). Obviously, if $|E| = 0$, then E is an ℓ^p-set of uniqueness, for all $1 \leqslant p \leqslant 2$. If $p = 2$, then the only sets of ℓ^2-uniqueness are the sets of measure zero. For $p < 2$, however, there are ℓ^p-sets of uniqueness which have a strictly positive measure; see paragraph IV.2.5. in Y. Katznelson [8] for a construction of such a set (observe that only closed sets are considered by the author). In fact, Y. Katznelson proved in [9] that there exist sets of Lebesgue measure arbitrarily close to 1, which are ℓ^p-sets of uniqueness for all $p < 2$. Observe that it follows from definition (as well as from our result and the implications holding between different levels of ℓ^p-linear independence) that ℓ^p-sets of uniqueness are also $\ell^{p'}$-sets of uniqueness for all $p' < p$. I. I. Hirschman and Y. Katznelson showed in [7] that for any $1 < q < p \leqslant 2$ there exists a set which is an ℓ^q-set of uniqueness, but is not an ℓ^p-set of uniqueness. Therefore, for any

$1 < q < p \leqslant 2$ there exists a Bessel system $\mathcal{B}_\psi$ which is ℓ^q-linearly independent but is not ℓ^p-linearly independent.

We are now ready to state and prove our main result. Observe that in the case $p = 2$ we obtain the condition given in (5).

THEOREM 3. *Let $1 \leqslant p \leqslant 2$ and $0 \neq \psi \in L^2(\mathbb{R})$ such that $\mathcal{B}_\psi$ is a Bessel system for $\langle\psi\rangle$. Then $\mathcal{B}_\psi$ is ℓ^p-linearly independent if and only if Z_ψ is an ℓ^p-set of uniqueness.*

PROOF. Let us recall first that $\langle\psi\rangle$ is isometrically isomorphic to $L^2(\mathbb{T}; p_\psi)$; the L^2 space on $\mathbb{T}$ with measure $p(\xi)\mathrm{d}\xi$; see [5] and the references therein for more details. Furthermore, the role of $\mathcal{B}_\psi$ is in $L^2(\mathbb{T}; p_\psi)$ played by the family of exponentials

$$\{e_k(\xi) = \exp(2\pi i \xi k); k \in \mathbb{Z}\}.$$

Hence, the partial sum $\sum_{|k| \leq n} c_k \psi_k$ corresponds to the sum $\sum_{|k| \leq n} c_k e_k$. Therefore, $\mathcal{B}_\psi$ is ℓ^p-linearly independent if and only if

$$\begin{aligned} \Big((c_k)_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}), \lim_{n\to\infty} \| \sum_{|k| \leqslant n} c_k e_k \|_{L^2(\mathbb{T}, p_\psi)} = 0 \\ \Rightarrow c_k = 0, \text{ for every } k \in \mathbb{Z}\Big). \end{aligned} \tag{7}$$

Let us also introduce the following imbedding operator: $I_\psi : L^2(\mathbb{T}) \to L^2(\mathbb{T}; p_\psi)$ given by

$$I_\psi(m) := m, \quad m \in L^2(\mathbb{T}). \tag{8}$$

Observe that, since $\mathcal{B}_\psi$ is a Bessel system for $\langle\psi\rangle$, then p_ψ is bounded above (see (6)), and therefore, I_ψ is a well-defined bounded linear operator. Furthermore, the kernel $\mathrm{Ker}(I_\psi)$ consists precisely of $m \in L^2(\mathbb{T})$ such that the set support (considered up to a.e.)

$$\operatorname{supp} m := \{\xi \in \mathbb{T} : m(\xi) \neq 0\} \subseteq Z_\psi. \tag{9}$$

Assume first that $\mathcal{B}_\psi$ is ℓ^p-linearly independent. We want to prove then that Z_ψ is an ℓ^p-set of uniqueness. Suppose to the contrary that there exists $0 \neq f \in L^2(\mathbb{T})$ such that $f|_{\mathbb{T}\setminus Z_\psi} = 0$ (i.e., $\operatorname{supp} f \subseteq Z_\psi$) and $(\widehat{f}(k))_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Since $f \neq 0$, the sequence $(\widehat{f}(k))_{k\in\mathbb{Z}}$ is non-trivial. Furthermore

$$\| \sum_{|k| \leq n} \widehat{f}(k) e_k - f \|_{L^2(\mathbb{T})} \to 0, \text{ as } n \to \infty.$$

Since I_ψ is continuous and $f \in \mathrm{Ker}(I_\psi)$ we obtain

$$\| \sum_{|k| \leq n} \widehat{f}(k) e_k \|_{L^2(\mathbb{T}; p_\psi)} \to 0, \text{ as } n \to \infty. \tag{10}$$

Since $(\widehat{f}(k))_{k\in\mathbb{Z}}$ is non-trivial, then (10) contradicts (7), and therefore, the ℓ^p-linear independence of $\mathcal{B}_\psi$.

Assume now that Z_ψ is an ℓ^p-set of uniqueness. We want to prove that then (7) is valid. Suppose to the contrary that there is a non-trivial sequence $(c_k)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ such that

$$\lim_{n\to\infty} \Big\| \sum_{|k|\leqslant n} c_k e_k \Big\|_{L^2(\mathbb{T};p_\psi)} = 0. \tag{11}$$

Since $\ell^p(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$, there exists $f \in L^2(\mathbb{T})$ such that $\widehat{f}(k) = c_k$, for every $k \in \mathbb{Z}$. Since (c_k) is non-trivial, we have $f \neq 0$. Therefore also

$$\Big\| \sum_{|k|\leqslant n} c_k e_k - f \Big\|_{L^2(\mathbb{T})} \to 0, \text{ as } n \to \infty.$$

Since I_ψ is continuous, using (11), we obtain $I_\psi(f) = 0 \in L^2(\mathbb{T};p_\psi)$, i.e., $\operatorname{supp} f \subseteq Z_\psi$. Hence, we have $0 \neq f \in L^2(\mathbb{T})$ such that $f|_{\mathbb{T}\setminus Z_\psi} = 0$ a.e., and $(\widehat{f}(k))_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. This is in contradiction with the assumption that Z_ψ is an ℓ^p-set of uniqueness.

□

If one examines the proof above carefully, then one sees that the key consequence of the Bessel system assumption is that, given $(c_k) \in \ell^p$, we obtain the convergence of the sum $\sum_{k\in\mathbb{Z}} c_k e_k$ in $L^2(\mathbb{T};p_\psi)$. Observe that in the case $p=1$, i.e., when $(c_k) \in \ell^1$, this is always true. Hence, we also have the following corollary.

COROLLARY 4. *$\mathcal{B}_\psi$ is ℓ^1-linearly independent if and only if Z_ψ is an ℓ^1-set of uniqueness.*

We expect that our main theorem is also valid without the restriction on Bessel systems, but this would require some additional technique (perhaps like in the case $p=2$). We leave this, as well as the issue of $p>2$, to future research.

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