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MIMO Performance Gains – A Signal Processing Point of View

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In order to wisely use the degrees of freedom which a multi-input multi-output (MIMO) offers, it is necessary to identify and quantify the possible gains in performance one can expect from MIMO signal processing. In this paper, we derive measures which allow to quantify the amount of antenna gain, diversity gain, and multiplexing gain of a MIMO system. It turns out that it is impossible to maximize all three gains, or any pair of two gains at the same time. The necessary trade-off between performance gains is demonstrated.

Key words: MIMO peformance measures, antenna gain, diversity gain, multiplexing gain

1 INTRODUCTION

The efficient use of available bandwidth is of paramount importance for wireless communication systems which need to satisfy the high data rate demands of emerging multimedia services. Usage of multiple antennas at both sides of the wireless link is now considered as a promising way to achieve the necessary bandwidth efficiency. These so-called multiple-input multiple-output (MIMO) systems offer new degrees of freedom, which have to be used carefully in order to maximize the benefit. To optimize the degrees of freedom, it is necessary to identify and quantify the possible gains in performance one can expect from a MIMO system. Moreover, quantification of performance gains can be used to communicate the current level of performance to higher layers of the protocol stack, like data transport mechanisms and service appliations.

In general, three fundamentally different performance gains can be distinguished: transmit power efficiency, link reliability, and bandwidth efficiency. An increase in transmit power efficiency means that the receive power is increased while the transmit power is kept constant. A MIMO system can achieve this gain in efficiency by transmit signal processing which is making use of MIMO antenna gain [1]. The link reliability is increased if the fluctuation of the receive power is reduced with respect to its mean value. An increase in link reliability can be achieved by a MIMO system with signal processing aiming at transmit and receive antenna diversity [2, 3]. The bandwidh efficiency is increased if information can be transfered at higher rate within the same bandwidth using the same transmit power. A MIMO system can achieve an increase in bandwith efficiency by transmit signal processing and channel coding aiming at the parallel transmission of independent information streams at the same time inside the same bandwidth. This parallel transmission is usually referred to as spatial multiplexing [4]. A MIMO system can achieve all these goals at least individually by proper signal processing and channel coding. The transmit power efficiency and the bandwidth efficiency are directly related to the antenna gain and multiplexing gain of a MIMO system, respectively, while the link reliability is controlled by diversity gain. All of the three gains depend on statistical properties of the MIMO channel and exhibit interdependencies. It is impossible to maximize all three gains at the same time. A trade-off between these performance gains is necessary.

In this paper, we derive measures which allow to quantify the amount of antenna gain, diversity gain, and multiplexing gain of a MIMO system for different amounts of transmit channel state information (TCSI). It turns out that the advantage in average channel capacity which is due to long-term average TCSI compared to no TCSI can be explained and quantified by the proposed measure for MIMO antenna gain. The definition for the diversity gain is based on the relative fluctuation of the received signal power. It can be decoupled into a transmit diversity gain and a receive diversity gain, and can be used to quantify how effectively the MIMO system is making use of the space diversity present in the MIMO channel. Finally, the proposed measure for multiplexing gain is based on the slope of instantaneous or average capacity. It reflects how effectively the MIMO system is making use of parallel information channels.

This paper is organized as follows. We first establish a MIMO system model in Section 2. Then we discuss and derive measures for the three performance gains in Section 3 to 5. Finally, we demonstrate the trade-off of them in Section 6.

2 WIRELESS MIMO SYSTEM MODEL

In a wireless MIMO system, several transmit and several receive antennas are interconnected by a time-varying wireless channel. The time variation leads to fading of the receive signal amplitude. The fading is modelled by a random process rather than being treated deterministically. For reasons of brevity, in this paper we focus on frequency flat fading[1]. In this way, the wireless MIMO channel can be described by a stochastic channel matrix $H \in \mathcal{C}^{M \times N}$, where the entry $h_{i,j}$ at the i-th row and the j-th column is a random variable which denotes the instantaneous[2] complex baseband channel coefficient between the i-th receive and the j-th transmit antenna. The channel coefficients $h_{i,j}$ usually exhibit certain correlation, which is due to the geometrical structure of scattering, reflecting and diffracting obstacles located around and between the receiver and the transmitter. In case of Rayleigh fading, the correlation can be described by the fading correlation matrix

$$\boldsymbol{R} = \mathrm{E}\left[\mathbf{vec}[\boldsymbol{H}]\mathbf{vec}[\boldsymbol{H}]^{\mathrm{H}}\right], \tag{1}$$

where $\mathbf{vec}[.]$ and $\mathrm{E}[.]$ denote the column stacking and the expectation operation, respectively, while $(.)^{\mathrm{H}}$ refers to the complex conjugate transpose operation. In the important special case where the receive side and the transmit side random fading processes are independent, the correlation matrix can be decomposed into the tensor product

$$\boldsymbol{R} = \frac{1}{\mathrm{tr}\,\boldsymbol{R}_{\mathrm{Tx}}} \boldsymbol{R}_{\mathrm{Tx}}^{\mathrm{T}} \otimes \boldsymbol{R}_{\mathrm{Rx}}, \tag{2}$$

of a receive fading correlation matrix

$$\boldsymbol{R}_{\mathrm{Rx}} = \mathrm{E}\left[\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}\right] \in \mathcal{C}^{M \times M},$$

and a transmit fading correlation matrix

$$\boldsymbol{R}_{\mathrm{Tx}} = \mathrm{E}\left[\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right] \in \mathcal{C}^{N \times N}.$$

[1] Note that a frequency selective wireless channel can be transformed into a set of frequency flat channels by application of a multi-carrier channel model.

[2] In a block fading model, the $h_{i,j}$ are treated as constant for the so-called decorrelation time and then take on new independent random values. This random realizations of $h_{i,j}$ are called *instantaneous* channel coefficients.

The symbols tr and $(.)^{\mathrm{T}}$ refer to the matrix trace and transpose operation, respectively, while $\otimes$ denotes the tensor product. The stochastic channel matrix can then be written as

$$\boldsymbol{H} = \frac{1}{\sqrt{\mathrm{tr}\,\boldsymbol{R}_{\mathrm{Tx}}}} \boldsymbol{R}_{\mathrm{Rx}}^{1/2} \boldsymbol{G} \boldsymbol{R}_{\mathrm{Tx}}^{1/2}, \tag{3}$$

where $\boldsymbol{G} \in \mathcal{C}^{M \times N}$ is a matrix with zero mean, unity variance, complex, circularly symmetric, i.i.d. Gaussian random entries. In case that $\boldsymbol{R}_{\mathrm{Rx}}$ is a scaled identity matrix while $\boldsymbol{R}_{\mathrm{Tx}}$ is not, we speek of a *semi-correlated* channel [5, 6]. Let us now write the received signal $\boldsymbol{r} \in \mathcal{C}^{M \times 1}$ as

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v}, \tag{4}$$

where $\boldsymbol{v} \in \mathcal{C}^{M \times 1}$ is a random vector, which contains noise samples for the M receive antennas, while $\boldsymbol{x} \in \mathcal{C}^{N \times 1}$ is the vector of symbols transmitted in parallel over the N transmit antennas. In this paper, we assume spatially white noise of power

$$\sigma_v^2, \text{ i.e. } \mathrm{E}\left[\boldsymbol{v}\boldsymbol{v}^{\mathrm{H}}\right] = \sigma_v^2 \mathbf{I}_M,$$

where $\mathbf{I}_M$ denotes the $M \times M$ identity matrix. It is convenient to write $\boldsymbol{x}$ in the form

$$\boldsymbol{x} = \boldsymbol{T}\boldsymbol{P}^{1/2}\boldsymbol{s}, \tag{5}$$

where the vector $\boldsymbol{s} \in \mathcal{C}^{L \times 1}$ contains complex Gaussian zero-mean, unity variance random symbols of L independent data streams which are to be transmitted over the MIMO channel in parallel, hence $\mathrm{E}\left[\boldsymbol{s}\boldsymbol{s}^{\mathrm{H}}\right] = \mathbf{I}_L$. The transmit power assigned to each data stream is collected in the diagonal matrix $\boldsymbol{P} \in \mathcal{R}^{L \times L}$, with $P_{\mathrm{T}} = \mathrm{tr}\,P$ being the total transmit power. Finally, the matrix $\boldsymbol{T} \in \mathcal{C}^{N \times L}$ defines the mapping from the L data streams onto the N transmit antennas and consists of L unity norm column vectors. In this way,

$$P_{\mathrm{T}} = \mathrm{E}_s\left[\|\boldsymbol{x}\|_2^2\right] = \mathrm{tr}\,(\boldsymbol{T}^{\mathrm{H}}\boldsymbol{T}\boldsymbol{P}) = \mathrm{tr}\,\boldsymbol{P}. \tag{6}$$

The mutual information between $\boldsymbol{s}$ and $\boldsymbol{r}$ is given by [7]

$$I(\boldsymbol{s},\boldsymbol{r}|\boldsymbol{T},\boldsymbol{P}) = \log_2 \det\left(\mathbf{I}_L + \frac{\boldsymbol{T}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\boldsymbol{T}\boldsymbol{P}}{\sigma_v^2}\right), \tag{7}$$

bits per channel use. The channel capacity is the maximum mutual information which can be obtained by proper choice of $\boldsymbol{T}$ and $\boldsymbol{P}$. To which extent this maximization can be carried out depends on how much the transmitter is aware of the channel matrix $\boldsymbol{H}$. This awareness is called transmit channel state information (TCSI). We distinguish between the following three cases.

– **Full TCSI (F-TSCI):** The matrix $\boldsymbol{H}$ is known to the transmitter. With the eigenvalue decomposition

$$\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{H}}, \tag{8}$$

the mutual information is maximized by setting $\boldsymbol{T}=\boldsymbol{U}$ [7] and choosing $\boldsymbol{P}$ by the waterfilling policy [8, 9] based on the eigenvalues λ_i of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$. The capacity becomes

$$C = \sum_{i=1}^{L} \log_2 \left(1 + \frac{P_i}{\sigma_v^2} \lambda_i \right), \tag{9}$$

with the transmit powers P_i chosen according to the waterfilling solution.

– **No TCSI:** The transmitter is completely unaware of the channel. The best it can do [7] is to set $\boldsymbol{T}=\mathbf{I}_N$ or to any unitary matrix, and $\boldsymbol{P} = (P_{\mathrm{T}}/N)\,\mathbf{I}_N$.

– **Long-term TCSI (LT-TCSI):** The transmitter is unaware of the matrix $\boldsymbol{H}$, but knows its statistical properties. In case of the stochastic channel model (3), the transmitter is aware of $\boldsymbol{R}_{\mathrm{Tx}}$ and possibly of $\boldsymbol{R}_{\mathrm{Rx}}$. In this way, it is possible to maximize the average mutual information. With the eigenvalue decomposition

$$\mathrm{E}\left[\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right] = \boldsymbol{R}_{\mathrm{Tx}} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{H}}, \tag{10}$$

the average mutual information is maximized by setting $\boldsymbol{T}=\boldsymbol{Q}$ [10]. The exact solution for the transmit power distribution $\boldsymbol{P}$, which also depends on $\boldsymbol{R}_{\mathrm{Rx}}$, was recently found analytically in [11]. A much simpler, yet sub-optimum solution is reported in [6, 1] which determines $\boldsymbol{P}$ by the waterfilling policy based on the eigenvalues of $\boldsymbol{R}_{\mathrm{Tx}}$. The difference in mutual information compared to the exact power distribution turns out to be negligible [11].

3 ANTENNA GAIN

A MIMO system can be used to increase the transmit power efficiency compared to a SISO system. This is due to the employment of antenna gain A_{ANT}, which generic definition is given by

$$A_{\mathrm{ANT}} = \frac{P_{\mathrm{Rx}}^{(\mathrm{MIMO})}}{P_{\mathrm{Rx}}^{(\mathrm{SISO})}}, \tag{11}$$

where $P_{\mathrm{Rx}}^{(\mathrm{MIMO})}$ and $P_{\mathrm{Rx}}^{(\mathrm{SISO})}$ are the receive powers for a MIMO system and for a SISO system, respectively, when the transmit power is the same in both cases. How much antenna gain is possible depends on how much the transmitter is aware of the channel. An attempt to define an antenna gain was already made in [12]. However, that definition was made with a single data stream ($L=1$) in mind, and does not cover the case of long-term TCSI. In the sequel, we follow the definition given by the authors in [1].

A. Antenna gain with full TCSI

For a given channel matrix $\boldsymbol{H}$, the received signal power in the MIMO case is given by

$$P_{\mathrm{Rx}}^{(\mathrm{MIMO})} = \mathrm{E}\left[\|\boldsymbol{r}\|_2^2 \,|\boldsymbol{H}\right] = \mathrm{tr}\,(\boldsymbol{T}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\boldsymbol{T}\boldsymbol{P}). \tag{12}$$

With the eigenvalue decomposition from (8) and the capacity achieving $\boldsymbol{T}=\boldsymbol{U}$, we obtain

$$P_{\mathrm{Rx}}^{(\mathrm{MIMO})} = \mathrm{tr}\,(\boldsymbol{\Lambda}\boldsymbol{P}), \tag{13}$$

where $\boldsymbol{\Lambda}$ is the eigenvalue matrix of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ from (8). If instead just a single pair of antennas, say the i-th receive and the j-th transmit antenna were used to form a SISO system, the corresponding receive power would be given by $P_{\mathrm{T}} \cdot |h_{i,j}|^2$. Taking the average over all possible pairs of receive and transmit antennas we obtain the received signal power for the SISO case

$$P_{\mathrm{Rx}}^{(\mathrm{SISO})} = \frac{P_{\mathrm{T}}}{NM} \sum_{j=1}^{N} \sum_{i=1}^{M} |h_{i,j}|^2 = \frac{(\mathrm{tr}\,\boldsymbol{\Lambda})(\mathrm{tr}\,\boldsymbol{P})}{NM}. \tag{14}$$

With (11), we arrive at the definition of antenna gain with full TCSI:

$$A_{\mathrm{ANT}}^{(\mathrm{F-TCSI})} = NM \frac{\mathrm{tr}\,(\boldsymbol{\Lambda}\boldsymbol{P})}{(\mathrm{tr}\,\boldsymbol{\Lambda})(\mathrm{tr}\,\boldsymbol{P})}. \tag{15}$$

The achievable antenna gain depends both on the channel matrix and on the transmit power distribution. The antenna gain is maximized if only the data stream associated with the largest eigenvalue $\lambda_{\max}$ of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ is powered up:

$$\hat{A}_{\mathrm{ANT}}^{(\mathrm{F-TCSI})} = \max_{P} A_{\mathrm{ANT}}^{(\mathrm{F-TCSI})} = NM \frac{\lambda_{\max}}{\mathrm{tr}\,\boldsymbol{\Lambda}}. \tag{16}$$

The absolute maximum value of $\hat{A}_{\mathrm{ANT}}^{(\mathrm{F-TCSI})} = NM$ occurs in situations when rank $[\boldsymbol{H}]=1$. This explains why rank deficient channels can outperform full rank channels, as is detailed in [1] and [13]. Moreover, it turns out that the average maximum antenna gain of an uncorrelated Gaussian MIMO channel grows much slower than the product of the number of transmit and receive antennas. This can be shown by application of a result from [14] which bounds the average maximum eigenvalue of a matrix $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ for the case that $\boldsymbol{H} \in \mathcal{C}^{M\times N}$ is a complex Gaussian matrix with zero mean, i.i.d. components:

$$\mathrm{E}\left[\frac{\lambda_{\max}}{\mathrm{tr}\,\boldsymbol{\Lambda}}\right] < \left(\frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}}\right)^2. \tag{17}$$

From this follows:

$$\mathrm{E}\left[\hat{A}_{\mathrm{ANT}}^{(\mathrm{F-TCSI})}\right] < \left(\sqrt{M}+\sqrt{N}\right)^2. \qquad (18)$$

B. Antenna gain with long-term TCSI

When the transmitter is aware of the channel on average only, it is reasonable to base the definition of antenna gain on the average receive power

$$\bar{P}_{\mathrm{Rx}}^{(\mathrm{MIMO})} = \mathrm{E}\left[\mathrm{E}\left[\|\boldsymbol{r}\|_2^2 \,|\, \boldsymbol{H}\right]\right] = \mathrm{tr}\,(\boldsymbol{T}^{\mathrm{H}}\boldsymbol{R}_{\mathrm{Tx}}\boldsymbol{T}\boldsymbol{P}). \qquad (19)$$

With the eigenvalue decomposition from (10) and the capacity achieving $\boldsymbol{T}=\boldsymbol{Q}$, we obtain

$$\bar{P}_{\mathrm{Rx}}^{(\mathrm{MIMO})} = \mathrm{tr}\,(\boldsymbol{D}\boldsymbol{P}), \qquad (20)$$

where $\boldsymbol{D}$ is the eigenvalue matrix of the transmit fading correlation matrix $\boldsymbol{R}_{\mathrm{Tx}}$ from (10). For the SISO case, the average received signal power is

$$\bar{P}_{\mathrm{Rx}}^{(\mathrm{SISO})} = \frac{P_{\mathrm{T}}}{NM}\sum_{j=1}^{N}\sum_{i=1}^{M} E\left[\left|h_{i,j}\right|^2\right] = \frac{(\mathrm{tr}\,\boldsymbol{D})(\mathrm{tr}\,\boldsymbol{P})}{NM}. \qquad (21)$$

With (11), we arrive at the definition of antenna gain with long-term TCSI:

$$A_{\mathrm{ANT}}^{(\mathrm{LT-TCSI})} = \underbrace{M}_{A_{\mathrm{Rx-ANT}}^{(\mathrm{LT-TCSI})}} \cdot \underbrace{N\frac{\mathrm{tr}\,(\boldsymbol{D}\boldsymbol{P})}{(\mathrm{tr}\,\boldsymbol{D})(\mathrm{tr}\,\boldsymbol{P})}}_{A_{\mathrm{Tx-ANT}}^{(\mathrm{LT-TCSI})}}. \qquad (22)$$

Notice that the second term in the product on the right hand side of (22) only depends on transmit side fading properties. Hence, in contrast to the instantaneous case, the long-term average antenna gain decomposes into a product of a receive side antenna gain and a transmit side antenna gain. The achievable long-term antenna gain depends both on the transmit fading correlation matrix and on the transmit power distribution. The antenna gain is maximized if only the data stream associated with the largest eigenvalue $D_{\max}$ of R_{Tx} is powered up:

$$\hat{A}_{\mathrm{ANT}}^{(\mathrm{LT-TCSI})} = \max_{P} A_{\mathrm{ANT}}^{(\mathrm{LT-TCSI})} = NM\frac{D_{\max}}{\mathrm{tr}\,\boldsymbol{D}}. \qquad (23)$$

The absolute maximum value of $\hat{A}_{\mathrm{ANT}}^{(\mathrm{LT-TCSI})} = NM$ occurs if rank $[\boldsymbol{R}_{\mathrm{Tx}}]=1$. This explains why channels with strong transmit fading correlation can outperform uncorrelated channels, as is detailed in [1] and [13].

C. Antenna gain with no TCSI

The uniform power distribution $\boldsymbol{P}=(P_{\mathrm{T}}/N)\mathbf{I}_N$ in case of no TCSI turns the antenna gain from (22) into

$$A_{\mathrm{ANT}}^{(\mathrm{No-TCSI})} = M. \qquad (24)$$

In this case, only receive antenna gain can be obtained.

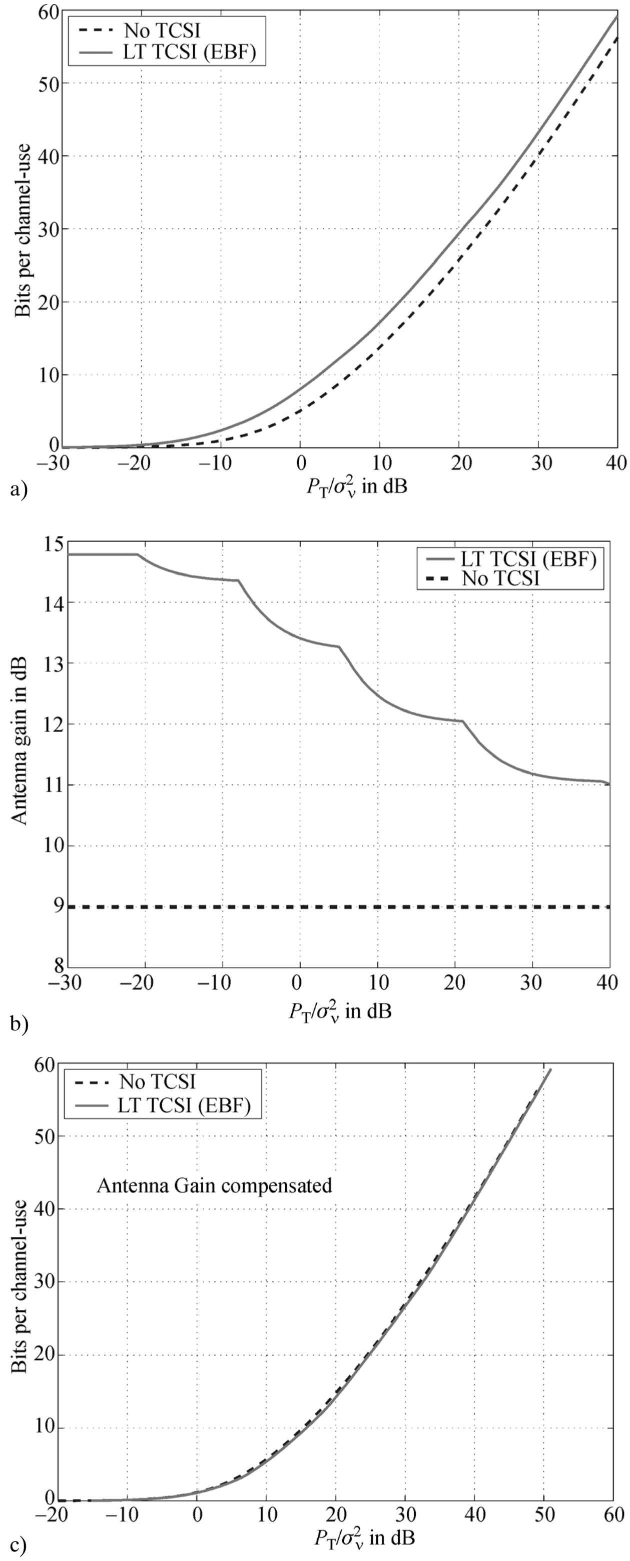

Fig. 1 a) Average channel capacity for a semi-correlated channel, with LT-TCSI and with no TCSI. b) Respective antenna gains in dB. c) Antenna-gain compensated curves of average channel capacity. The compensation is performed by adding the respective antenna gains to the transmit power (all in dB)

D. Antenna gain and channel capacity

It turns out by numerical analysis, that the antenna gain defined in (15) and (22), respectively, describes the capacity advantage of longterm TCSI with respect to no TCSI in a semi-correlated channel very well. To illustrate this, let us have a look at a MIMO system with $M = 8$ receive and $N = 8$ transmit antennas. The channel is semi-correlated, such that there is one remote scatterer which is illuminated by the transmit antenna array (half wavelength spaced uniform linear array) with an angle-spread of 30°. On Figure 1.a), we can see the average channel capacity as function of transmit power with long-term and no TCSI. The benefit of the long-term TCSI with respect to no TCSI is due to antenna gain. To support this assertion, we add the respective antenna gains (in dB) to the ratio P_T/σ_v^2 (in dB). These antenna gains are shown for the case of long-term TCSI and for no TCSI in Figure 1.b). In this way, we compensate for the respective antenna gains, such that the curves on Figure 1.c) show the average channel capacity as if there were no antenna gains. As we can see, the curves of the average channel capacities now lie almost on top of each other for all transmit powers. This shows that it is indeed the *antenna gain* which is making the difference in average channel capacity when having no or long-term average TCSI available. This demonstrates the applicability of the defined antenna gain as a MIMO performance measure.

4 MULTIPLEXING GAIN

An important feature of MIMO systems is the possibility to transfer $L > 1$ data streams in parallel at the same time inside the same bandwidth. From (9), we can see that for high values of P_T the capacity increases by L bits per channel-use when we double the transmit power (increase by 3 dB). The existence of parallel channels therefore manifests itself in the *slope* of the capacity as function of the (logarithmic) transmit power P_T or the dimensionless ratio P_T/σ_v^2. Therefore, it makes sense to define a measure for the amount of multiplicity of subchannels by the slope of the capacity C as function of $\log_2(P_T/\sigma_v^2)$. Similar to the antenna gain, we distinguish among different amount of TCSI.

A. Multiplexing gain with full TCSI

We define the multiplexing gain with full TCSI as

$$A_{\mathrm{MUX}}^{(\mathrm{F-TCSI})} = \frac{\mathrm{d}C}{\mathrm{d}\log_2\left(\frac{P_T}{\sigma_v^2}\right)}. \qquad (25)$$

It can be shown [13] that (25) can be written in terms of the eigenvalue matrix $\mathbf{\Lambda}$ from (8) and the matrix of transmit power $\boldsymbol{P}$ in the following way

$$A_{\mathrm{MUX}}^{(\mathrm{F-TCSI})} = \mathrm{tr}(\mathbf{\Lambda}\boldsymbol{P}(\sigma_v^2\mathbf{I}_N + \mathbf{\Lambda}\boldsymbol{P})^{-1}). \qquad (26)$$

The maximum multiplexing gain $\hat{A}_{\mathrm{MUX}}^{(\mathrm{F-TCSI})}$ bis achieved if the transmit power P_T is shared between the eigenmodes in the following way:

$$\frac{P_i}{\sigma_v^2} = \max\left(0, \frac{\zeta}{\sqrt{\lambda_i}} - \frac{1}{\lambda_i}\right), \qquad (27)$$

where $\zeta > 0$ is chosen such that $\sum_{i=1}^{N} P_i = P_T$. This is different from the waterfilling policy which shows that in general the maximum mutual information is not achieved when the multiplexing gain is at its peak[3]. This is not surprising, since in maximizing mutual information there is already a trade-off between multiplexing and antenna gain involved. Notice that $0 \le A_{\mathrm{MUX}}^{(\mathrm{F-TCSI})} \le \mathrm{rank}[\boldsymbol{H}]$.

B. Multiplexing gain with long-term TCSI

When the channel is known only on average to the transmitter, it is consequent to define the multiplexing gain as the slope of the average capacity as function of the logarithmic transmit power:

$$A_{\mathrm{MUX}}^{(\mathrm{avr})} = \frac{\mathrm{dE}[C]}{\mathrm{d}\log_2\left(\frac{P_T}{\sigma_v^2}\right)}. \qquad (28)$$

However, the evaluation of the expectation is rather involved for MIMO systems with semi-correlated fading [11]. The authors therefore propose [13] a simpler definition of multiplexing gain for longterm TCSI, which is based on the average channel. That is, we replace $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ by the transmit fading correlation matrix $\boldsymbol{R}_{\mathrm{Tx}} = \mathrm{E}[\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}]$. In this way, we obtain

$$A_{\mathrm{MUX}}^{(\mathrm{LT-TCSI})} = \mathrm{tr}(\boldsymbol{D}\boldsymbol{P}(\sigma_v^2\mathbf{I}_N + \boldsymbol{D}\boldsymbol{P})^{-1}). \qquad (29)$$

Due to Jensen's inequality, the long-term multiplexing gain from (29) provides an upper bound on the average multiplexing gain from (28):

$$A_{\mathrm{MUX}}^{(\mathrm{avr})} \le A_{\mathrm{MUX}}^{(\mathrm{LT-TCSI})}. \qquad (30)$$

[3] It is interesting to note that for maximizing multiplexing gain, sometimes a data stream with lower associate eigenvalue can be assigned more transmit power than a data stream with larger associated eigenvalue. This never happens with the Waterfilling policy.

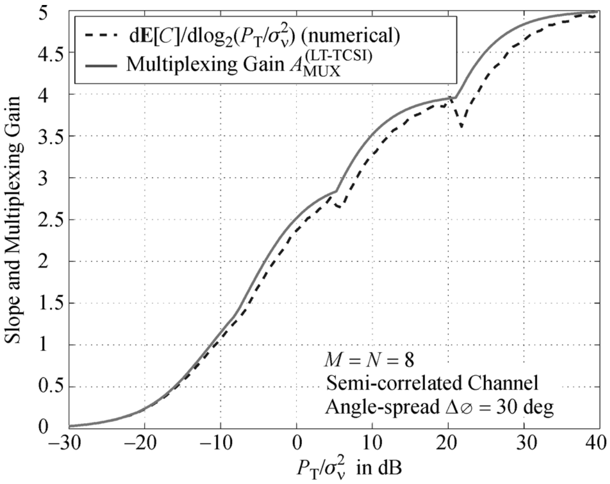

Fig. 2 Numerically evaluated slope of average capacity with respect to $\log_2(P_T/\sigma_v^2)$ and the long-term multiplexing gain

This upper bound is usually fairly tight, so that we can have $A_{\text{MUX}}^{(\text{avr})} \approx A_{\text{MUX}}^{(\text{LT-TCSI})}$ as illustrated in Figure 2. Notice that we have $0 \le A_{\text{MUX}}^{(\text{LT-TCSI})} \le \text{rank}[\boldsymbol{R}_{Tx}]$.

C. Multiplexing gain with no TCSI

The uniform power distribution $\boldsymbol{P} = (PT/N)\mathbf{I}_N$ in case of no TCSI turns the multiplexing gain from (29) into

$$A_{\text{MUX}}^{(\text{no-TCSI})} = \text{tr}\left(\boldsymbol{D}\left(\frac{N\sigma_v^2}{P_T}\mathbf{I}_N + \boldsymbol{D}\right)^{-1}\right). \tag{31}$$

Notice that for $P_T \to \infty$ we have $A_{\text{MUX}}^{(\text{no-TCSI})} = N$ in case of a full rank transmit fading correlation matrix.

5 DIVERSITY GAIN

The less correlation is present between the random variables which make up the MIMO channel matrix, the less variation of the received signal power and hence, link quality can be expected. We define the diversity gain which quantifies the amount of this variance by means of the so-called diversity measure.

A. Diversity measure

For a Rayleigh fading MIMO channel matrix $\boldsymbol{H}$ with fading correlation matrix $\boldsymbol{R}$ from (1), the diversity measure is defined as [15]:

$$\Psi(\boldsymbol{R}) = \frac{(\text{tr}\,\boldsymbol{R})^2}{\text{tr}(\boldsymbol{R}^2)}. \tag{32}$$

This definition[4)] has the property

$$\Psi(\boldsymbol{R}) = \frac{(E[\gamma])^2}{\text{var}[\gamma]}, \text{ where } \gamma = \|\boldsymbol{H}\|_F^2. \tag{33}$$

Herein $\|.\|_F^2$ denotes the squared Frobenius norm, i.e. the sum of squared magnitudes of all entries of the matrix passed as its argument. In this way, the diversity measure quantifies the relative fluctuation of the channel energy. The higher the value of $\Psi(\boldsymbol{R})$, the lower the fluctuation, i.e. the less correlation is present between the channel coefficients, and hence, more diversity is available. In case the correlation matrix $\boldsymbol{R}$ can be decomposed according to (2), the diversity measure conveniently factors into the product of the receice and the transmit diversity measure:

$$\Psi(\boldsymbol{R}) = \Psi(\boldsymbol{R}_{\text{Rx}}) \cdot \Psi(\boldsymbol{R}_{\text{Tx}}). \tag{34}$$

The diversity measure can for instance be used to decide which of two MIMO channels has stronger fading correlation or equivalently provides more diversity[5)]. Another application of the diversity measure is the construction of equivalence classes of MIMO channels. It turns out, that channel matrices which have the same diversity measure perform essentially the same with respect to channel capacity or throughput [15]. In this paper, we use the diversity measure to define the diversity gain. However, we restrict the definition to the case of long--term and no TCSI.

B. Diversity gain with long-term TCSI

We can write the received signal from (4) and (5) also as $r = \boldsymbol{Z}\boldsymbol{s} + v$, where

$$\boldsymbol{Z} = \boldsymbol{H}\boldsymbol{T}\boldsymbol{P}^{\frac{1}{2}} \tag{35}$$

is the effective channel matrix which includes the transmit signal processing. By setting the capacity achieving $\boldsymbol{T} = \boldsymbol{Q}$ with $\boldsymbol{Q}$ from (10), we define the long-term transmit diversity gain as [13]:

$$A_{\text{Tx-DIV}}^{(\text{LT-TCSI})} = \Psi\left(\text{E}\left[\boldsymbol{Z}^{\text{H}}\boldsymbol{Z}\right]\right) = \frac{(\text{tr}(\boldsymbol{D}\boldsymbol{P}))^2}{\text{tr}(\boldsymbol{D}^2\boldsymbol{P}^2)}, \tag{36}$$

with $\boldsymbol{D}$ from (10). The long-term receive diversity gain is defined as

$$A_{\text{Rx-DIV}}^{(\text{LT-TCSI})} = \Psi\left(\text{E}\left[\boldsymbol{Z}\boldsymbol{Z}^{\text{H}}\right]\right) = \frac{(\text{tr}\,\boldsymbol{R}_{\text{Rx}})^2}{\text{tr}(\boldsymbol{R}_{\text{Rx}}^2)}. \tag{37}$$

4) A similar definition of a diversity measure has also been made in [16] for the SIMO (single-input multiple-output) case. The authors of [16] base their definition however on the signal to noise ratio after maximum ratio combining. Nevertheless, their result is a special case of (32).

5) This method is compatible to the use of majorization [17] proposed in [18, 19] for the same purpose.

With (34), the total diversity gain is then the product of the receive and the transmit diversity gains:

$$A_{\mathrm{DIV}}^{(\mathrm{LT-TCSI})} = A_{\mathrm{Rx-DIV}}^{(\mathrm{LT-TCSI})} \cdot A_{\mathrm{Tx-DIV}}^{(\mathrm{LT-TCSI})}. \quad (38)$$

It turns out that the diversity gain $A_{\mathrm{DIV}}^{(\mathrm{LT-TCSI})}$ in (38) quantifies the relative fluctuation of the received signal power, i.e.

$$A_{\mathrm{DIV}}^{(\mathrm{LT-TCSI})} = \frac{\left(\mathrm{E}\left[\|\boldsymbol{r}\|_2^2\right]\right)^2}{\mathrm{var}\left[\|\boldsymbol{r}\|_2^2\right]}, \quad (39)$$

while

$$A_{\mathrm{Tx-DIV}}^{(\mathrm{LT-TCSI})} \text{ and } A_{\mathrm{Rx-DIV}}^{(\mathrm{LT-TCSI})}$$

quantify how much influence the transmit and the receive side have on this relative fluctuation. The largest value of $A_{\mathrm{DIV}}^{(\mathrm{LT-TCSI})}$ is given by the product

$$\mathrm{rank}[\boldsymbol{R}_{\mathrm{Rx}}] \cdot \mathrm{rank}[\boldsymbol{R}_{\mathrm{Rx}}]$$

and is achieved for a transmit power distribution which satisfies

$$P_1 \cdot D_1 = P_2 \cdot D_2 = \cdots = P_{\mathrm{rank}[\boldsymbol{R}_{\mathrm{Tx}}]} \cdot D_{\mathrm{rank}[\boldsymbol{R}_{\mathrm{Tx}}]}. \quad (40)$$

In this way, all data streams are received with the same average power. Dividing (36) by $\mathrm{rank}[\boldsymbol{R}_{\mathrm{Tx}}]$ we can obtain a figure of how effectively the transmitter is exploiting the available diversity of the channel.

C. Diversity gain with no TCSI

With equal power distribution $\boldsymbol{P} = (P_{\mathrm{T}}/N)\mathbf{I}_N$, it follows from (36) and (37)

$$A_{\mathrm{Tx-DIV}}^{(\mathrm{no-TCSI})} = \Psi(\boldsymbol{R}_{\mathrm{Tx}}) \text{ and } A_{\mathrm{Rx-DIV}}^{(\mathrm{no-TCSI})} = \Psi(\boldsymbol{R}_{\mathrm{Rx}}), \quad (41)$$

while the total diversity measure becomes

$$A_{\mathrm{DIV}}^{(\mathrm{no-TCSI})} = \Psi(\boldsymbol{R}_{\mathrm{Rx}}) \cdot \Psi(\boldsymbol{R}_{\mathrm{Tx}}) = \Psi(\boldsymbol{R}). \quad (42)$$

Without any TCSI, the diversity gain is soley determined by the correlation properties of the channel matrix. Except for the case of uncorrelated fading, the achievable diversity gain is always less than that with long-term average TCSI.

6 FUNDAMENTAL GAIN TRADE-OFF

Antenna gain, diversity gain and multiplexing gain are maximized by different transmit power distributions. By focusing on the long-term TCSI version of those gains we have

1. Long-term transmit antenna gain

$$P_1 = P_{\mathrm{T}},\ P_2 = P_3 = \ldots = P_N = 0 \quad (43)$$

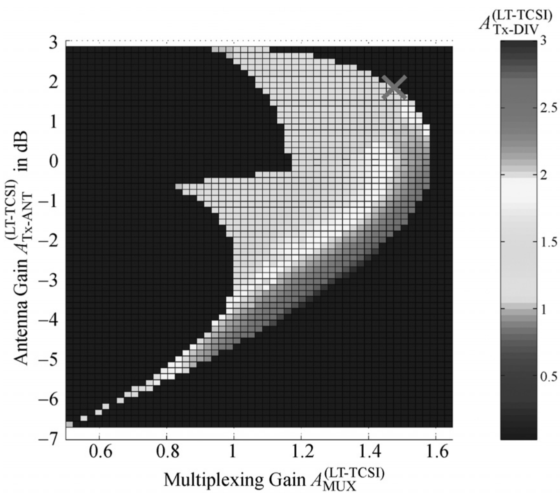

Fig. 3 $A_{\mathrm{Tx-DIV}}^{\mathrm{LT-TCSI}}$ as function of pairs of $A_{\mathrm{Tx-ANT}}^{\mathrm{LT-TCSI}}$ and $A_{\mathrm{MUX}}^{\mathrm{LT-TCSI}}$. The areas which are assigned a diversity gain of 0 correspond to regions where there is no solution, i.e. the given pair of antenna gain and multiplexing gain cannot be implemented. The cross marks the gain trade off which is maximizing average mutual information

2. Long-term multiplexing gain

$$P_i = \sigma_v^2 \cdot \max\left(0, \frac{\zeta}{\sqrt{D_i}} - \frac{1}{D_i}\right), \quad i \in \{1, 2, \ldots, L\}, \quad (44)$$

and $\zeta > 0$ chosen such that $\sum_{i=1}^{L} P_i = P_{\mathrm{T}}$.

3. Long-term diversity gain

$$P_i = \frac{\frac{P_{\mathrm{T}}}{D_i}}{\mathrm{tr}(\boldsymbol{D}^{-1})}, \quad i \in \{1, 2, \ldots, N\}. \quad (45)$$

There is no way to maximize all gains or even any pair of gains at the same time. Therefore, there is an elementary trade-off among these three performance gains. In order to illustrate this trade-off, let us look at an example case assuming a $N = M = 3$ semi-correlated channel with long-term eigenvalues given by $(D_1, D_2, D_3) = (9, 4, 1)$ and transmit power $P_{\mathrm{T}} = \sigma_v^2$. Figure 3 depicts the long-term transmit diversity gain which can be achieved for a given combination of long-term transmit antenna gain and long-term multiplexing gain. The combinations where there exists no solution[6)] are marked as diversity gain of zero (dark area in Figure 3). One can see that the highest amount of diversity gain occurs at a combination of moderate multiplexing gain and negative (in dB) antenna gain. Maximizing multi-

6) i.e. there is no transmit power distribution with which the given pair of antenna and multiplexing gain could be implemented.

plexing gain requires that the diversity gain gets down. Similarly, maximization of antenna gain costs both multiplexing and diversity gain. However, note that for a fixed long-term transmit antenna gain the diversity gain is increasing with increasing multiplexing gain. The maximum value of the diversity gain for a given antenna gain occurs very close to the right hand border of the valid gain region. The gain trade-off which is achieving average capacity is located on this right hand border (point marked with a cross in Figure 3).

7 SUMMARY

Three performance measures are proposed and derived which allow quantification of the amount of antenna gain, diversity gain, and multiplexing gain of a wireless MIMO system. These three gains reflect the performance of a MIMO system with respect to its transmit power efficiency, the link realibility and its usage of the space dimension. It is shown that these three performance gains can not be maximized all at the same time. In practice a gain trade-off is unavoidable. Such a trade-off between performance gains is demonstrated in an example case.

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Dobitci MIMO sustava sa stanovišta obrade signala. Da bi se što bolje iskoristili svi stupnjevi slobode koje nude MIMO sustavi, potrebno je prepoznati i vrednovati moguće dobitke koji se mogu očekivati od MIMO obrade signala. U ovom radu su izvedeni postupci vrednovanja dobitka antene, dobitka diverzitija te dobitka multipleksiranja MIMO sustava. Pokazana je nemogućnost istovremenog maksimiranja sva tri dobitka ili bilo kojeg para dobitaka. Prikazan je nužni kompromis među pojedinim dobitcima sustava.

Ključne riječi: mjera kvalitete MIMO sustava, dobitak antene, dobitak diverzitija, dobitak multipleksiranja

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