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PROPERTY A AND ASYMPTOTIC DIMENSION

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Abstract. The purpose of this note is to characterize the asymptotic dimension asdim(X) of metric spaces X in terms similar to Property A of Guoliang Yu. We prove that for a metric space (X, d) and $n \geq 0$ the following conditions are equivalent:

a. asdim$(X, d) \leq n$.
b. For each $R, \epsilon > 0$ there is $S > 0$ and finite non-empty subsets $A_x \subset B(x, S) \times \mathbb{N}$, $x \in X$, such that $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$ if $d(x, y) < R$ and the projection of A_x onto X contains at most $n + 1$ elements for all $x \in X$.
c. For each $R > 0$ there is $S > 0$ and finite non-empty subsets $A_x \subset B(x, S) \times \mathbb{N}$, $x \in X$, such that $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{1}{n+1}$ if $d(x, y) < R$ and the projection of A_x onto X contains at most $n + 1$ elements for all $x \in X$.

1. Introduction

Property A was introduced by G.Yu in [6]. We adopt the following definition from [3] (see also [5]):

Definition 1.1. *A discrete metric space (X, d) has property A if for all $R, \epsilon > 0$, there exists a family $\{A_x\}_{x \in X}$ of finite, non-empty subsets of $X \times \mathbb{N}$ such that:*

- *for all $x, y \in X$ with $d(x, y) \leq R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$, where $A_x \Delta A_y = (A_x \cup A_y) - (A_x \cap A_y)$ is the simetric difference of the sets,*
- *there exists $S > 0$ such that for each $x \in X$, if $(y, n) \in A_x$, then $d(x, y) \leq S$.*

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Asymptotic dimension was introduced by M. Gromov in [1] (see section 1.E) as a large-scale analogue of the classical notion of topological covering dimension. It is a coarse invariant that has been extensively investigated (see chapter 9 of [4] for some results and further references).

Definition 1.2. *A metric space (X, d) is said to have finite asymptotic dimension if there exists $k \geq 0$ such that for all $L > 0$ there exists a uniformly bounded cover of X (i.e., there exists $S > 0$ such that all elements of the cover are of diameter at most S) of Lebesgue number at least L (i.e., every L-ball $B(x, L)$ is contained in some element of the cover) and multiplicity at most $k + 1$ (i.e., each point of X belongs to at most $k + 1$ elements of the cover). The least possible such k is the* asymptotic dimension *of X.*

One of the basic results is that spaces of finite asymptotic dimension have property A and known proofs of it use Higson-Roe characterization of Property A (see [2] and [5]). The purpose of this note is to provide a simple proof of that result and prove the following connection between Property A and asymptotic dimension.

2. The main theorem

Theorem 2.1. *If (X, d) is a metric space and $n \geq 0$, then the following conditions are equivalent:*

a. $\operatorname{asdim}(X, d) \leq n$.
b. *For each $R, \epsilon > 0$ there is $S > 0$ and finite non-empty subsets $A_x \subset B(x, S) \times \mathbb{N}$, $x \in X$, such that $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$ if $d(x, y) < R$ and the projection of A_x onto X contains at most $n+1$ elements for all $x \in X$.*
c. *For each $R > 0$ there is $S > 0$ and finite non-empty subsets $A_x \subset B(x, S) \times \mathbb{N}$, $x \in X$, such that $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{1}{n+1}$ if $d(x, y) < R$ and the projection of A_x onto X contains at most $n+1$ elements for all $x \in X$.*

Proof. a) $\Longrightarrow$ b). Suppose $\operatorname{asdim}(X, d) \leq n$ and $R, \epsilon > 0$. Pick a uniformly bounded cover $\mathcal{U}$ of X of multiplicity at most $n + 1$ and Lebegue number at least $L = 2R + \frac{(2n+1)R}{\epsilon}$. Let S be a number such that $\operatorname{diam}(U) < S$ for each $U \in \mathcal{U}$. For every $U \in \mathcal{U}$ pick an element $a_U \in U$. We call a finite sequence $x_0, \ldots, x_n$ of points in X an R-chain from x_0 to x_n if $d(x_i, x_{i-1}) < R$ for $i = 1, \ldots, n$. For $x \in X$ and $U \in \mathcal{U}$ let $l_U(x)$ denote the length of the shortest R-chain joining x and a point outside of U, if there is no such chain, we put $l_U(x)$ equal to the integer part of $\frac{L}{R}$. Then let A_x be the following union over all elements from $\mathcal{U}$ containing x.

$$A_x = \bigcup_{U \ni x} \{a_U\} \times \{1, \ldots, l_U(x)\}$$

These sets are either empty (and we ignore them) or they are finite non-empty sets and $A_x \subset B(x, S) \times \mathbb{N}$. If $d(x, y) < R$, then $|l_U(x) - l_U(y)| \leq 1$, and because $\mathcal{U}$ is a cover of Lebesque number greather than R and of multiplicity at most $n+1$ the total number of elements of $\mathcal{U}$ containing at least one of x or y is at most $2n+1$. Therefore $|A_x \Delta A_y| \leq 2n+1$. There exists $U_0 \in \mathcal{U}$ such that $B(x,L) \subset U_0$. Every R-chain joining x or y to $X \setminus U$ must have at least $\frac{L-R}{R}$ elements. If there is no R-chain from x to $X \setminus U$, there is also no R-chain from y to $X \setminus U$, hence $\{a_U\} \times \{1, \ldots, [\frac{L}{R}]\} \subset A_x, A_y$. In any case we have $|A_x \cap A_y| > \frac{L-R}{R} - 1$. Consequently

$$\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{(2n+1) \cdot R}{L-2R} = \epsilon.$$

c) $\Longrightarrow$ a). Given $R > 0$ pick $S > 0$ and finite subsets $A_x \subset B(x,S) \times \mathbb{N}$, $x \in X$, such that

$$\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{1}{n+1}$$

if $d(x,y) < R$ and the projection of A_x onto X contains at most $n+1$ elements for all $x \in X$. Define sets U_x as consisting precisely of $y \in X$ such that $(\{x\} \times \mathbb{N}) \cap A_y \neq \emptyset$. The multiplicity of the cover $\{U_x\}_{x \in X}$ of X is at most $n+1$ as $z \in \bigcap\limits_{i=1}^{k} U_{x_i}$ implies x_i belongs to the projection of A_z, so $k \leq n+1$.

Let us show that $\{U_x\}_{x \in X}$ has Lebesgue number at least R. Given $x \in X$ choose $z \in X$ so that $|(\{z\} \times \mathbb{N}) \cap A_x|$ maximizes all $|(\{y\} \times \mathbb{N}) \cap A_x|$. In particular

$$|(\{z\} \times \mathbb{N}) \cap A_x| \geq \frac{|A_x|}{n+1}.$$

Let $d(x,y) < R$ then $y \notin U_z$ implies $|A_x \Delta A_y| \geq \frac{|A_x|}{n+1}$, so $\frac{|A_x \Delta A_y|}{|A_x|} \geq \frac{1}{n+1}$, a contradiction. Therefore $y \in U_z$, hence $B(x,R) \subset U_z$. ∎

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