ECONOMETRIC APPROACH TO DIFFERENCE EQUATIONS MODELING OF EXCHANGE RATES CHANGES

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Abstract

Time series models that are commonly used in econometric modeling are autoregressive stochastic linear models (AR) and models of moving averages (MA). Mentioned models by their structure are actually stochastic difference equations. Therefore, the objective of this paper is to estimate difference equations containing stochastic (random) component. Estimated models of time series will be used to forecast observed data in the future. Namely, solutions of difference equations are closely related to conditions of stationary time series models. Based on the fact that volatility is time varying in high frequency data and that periods of high volatility tend to cluster, the most successful and popular models in modeling time varying volatility are GARCH type models and their variants. However, GARCH models will not be analyzed because the purpose of this research is to predict the value of the exchange rate in the levels within conditional mean equation and to determine whether the observed variable has a stable or explosive time path. Based on the estimated difference equation it will be examined whether Croatia is implementing a stable policy of exchange rates.

Key words: *difference equations, time series, econometric estimation, exchange rate*

1. INTRODUCTION

Time series models that are commonly used in econometric modeling are autoregression stochastic linear models (AR) and moving average models (MA). By their structure, these models are actually stochastic difference equations. The objective of evaluation of time series models is based on analysis of difference equations containing stochastic (random) components, in order to forecast the observed phenomena in the future. Whether the observed variables have a stable or an explosive time path is closely related to conditions of stationary time series models. Namely, if for example AR(p) model is not stationary then it can't be used in forecasting purposes.

The paper will show that the deterministic part of the particular solutions in econometric sense is the same as expectation of stochastic processes. Whatsoever, to determine the order of difference equations and autoregression models the autocorrelation functions and partial autocorrelation functions are usually estimated. An example will be illustrated on the basis of the movement of exchange rates including monthly observations. The most basic objective of monetary policy in liberal democracies, including Croatia, is price stability, respectively in Croatia the stability of exchange rates to control inflationary expectations. Based on the estimated difference equation it will be examined whether Croatia is implementing a stable policy of exchange rates. Evidence of implementation of stable policy of exchange rate also means the successful implementation of the primary objective of Croatian monetary policy.

The structure of the paper is organized as follows. In the second section time series models are described as a type of difference equation in their most general form, containing stochastic component (random or irregular disturbance). In the third section difference equations and their solutions are examined with implementation on modeling of exchange rate changes in Croatia. Last section provides concluding remarks of empirical findings.

2. TIME SERIES MODELS

In the most general form difference equations of order *p* can be written as follows

$$y_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + x_t \, , \tag{1}$$

where component x_t is the random process. In econometrics x_t is important as a special case when

$$x_t = \sum_{j=0}^{\infty} \beta_j \varepsilon_{t-j} \, , \tag{2}$$

where $\{\varepsilon_t\}$ is *white noise* process.[1]

Linear stochastic ARMA(*p*,*q*) model of observed time series $\{y_t\}$ contains the following analytical expression (Enders, 2004)

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \ldots + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2} + \ldots + \beta_q \varepsilon_{t-q} \, . \tag{3}$$

In this autoregression model, i.e. AR (*p*)

$$\alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} \, , \tag{4}$$

is a part of the model moving average model, i.e. MA(*q*)

[1] Stochastic process is a "white noise" process whose random variables are non-correlated with expectation equal to zero, constant variances and covariance equal to zero for any lag $k \neq 0$.

$$\sum_{j=o}^{q} \beta_j \varepsilon_{t-j} . \tag{5}$$

Taking into account equation (5) it can be noted that the special case of equation (2) is actually MA(∞) process.

If it is assumed ARMA(*p*,*q*) model in which p = 1 and q = 0, we get

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t . \tag{6}$$

It can be seen that the equation (6) is a difference equation of the first order with a stochastic term ε_t. Solving stochastic difference equation (6) by iterative method (continuous substitution for one period in advance of starting time $t = 0$) and providing initial value y_0, we get the following solution

$$y_t = \alpha_0 \sum_{i=0}^{t-1} \alpha_1^i + \alpha_1^t y_0 + \sum_{i=0}^{t-1} \alpha_1^i \varepsilon_{t-i} . \tag{7}$$

If the initial value y_0 is not set, then by continuous substitution for *m* period from the moment $t = 0$ follows

$$y_t = \alpha_0 \sum_{i=0}^{t+m} \alpha_1^i + \sum_{i=0}^{t+m} \alpha_1^i \varepsilon_{t-i} + \alpha_1^{t+m+1} y_{-m-1} . \tag{8}$$

If we assume that in equation (8) $|\alpha_1| < 1$, then for a very large shift of *m* periods we can obtain

$$y_t = \frac{\alpha_0}{1-\alpha_1} + \sum_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i} , \tag{9}$$

as an infinite geometric sum $\left(1 + \alpha_1 + \alpha_1^2 + \alpha_1^3 + ...\right)$ converges to $\frac{1}{1-\alpha_1}$, as component α_1^{t+m+1} tends to zero when $m \to \infty$ (Hamilton, 1994). The solution in equation (9) is a particular solution of stochastic difference equation of the first order, while the component

$$\frac{\alpha_0}{1-\alpha_1} , \tag{10}$$

is deterministic part of the particular solutions of stochastic difference equations. [2]

Specifically, the solution obtained in equation (9) is not unique, because for any arbitrary constant A there is a solution

$$y_t = A\alpha_1^t + \frac{\alpha_0}{1-\alpha_1} + \sum_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i} . \tag{11}$$

So, if we take a homogeneous part of the equation (6)

[2] Deterministic part of particular solution in terms of econometrics actually corresponds to the mean value of the process (when taking mathematical expectation of the expression (9)), i.e. $\mu = \frac{\alpha_0}{1-\alpha_1}$.

$$y_t = \alpha_1 y_{t-1}, \tag{12}$$

which solution is equally to

$$y_t = \alpha_1^t y_0, \tag{13}$$

then it can be shown that for any constant A solution

$$y_t = A\alpha_1^t, \tag{14}$$

is homogeneous solution.[3]

Since the $A\alpha_1^t$ is homogeneous solution of stochastic difference equation of the first order, then the solution in equation (11) is called a general solution (the sum of particular solution and homogeneous solution), whereby the arbitrary constant A can be eliminated by introducing the initial condition y_0 (Chiang, 1994).

If we introduce the initial value of y_0 then (11) must hold for each t, and also for $t = 0$

$$y_0 = A + \frac{\alpha_0}{1-\alpha_1} + \sum_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i}. \tag{15}$$

Then the value of A equal

$$A = y_0 - \frac{\alpha_0}{1-\alpha_1} - \sum_{i=0}^{\infty} \alpha_1^i \varepsilon_{t-i}, \tag{16}$$

from which follows the solution

$$y_t = \left(y_0 - \frac{\alpha_0}{1-\alpha_1} \right) \alpha_1^t + \frac{\alpha_0}{1-\alpha_1} + \sum_{i=0}^{t-1} \alpha_1^i \varepsilon_{t-i}. \tag{17}$$

3. DIFFERENCE EQUATIONS MODELLING OF EXCHANGE RATES AND FORECASTING

The most basic objective of monetary policy in liberal democracies, including Croatia, is price stability, respectively in Croatia the stability of exchange rates to control inflationary expectations. Price stability and growth financing through import foreign savings are features of the transition model of development. The mechanism that allows movement of savings from developed countries into developing countries in the form of investment is international capital mobility. As the country implements reforms in the process of association in EU, than appreciation pressures, typical for this model, increase (Tica et al., 2007). On appreciation pressures, the monetary policy in Croatia responded whit buying out foreign currency to defend exchange rate and thus simultaneously increased the amount of kuna in circulation and currency reserves. In the context of the global crisis, appreciation pressures are reduced and priority of the Croatian National Bank is to prevent excessive exchange rate volatility in order to preserve price stability.

[3] The solution of $y_t = \alpha_1 y_{t-1}$ it is actually $A\alpha_1^t$ because of equality $A\alpha_1^t = \alpha_1 A\alpha_1^{t-1}$.

Therefore, in this paper difference equation will be estimated for describing the movements of exchange rates HRK/EUR based on monthly observations from January 1999 to June 2010.

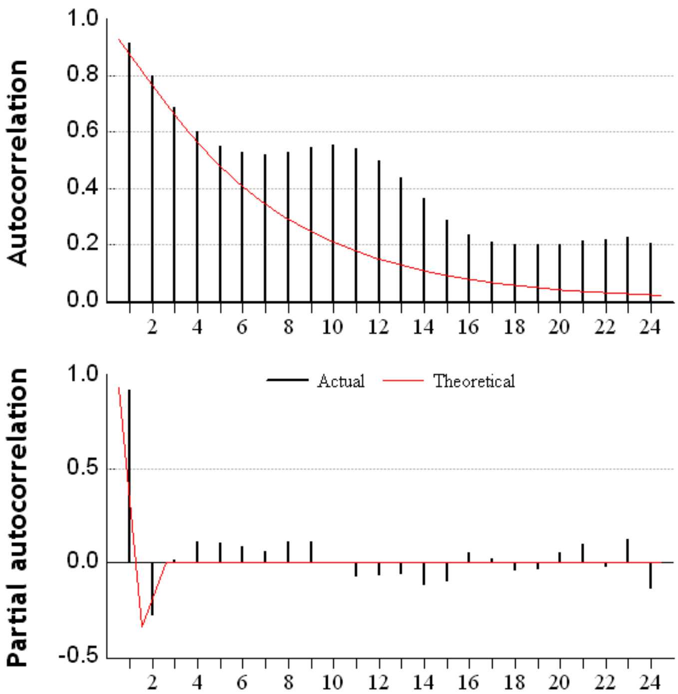

Figure 1:. Autocorrelation and partial autocorrelation functions from the sample.

The order of difference equation is determined according to the autocorrelation and partial autocorrelation functions from the sample. Econometric approach in time series modeling depends on the significance of coefficients of autocorrelation function (ACF) and partial autocorrelation functions (PACF). From Figure 1 it can be shown that ACF decreases exponentially, while the PACF dies out after the second time lag, so the estimated second-order difference equations is evaluated (Hamilton, 1994). Therefore, using Eviews software the second order difference equation, which corresponds to the autoregression model AR(2), was estimated.

In Table 1 estimated coefficients with usual statistical tests and diagnostics measures are presented. Namely, second-order difference equation is estimated using method of least squares.

Based on estimated results from January 1999 to June 2010, stochastic difference equation for the variable exchange rate can be expressed as follows

$$y_t = 7.412164 + 1.245431 y_{t-1} - 0.340068 y_{t-2} + \varepsilon_t . \tag{18}$$

There are two solutions to the homogeneous difference equation of the second order. Homogeneous part of equation (18) equals

$$y_t - 1.245431 y_{t-1} + 0.340068 y_{t-2} = 0 . \tag{19}$$

If possible solutions of the equation (19) is $\mathrm{y}_t = \mathrm{a}^t$, then by substitution we get

$$a^t - 1.245431a^{t-1} + 0.340068a^{t-2} = 0 . \quad (20)$$

Table 1: Estimated coefficients of the second-order difference equation with statistical tests and diagnostics measures

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	7.412164	0.047610	155.6841	0.0000
AR(1)	1.245431	0.079928	15.58190	0.0000
AR(2)	-0.340068	0.080231	-4.238589	0.0000

R-squared	0.875684	Mean dependent var	7.427235
Adjusted R-squared	0.873814	S.D. dependent var	0.146842
S.E. of regression	0.052162	Akaike info criterion	-3.047109
Sum squared resid	0.361878	Schwarz criterion	-2.982859
Log likelihood	210.2034	F-statistic	468.4250
Durbin-Watson stat	1.992616	Prob(F-statistic)	0.000000
Inverted AR Roots	.84	.40	

Source: Croatian National Bank [online], http://www.hnb.hr/tecajn/htecajn.htm [Accessed13/09/.]

If we divide the equation (20) by a^{t-2}, the problem is to find the values of a that satisfy

$$a^2 - 1.245431a + 0.340068 = 0 . \quad (21)$$

The quadratic equation in (21) is called the characteristic equation, whose solutions are characteristic roots. Since the discriminant $d = 0.19082$ is positive than the characteristic roots are two distinct real numbers

$$a_{1/2} = \frac{1.245431 \pm \sqrt{0.190826}}{2}; \quad a_1 = 0.8411; \quad a_2 = 0.4043 . \quad (22)$$

However, these solutions are not unique. In fact, for any two arbitrary constants A_1 and A_2 the linear combination of homogeneous solutions also solves second order difference equation

$$A_1(0.8411)^t + A_2(0.4043)^t . \quad (23)$$

Since both roots are less then unity in absolute value, the homogeneous solution is convergent. From Figure 2 it can be seen that both homogeneous solutions converges directly because $0 < a_1, a_2 < 1$. Therefore, the solution is stable, i.e. the stability conditions are satisfied which means that characteristics roots lie within the unit circle. Figure 2 shows the time path of this solution for the case in which the arbitrary constants equal unity for 20 time units.

Terms of stability for second order difference equation are: (a) the largest characteristic root must be less than 1 and (b) the smallest characteristic root must be greater than -1 (Enders, 2004). Taking these conditions of stability into account with the assumption of positive discriminant value, we get the following relations

$$\begin{aligned} &\alpha_1 + \alpha_2 < 1 \\ &\alpha_2 - \alpha_1 < 1 . \\ &-1 < \alpha_2 < 1 \end{aligned} \quad (24)$$

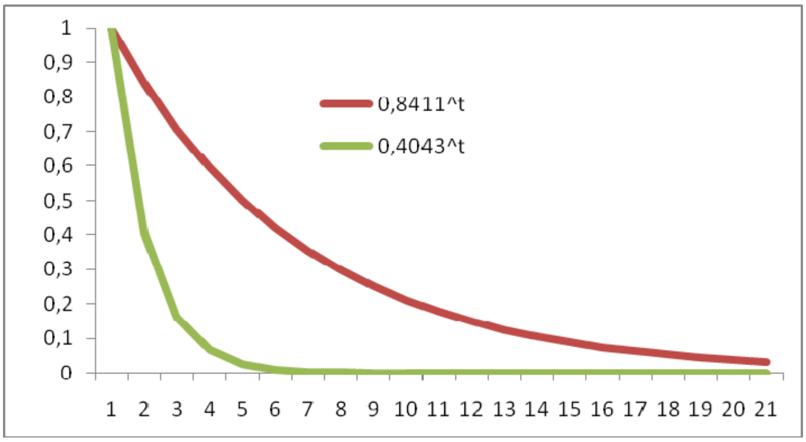

Figure 2:. *Convergence of the homogeneous solutions when* $A_1 = A_2 = 1$.

The above conditions describe the stationary conditions of second order autoregression AR(2). This means that the same conditions are fulfilled if the roots of the characteristic equation $a_1 = 0.841135$ and $a_2 = 0.4042975$ lie within the unit circle. It is easy to confirm that the conditions in (24) are satisfied for $\alpha_1 = 1.245431$ and $\alpha_2 = -0.340068$.

Each stochastic difference equation can be written using the lag operator

$$L^k y_t = y_{t-k}. \qquad (25)$$

Using the lag operator stochastic difference equation of second order can be written as follows

$$(1-\alpha_1 L - \alpha_2 L)y_t = \alpha_0 + \varepsilon_t, \qquad (26)$$

where $\left(1-\sum_{k=1}^{2}\alpha_k L^k\right)$ is the lag operator polynomial of the autoregression model.

According to results in Table 1 lag operator polynomial of AR(2) model has the form

$$(1-1.245431L + 0.340068L^2) = 0. \qquad (27)$$

Autoregression roots of a polynomial in the expression (27) are $L_1 = 1.1888698$ and $L_2 = 2.473426$. It is easy to confirm that the inverse roots of autoregression polynomial corresponds to the characteristic roots of the difference equations in expression (21)

$$\frac{1}{L_1} = 0.8411; \quad \frac{1}{L_2} = 0.4043. \qquad (28)$$

This means that the conditions of stability are satisfied if the roots of autoregression polynomial lie outside the unit circle, which is equivalent to the state that the roots of the characteristic equation lie inside the unit circle. Therefore, in Table 1 inverse roots of the autoregression polynomial in the last row (*Inverted AR Roots*) are presented. Inverted AR(2) roots are presented on Figure 3.

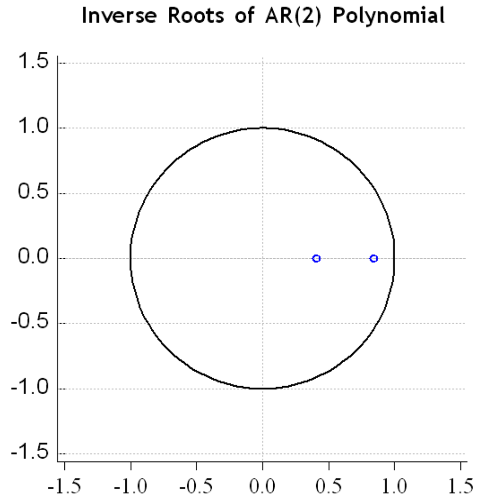

Figure3:. *Inverse roots of AR(2) polynomial*

So, if the inverse roots of autoregression polynomial lie outside the unit circle, then we say that the stationary condition is satisfied. If the condition is met then the stationary time series is suitable for forecasting.

Specifically, we have already shown that the solution in (23) is homogeneous solution of stochastic difference equation in (18), while the expression (9) shows how the particular solution can depend only on constants and elements of $\{\varepsilon_t\}$ if $|\alpha_1| < 1$. Therefore, particular solution of second order difference equation in which the characteristic roots are inside the unit circle is

$$y_t = b_0 + c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2}....... \tag{29}$$

In equation (29) the constant term corresponds to deterministic part of the particular solution

$$b_0 = \frac{\alpha_0}{1-\alpha_1-\alpha_2} = 78.322052. \tag{30}$$

To find a stochastic part of the particular solution it is necessary to check whether a possible solution

$$y_t = \sum_{i=0}^{\infty} c_i\varepsilon_{t-i}, \tag{31}$$

indeed is a solution (Enders, 2004). Thus, after substitution of expression (31) into (18) it is necessary to examine under which conditions the following equality is valid

$$\begin{aligned} c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + ... = 1.24543\cdot&\left[c_0\varepsilon_{t-1} + c_1\varepsilon_{t-2} + c_2\varepsilon_{t-3} + ...\right] \\ -0.340068\cdot&\left[c_0\varepsilon_{t-2} + c_1\varepsilon_{t-3} + c_2\varepsilon_{t-4} + ...\right] + \varepsilon_t \end{aligned}. \tag{32}$$

Since equality in (32) must hold for all $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, ...$, then it is necessary to satisfy the following relations

$$\begin{aligned} &c_0 = 1 \\ &c_1 = 1.24543c_0 \\ &\vdots \\ &c_i = 1.24543c_{i-1} - 0.340068c_{i-2} \quad \forall\, i \ge 2 \end{aligned}. \tag{33}$$

Coefficients c_i can easily be obtained iteratively, $c_0 = 1, c_1 = 1.24543, c_2 = 1.211028, \ldots$. In other words, we seek the solution of second order difference equation $c_i = 1.24543 c_{i-1} - 0.340068 c_{i-2}$ with initial conditions $c_0 = 1$ and $c_1 = 1.24543$. Solution of difference equations is

$$c_i = A_1 (0.84)^i + A_2 (0.40)^i . \tag{34}$$

Introducing initial conditions in equation (34), it is easy to solve the system

$$\begin{aligned} 1 &= A_1 + A_2 \\ 1.24543 &= A_1 0.84 + A_2 0.40 \end{aligned} . \tag{35}$$

Introducing solutions of (35) in (34) yields a solution

$$c_i = 1.92 (0.84)^i - 0.92 (0.40)^i . \tag{36}$$

Thus, the sum of a linear combination of homogeneous solutions and deterministic and stochastic particular solutions equals

$$y_t = 78.322 + A_1 \left(0.84\right)^t + A_2 \left(0.4\right)^t + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i} , \tag{37}$$

where c_i is defined in equation (36). Specifically, in our case it is interesting that in the long term mean value of the exchange rate equals to 7.412 HRK/EUR. It is also possible to calculate the average deviation from the expected rate in the long run, i.e. the standard deviation of AR(2) model of time series. The average deviation from the expected value in the long run equals to 0.4593HRK/EUR. This means that in the future we can expect a variation rate in the range of $\pm$ 6.20%. The results confirm that the Croatian National Bank conducted a stable policy of exchange rate. Evidence of implementation of stable policy of exchange rate also means the successful implementation of the primary objective of Croatian monetary policy.

4. CONCLUDING REMARKS

Linear stochastic time series models by their structure are actually difference equations containing random component which follows white noise process. Solutions of difference equations are closely related to the stationary conditions of time series models. It can be shown that the conditions of stability are satisfied if the roots of autoregression polynomial lie outside the unit circle, which is equivalent to the state that the roots of the characteristic equation lie inside the unit circle. To determine whether the observed variable has a stable or explosive time path means that the time series converges or diverges in the long run. From estimated AR(2) model of exchange rate changes it can be seen that both homogeneous solution converges directly. We can also conclude that deterministic part of particular solution in terms of econometrics corresponds to the expected value of the process, while the stochastic part of particular solution follows MA(∞) process. The average deviation from the expected value of exchange rate HRK/EUR in the range of $\pm$ 6.20% in the long run confirms stable exchange rate policy.

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