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Temel Ermiş
Özcan Gelişgen
Rustem Kaya

# On Taxicab Incircle and Circumcircle of a Triangle

**On Taxicab Incircle and Circumcircle of a Triangle**

**ABSTRACT**

In this work, we study existence of taxicab incircle and circumcircle of a triangle in the taxicab plane and give the functional relationship between them in terms of slope of sides of the triangle. Finally, we show that the point of intersection of taxicab inside angle bisectors of a triangle is the center of taxicab incircle of the triangle.

**Key words:** taxicab distance, taxicab circle, taxicab incircle, taxicab circumcircle, taxicab plane and taxicab geometry

**MSC 2010:** 51K05, 51K99, 51N20

**O taxicab upisanoj i opisanoj kružnici trokuta**

**SAŽETAK**

U ovom radu promatramo postojanje taxicab upisane i opisane kružnice trokuta u taxicab ravnini i dajemo njihov odnos s obzirom na koeficijente smjera stranica trokuta. Naposljetku, pokazujemo da je sjecište taxicab unutarnjih simetrala kuta trokuta središte njegove taxicab upisane kružnice.

**Ključne riječi:** taxicab udaljenost, taxicab kružnica, taxicab upisana kružnica, taxicab opisana kružnica, taxicab ravnina and taxicab geometrija

## 1 Introduction

Taxicab plane geometry introduced by Menger and developed by Krause using the metric $d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|$ instead of the well-known Euclidean metric $d_E(P,Q) = \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right)^{1/2}$ for the distance between any two points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ in the analytical plane $\mathbb{R}^2$. According to definition of taxicab distance function, the path between two points in the plane is union of two line segments which each of line segments is parallel to one of coordinate axis. Thus, taxicab distance is the sum of Euclidean lengths of these two line segments. Taxicab geometry have studied and developed in different aspects by mathematicians. One can see for some of these in [2], [3], [5], [6], [7], [9]. The linear structure of the taxicab plane is almost the same as Euclidean plane. There is one different aspect. Euclidean and taxicab planes have different distance functions. So it seems interesting to study the taxicab analogues of topics that are related with the concept of distance.

It is well known that there exist a unique incircle and a unique circumcircle for a given triangle in the Euclidean geometry. Also the distance $d$ between the centers of incircle and circumcircle of a triangle is $d = \sqrt{R(R-2r)}$ where $r$ and $R$ are radii of incircle and of circumcircle, respectively. Here, we study and extend these three properties to the taxicab plane.

## 2 Taxicab incircles and circumcircles of a given triangle

The **incircle** of a triangle is the circle contained in the triangle and touches to (*is tangent to*) each of the three sides at one point. A circle passing through all three vertices of a triangle is called **circumcircle** of the triangle.

Let $C$ be a point in the taxicab plane, and $r$ be a positive real number. The set of points $\{X : d_T(C,X) = r\}$ is called **taxicab circle**, the point $C$ is called *center of the taxicab circle*, and $r$ is called the *length of the radius* or simply *radius* of the taxicab circle. Every taxicab circle in the taxicab plane is an Euclidean square having sides with slopes $\pm 1$.

Let $l$ be a line with slope $m$ in the taxicab plane. $l$ is called a **gradual line**, a **steep line,** a **separator** if $|m| < 1$, $|m| > 1$, $|m| = 1$, respectively (see Figure 1). In particularly, a gradual line is called **horizontal** if it is parallel to $x$-axis, and a steep line is called **vertical** if it is parallel to $y$-axis.

The isometry group of taxicab plane is the semi direct product of $D(4)$ and $T(2)$ where $D(4)$ is the symmetry group of Euclidean square and $T(2)$ is the group of all translations in the plane. That is, the rotations $\theta = k\pi/2$, $k \in \mathbb{Z}$, the reflections by lines with horizontal, vertical and separators and all translations are isometries in the taxicab plane (see [5], [8]).

The distance from the point $P = (x_0, y_0)$ to a line $l$ with the equation
$ax + by + c = 0$ is $d_T(P,l) = |ax_0 + by_0 + c| \,/\, \max\{|a|, |b|\}$ (see [6]).

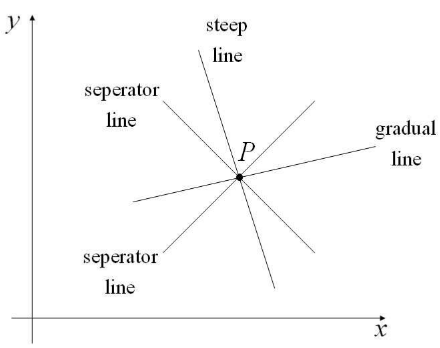


Figure 1

In the taxicab plane, the triangles can be classified as eight groups according to slopes of the sides of triangles:

**i)** All sides of the triangle lie on gradual (*steep*) lines.

**ii)** Two sides of the triangle lie on gradual (*steep*) lines, the other side lies on a steep (*gradual*) line.

**iii)** Two sides of the triangle lie on separator lines, the other side lies on a gradual (*steep*) or a horizontal (*vertical*) line.

**iv)** A side of the triangle lies on a separator line, two sides of the triangle lie on gradual (*steep*) lines.

**v)** A side of the triangle lies on a separator line, the other side lies on a gradual line and the third side lies on a steep line.

**vi)** A side of the triangle lies on a vertical line, the other side lies on a horizontal line and third side lies on a gradual (*steep*) line or separator line.

**vii)** A side of the triangle lies on a vertical (*horizontal*) line, two sides of the triangle lie on gradual (*steep*) lines.

**viii)** A side of the triangle lies on a vertical (*horizontal*) line, the other side lies on a gradual line and the third side lies on a steep line.

The next theorem gives an explanation about whether there exist an incircle and a circumcircle of a triangle or not.

**Theorem 1** *A triangle has a taxicab circumcircle and incircle if and only if two sides of the triangle lie on gradual (steep) lines and remaining side lies on a steep (gradual) line.*

**Proof.** Let $\overset{\triangle}{ABC}$ be a triangle having a taxicab circumcircle and a taxicab incircle and let $m_a$, $m_b$ and $m_c$ denote the slopes of lines $BC$, $AC$ and $AB$, respectively. A taxicab circumcircle compose of the line segments which are on lines with slopes $\mp 1$, passing through the vertices of a triangle. That is, there is a line with slope 1 or $-1$ passing through every vertex of the triangle, and the triangle is completely inside the taxicab circumcircle. Every vertex of a triangle is a point intersection of two sides of this triangle. Since three vertices of the triangle are on different sides of the taxicab circle, vertices of two sides of the triangle are on neighbour sides of the taxicab circle. So two vertices of the triangle are on lines with same slope 1, and the third vertex is on a line with slope $-1$ or vice versa.

Suppose that vertices $A$, $B$, $C$ of triangle are on lines with slopes $-1$, $+1$, $-1$, respectively. Thus $m_a < 1$ and $m_a > -1$. That is, $|m_a| < 1$. Similarly one can obtain $m_c < 1$ and $m_c > -1$. That is, $|m_c| > 1$. Third side $AC$ is on a gradual or a steep line. Other cases can be proven similarly.

Since the triangle $ABC$ has a taxicab incircle, three vertices of the circle are on sides of the triangle. Three vertices of the incircle are on lines slopes $\mp 1$ and 0 or $\infty$. So there are two different cases (see Figure 2).

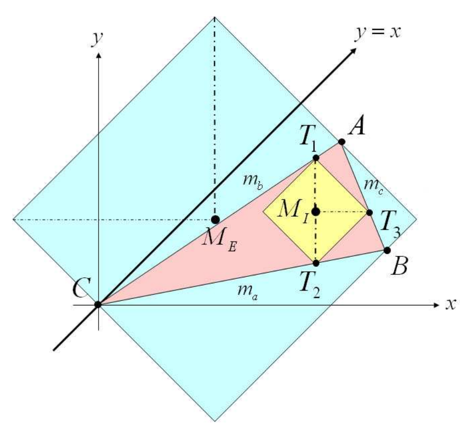


Figure 2

**Case I:** Let the vertices of the incircle be on lines with slopes $-1$, 1 and $\infty$. And let $T_1$, $T_2$, $T_3$ denote these points such that $T_1$, $T_2$ and $T_3$ are on sides $AC$, $BC$ and $AB$, respectively. If $T_1$ and $T_2$ are on a vertical line, then $|m_a| < 1$ and $|m_b| < 1$. Otherwise, a part of the sides of the incircle lie outside of the triangle. Also $|m_c| > 1$ since incircle must be inside the triangle.

**Case II:** Let the vertices of the incircle be on lines with slopes $-1$, $1$ and $0$. This case is proven similarly as case I.

Conversely, let two sides of the triangle be on gradual lines and other side be on a steep line. We may draw the lines of slopes 1 and $-1$ passing through vertices of the triangle. By the definition of the circumcircle, these lines must not pass through inner region of the triangle. Thus we may construct three sides of a taxicab circumcircle encircled the triangle. The fourth side of the taxicab circumcircle properly can be found.

Since three vertices of incircle of the triangle are on each sides of the triangle, we may construct incircle alike circumcircle case. □

In the rest of the article, we assume that the vertex $C$ of a triangle $\overset{\triangle}{ABC}$ is the intersection point of gradual (*steep*) sides of the triangle. Without lose of generality, a vertex of the triangle can be taken at origin since all translations of the analytical plane are isometries of the taxicab plane. So we take the vertex $C$ is at origin. Notice that this assumption about the position of the triangle does not loose the generality. If the role of vertices $A$ and $B$ of the triangle replace with each other, then the slopes $m_a$ and $m_b$ must be replaced with each other in all functional relations through the article.

The following theorem gives a relation between diameters of the circumcircle and incircle and the slopes of sides of a triangle in the taxicab plane.

**Theorem 2** *Let slopes of sides $AB, BC, CA$ of a triangle $\overset{\triangle}{ABC}$ be $m_c, m_a$ and $m_b$, respectively and $a, b, c$ be the taxicab lengths of sides of $BC, CA, AB$ of the triangle $\overset{\triangle}{ABC}$. If the triangle has a taxicab incircle with radius* $\mathbf{r}$ *and a circumcircle with radius* $\mathbf{R}$ *then*

$$\frac{\mathbf{r}}{2\mathbf{R}} = \begin{cases} \rho(m_a, m_b, m_c), & |m_a|<1,\ |m_b|<1 \text{ and } |m_c|>1 \\ \rho\left(-m_a^{-1}, -m_b^{-1}, -m_c^{-1}\right), & |m_a|>1,\ |m_b|>1 \text{ and } |m_c|<1 \end{cases}$$

*where* $\rho(m_a, m_b, m_c) =$

$$\frac{|m_b - m_a|\,|\delta(m_a, m_b) - m_c|}{\max\{|1+\delta(m_a,m_b)|, |1-\delta(m_a,m_b)|\}\,|\max\{m_a,m_b\}(1-m_c) + \min\{m_a,m_b\}(1+m_c) - 2m_c|}$$

*with*

$$\delta(m_a, m_b) = \begin{cases} m_a, & a > b \\ m_b, & a < b. \end{cases}$$

**Proof.** Let $\overset{\triangle}{ABC}$ be a triangle such that $|m_a| < 1$, $|m_b| < 1$ and $|m_c| > 1$ where $m_a$, $m_b$ and $m_c$ are slopes of sides $BC, AC$ and $AB$, respectively (see Figure 2). There are two main positions of gradual sides of the triangle:

**i**) Gradual sides are on same quadrant of the plane (see Figure 2)

**ii**) Gradual sides are on neighbour quadrants of the plane (see Figure 3).

Suppose that $A=(x_a, y_a)$, $B=(x_b, y_b)$ are points such that $x_b > x_a > 0$ and $y_a > y_b > 0$ (see Figure 2). Therefore, center and radius of the taxicab circumcircle of triangle $\overset{\triangle}{ABC}$ are

$M_E = \left(\frac{x_b - y_b}{2}, \frac{x_a - x_b + y_a + y_b}{2}\right)$ and

$\mathbf{R} = \frac{x_a + y_a}{2} = \frac{x_a(1+m_b)}{2}$.

Three vertices of the taxicab incircle are on lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ with the equations $y = m_c(x - x_a) + y_a$, $y = m_a x$ and $y = m_b x$, respectively. Let $T_1$ and $T_2$ be on lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, respectively. So $T_1 = (t, m_b t)$ and $T_2 = (t, m_a t)$ for $t \in \mathbb{R}^+$. Thus center and radius of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$ are

$$M_I = \left(t, m_a t + \frac{(m_b - m_a)t}{2}\right) \text{ and } \mathbf{r} = \frac{(m_b - m_a)t}{2}.$$

Since $T_3 = \left(t + \frac{(m_b - m_a)t}{2}, m_a t + \frac{(m_b - m_a)t}{2}\right)$ and $T_3$ is on the line with the equation $y = m_c(x - x_a) + y_a$, one can easily compute that

$$t = \frac{2x_a(m_b - m_c)}{m_a + m_b - 2m_c + m_c(m_a - m_b)}.$$

Thus the relation between $\mathbf{R}$ and $\mathbf{r}$ is

$$\mathbf{r} = \frac{2\mathbf{R}(m_b - m_c)(m_b - m_a)}{(1+m_b)[m_a(1+m_c) + m_b(1-m_c) - 2m_c]}$$

Similarly, let $\overset{\triangle}{ABC}$ be a triangle such that $|m_a| < 1$, $|m_b| < 1$ and $|m_c| > 1$ (see Figure 3a).

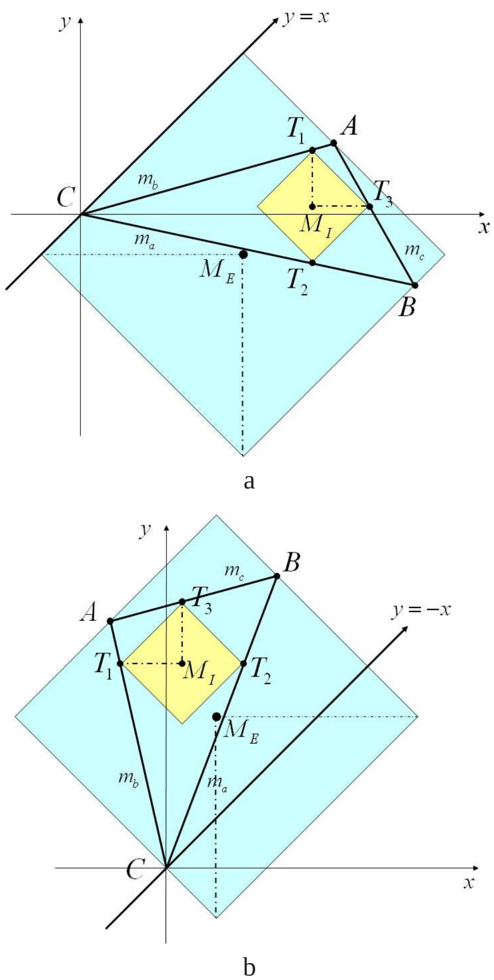


Figure 3

Suppose that $A=(x_a,y_a)$, $B=(x_b,y_b)$ are two points such that $x_b > x_a > 0$ and $y_a > 0 > y_b$ (see Figure 3a). Then, center and radius of the taxicab circumcircle of the triangle $\overset{\triangle}{ABC}$ are

$$M_E = \left(\frac{x_a+y_a}{2}, \frac{x_a-x_b+y_a+y_b}{2}\right)$$

and

$$\mathbf{R} = \frac{x_b-y_b}{2} = \frac{x_b(1-m_a)}{2}.$$

Three vertices of the taxicab incircle are on lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ with the equations $y=m_c(x-x_a)+y_a$, $y=m_a x$ and $y=m_b x$, respectively. Let $T_1$ and $T_2$ be on lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, respectively. So $T_1=(t,m_b t)$ and $T_2=(t,m_a t)$ for $t\in\mathbb{R}^+$. Thus the center and the radius of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$ are

$$M_I = \left(t, \frac{(m_a+m_b)t}{2}\right) \text{ and } \mathbf{r} = \frac{(m_b-m_a)t}{2}.$$

Since $T_3 = \left(t+\frac{(m_b-m_a)t}{2}, m_b t-\frac{(m_b-m_a)t}{2}\right)$ and point $T_3$ is on the line with the equation $y = m_c(x-x_b)+y_b$, one can easily compute that $t = \dfrac{2x_b(m_a-m_c)}{m_a+m_b-2m_c-m_c(-m_a+m_b)}$. Thus the relation between $\mathbf{R}$ and $\mathbf{r}$ is

$$\mathbf{r} = \frac{2\mathbf{R}(m_a-m_c)(m_b-m_a)}{(1-m_a)[m_a(1+m_c)+m_b(1-m_c)-2m_c]}. \tag{2.1}$$

Notice that if the triangle $\overset{\triangle}{ABC}$ located as in Figure 3a is rotated with $\frac{\pi}{2}$ about the origin, then the triangle will be in the position in Figure 3b. In this case, the relation between $\mathbf{R}$ and $\mathbf{r}$ is

$$\mathbf{r} = \frac{2\mathbf{R}(m_a-m_c)(m_b-m_a)}{(1+m_a)[-m_a(1+m_c)+m_b(1-m_c)+2m_a m_b]}. \tag{2.2}$$

In fact, if $m_a$, $m_b$ and $m_c$ are replaced with $-m_a^{-1}$, $-m_b^{-1}$ and $-m_c^{-1}$, respectively, in equation (2.1), then we have relation (2.2). By this result, the rotation with $\frac{\pi}{2}$ is an isometry of the taxicab plane. Also the triangle $\overset{\triangle}{ABC}$ such that $|m_a|<1$, $|m_b|<1$ and $|m_c|>1$ is mapped to the triangle $\overset{\triangle}{ABC}$ with $|m_a|>1$, $|m_b|>1$ and $|m_c|<1$ under the rotation with $\frac{\pi}{2}$. Thus, one can find the relations about all positions of the triangle $\overset{\triangle}{ABC}$ if $m_a$, $m_b$ and $m_c$ are replaced with $-m_a^{-1}, -m_b^{-1}$ and $-m_c^{-1}$ in the relations for the triangles with $|m_a|<1$, $|m_b|<1$ and $|m_c|>1$. So the relations about all positions of the triangle can be easily generalized. □

The next theorem gives a relation about distance between centers of the incircle and circumcircle of the triangle in terms of $\mathbf{R}$, $\mathbf{r}$ and slopes of sides of the triangle.

**Theorem 3** *Let slopes of sides $AB$, $BC$, $CA$ of a triangle $\overset{\triangle}{ABC}$ be $m_c$, $m_a$, $m_b$, respectively. Let $M_I$ and $M_E$ be centers of the taxicab incircle with radius $\mathbf{r}$ and taxicab circumcircle with radius $\mathbf{R}$ of the triangle, respectively. Then*

$$d_T(M_I,M_E) = \begin{cases} \psi(m_a,m_b,m_c), & |m_a|<1,\ |m_b|<1,\ |m_c|>1 \\ \psi(-m_a^{-1},-m_b^{-1},-m_c^{-1}), & |m_a|>1,\ |m_b|>1,\ |m_c|<1 \end{cases}$$

*where*

$$\psi(m_a,m_b,m_c) = \min\left\{ \left|\frac{m_a+m_b-2sgn(m_c)}{(m_b-m_a)}\mathbf{r}-\mathbf{R}\right|, \left|\frac{m_a+m_b-2sgn(m_c)}{(m_b-m_a)}\mathbf{r}-\left(1-\frac{2[(1-m_a)sgn(m_c)][m_b-m_c]}{[(1-m_b)sgn(m_c)][m_b-m_a]}\right)\mathbf{R}\right| \right\}.$$

**Proof.** Suppose that $\overset{\triangle}{ABC}$ is a triangle with vertices $A=(x_a,y_a)$, $B=(x_b,y_b)$ and $C=(0,0)$ such that $x_b > x_a > 0$ and $y_a > y_b$. If the triangle $\overset{\triangle}{ABC}$ has a taxicab circumcircle and incircle, then the two sides of the triangle are on gradual (*steep*) lines and third side is on a steep (*gradual*) line by Theorem 1 Therefore, there are two possible cases:

**Case I :** If $|m_c|>1$, $|m_a|<1$ and $|m_b|<1$, respectively (see Figure 2), then center and radius of the taxicab circumcircle of the triangle $\overset{\triangle}{ABC}$ are $M_E=\left(\frac{x_b\text{-}y_b}{2},\frac{x_a-x_b+y_a+y_b}{2}\right)$ and $\mathbf{R}=\frac{x_a+y_a}{2}$,respectively. Let vertices of the taxicab incircle on sides $CA$ and $CB$ be $T_1=(t,m_bt)$ and $T_2=(t,m_at)$ for $t\in\mathbb{R}^+$, respectively. Thus center and radius of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$ are $M_I=\left(t,m_at+\frac{(m_b\text{-}m_a)\,t}{2}\right)$ and $\mathbf{r}=\frac{(m_b-m_a)\,t}{2}$, where $t=\frac{2x_a\,(m_b-m_c)}{m_a+m_b-2m_c+m_c\,(m_a-m_b)}$. Thus distance between centers of the taxicab incircle and circumcircle of the triangle $\overset{\triangle}{ABC}$ is

$$\begin{aligned} d &= d_T\,(M_E,M_I)\\ &=\left|t-\frac{x_b-y_b}{2}\right|+\left|m_at+\frac{(m_b-m_a)\,t}{2}-\frac{x_a-x_b+y_a+y_b}{2}\right|\\ &=\left|\frac{2}{m_b\text{-}m_a}\mathbf{r}\text{-}\frac{(1\text{-}m_a)\,(m_b\text{-}m_c)}{(1\text{+}m_b)\,(m_c\text{-}m_a)}\mathbf{R}\right|\\ &\quad+\left|\frac{m_a\text{+}m_b}{m_b\text{-}m_a}\mathbf{r}\text{-}\mathbf{R}\text{+}\frac{(1\text{-}m_a)\,(m_b\text{-}m_c)}{(1\text{+}m_b)\,(m_c\text{-}m_a)}\mathbf{R}\right| \end{aligned}$$

Similarly the relations about all positions of the triangle can be easily generalized. □

In Euclidean geometry it is well known that intersection point of inside angle bisectors of a triangle is the center of incircle of the triangle. Each point of an angle bisector is equidistant from the sides of the angle. Since taxicab distance is different from Euclidean distance, taxicab bisector of a angle is different from Euclidean bisector of the angle. Thus the intersection point of taxicab inside angle bisectors of a triangle is, generally different from the Euclidean case. The following theorem expresses that the above property is also valid in the taxicab plane.

**Theorem 4** *The intersection point of the taxicab inside angle bisectors of a triangle with a taxicab incircle is the center of taxicab incircle of the triangle.*

**Proof.** A triangle $\overset{\triangle}{ABC}$ has a taxicab incircle if two sides of the triangle lie on gradual lines and the remaining side of its lies on a steep line by Theorem 1.

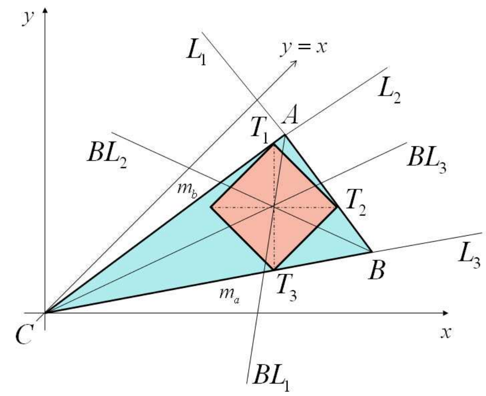


Figure 4

Equation of lines $L_1$, $L_2$ and $L_3$ containing the sides $AB$, $AC$ and $BC$ of the triangle are $y=m_c\,(x-x_a)+y_a$, $y=m_bx$ and $y=m_ax$, respectively. One can find the taxicab inside angle bisector lines $BL_1$, $BL_2$ and $BL_3$, by using definition of distance from a point to a line in the taxicab plane, as follows:

Since $BL_1=\left\{X=(x,y)\in\mathbb{R}^2 : d_T\,(X,L_1)=d_T\,(X,L_2)\right\}$; and

$$d_T\,(X,L_1)=\frac{|y+m_c\,(x_a-x)-y_a|}{\max\{|-m_c|\,,|1|\}}=\frac{y+m_c\,(x_a-x)-y_a}{m_c},$$

$$d_T\,(X,L_2)=\frac{|-m_bx+y|}{\max\{|-m_b|\,,|1|\}}=m_bx-y,$$

one can obtain the following equation:

$$\begin{aligned} d_T\,(X,L_1)=d_T\,(X,L_2)\Rightarrow \frac{y+m_c\,(x_a-x)-y_a}{m_c}&=m_bx-y\\ \Rightarrow y=\frac{m_c\,(1+m_b)}{1+m_c}x+\frac{y_a-m_cx_a}{1+m_c}&. \end{aligned}$$

Thus, equation of line $BL_1$ is $y=\frac{m_c\,(1+m_b)}{1+m_c}x+\frac{y_a-m_cx_a}{1+m_c}$.

Similarly the equations of lines $BL_2$ and $BL_3$ are

$$y=\frac{m_c\,(1-m_a)}{1-m_c}x+\frac{y_a-m_cx_a}{1-m_c}\quad\text{and}\quad y=\frac{m_a+m_b}{2}x,$$

respectively. If the system of linear equations consisting of equations of $BL_1$, $BL_2$ and $BL_3$ is solved, then it is seen that there is a unique intersection point of lines $BL_i$, $i=1,2,3$. Also this point is the center of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$. If two sides of the triangle lie on steep lines and the remaining side of its lies on a gradual line, the proof can be given, similarly. □

## 3 Taxicab incircles and circumcircles of a given triangle with at least a side on a separator line

In Euclidean geometry, it is well-known that the number of intersection points of a circle and a line is 0, 1 or 2. In the taxicab plane, this number is 0, 1, 2 or $\infty$. (see Figure 5)

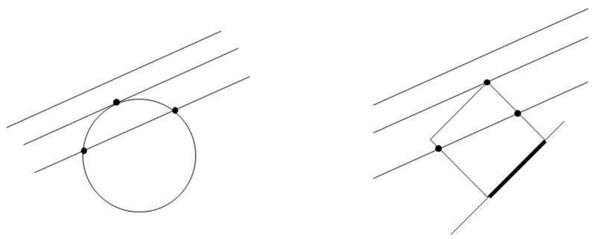

Figure 5

So far, we have considered that a incircle touches (is tangent to) each of the three sides of the triangle at only one point and circumcircle passes through all vertices of triangle. In this section we examine the incircle and circumcircle of a triangle by using following definition of concept of touching (tangent **!**). In the previous section, we have mentioned that a steep or a gradual line tangent to a taxicab circle if the taxicab circle and the steep or gradual line have common only one point. But sides of a taxicab circle always lie on separator lines. If slope of the a side of a triangle is $+1$ or $-1$ then that side coincides with a side of the taxicab incircle or circumcircle along a line segment. In this section, we consider the concept of **tangent along a line segment** if a line segment completely or partially lie on one side of the taxicab circle. This assumption increases the number of cases about existence of incircle and circumcircle for a given triangle in the taxicab plane. We only consider cases consisting infinite intersection points in this section. Because of this assumption, at least one side of the triangle must be on separator line. (see Figure 6).

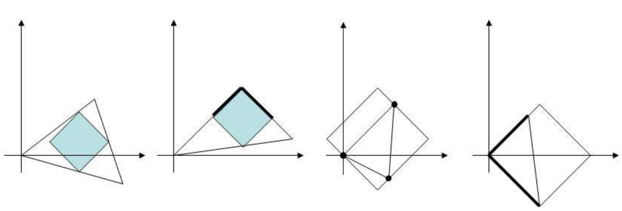

Figure 6

Next two theorems give some conditions about existence of incircle and circumcircle for a triangle with one or two sides on separator lines.

**Theorem 5** *A triangle with at least one side on a separator line has always a taxicab circumcircle and incircle.*

**Proof.** Let $\overset{\triangle}{ABC}$ be a triangle which at least one side of it is on a separator line. Then there are two main cases:

**1)** A side of the triangle $\overset{\triangle}{ABC}$ is on a separator line. (see Figure 7).

**2)** Two sides of the triangle $\overset{\triangle}{ABC}$ are on separator lines. (see Figure 8).

**Case 1:** Let a side of the triangle $\overset{\triangle}{ABC}$ be on a separator line in the taxicab plane. Suppose that vertex $C$ is at origin and $m_a$ is $-1$ (or 1), that is, side $BC$ lies on $y=-x$ (see Figure 7). A taxicab circumcircle composes of the line segments which are on lines with slopes $\mp 1$, passing through the vertices of a triangle. Since $m_a=-1$, and the triangle is completely inside the taxicab circumcircle, side with slope $-1$ of the taxicab circumcircle coincides with the side $BC$ of the triangle. So there are two subcases according to position of vertex $A$ :

**i)** Vertex $A$ is on opposite side of the circle according to side containing $BC$.

**ii)** Vertex $A$ is on neighbour side of the circle according to side containing $BC$.

Infinite circumcircles can be drawn for both of cases. In subcase I, all circumcircles have the same radius. Note that the circumcircle moves along the line segment $BC$. In subcase II, all circumcircles have different radii. Note that the center of the circumcircle moves along a horizontal or a vertical line. Distance between centers of two of circumcircles is called as **quantity of change**, and it is shown by $t$ (see Figure 7). If vertex $A$ is on opposite side of the circle to side containing $BC$, then one of $AC$ and $AB$ is on a gradual line, and the other is on steep line. If vertex $A$ is on neighbour side to side containing $BC$, then $AC$ and $AB$ are on gradual lines or steep lines.

A side of the taxicab incircle is on $BC$ since the taxicab incircle is a taxicab circle, and it is tangent to all three sides. There are two subcases according to position of side of the taxicab incircle :

**i)** Side of the taxicab incircle doesn't pass through vertex $B$ or $C$.

**ii)** Side of the taxicab incircle pass through vertex $B$ or $C$.

In subcase I, two lines with slope 1 can be drawn at end point of side on $BC$. Since the incircle is completely inside the triangle, one of side of the triangle is on a steep line, and the other is on a gradual line.

In subcase II, side of the taxicab incircle on $BC$ is tangent to two sides of the triangle. So sides of triangle are on gradual lines or steep lines (see Figure 7).

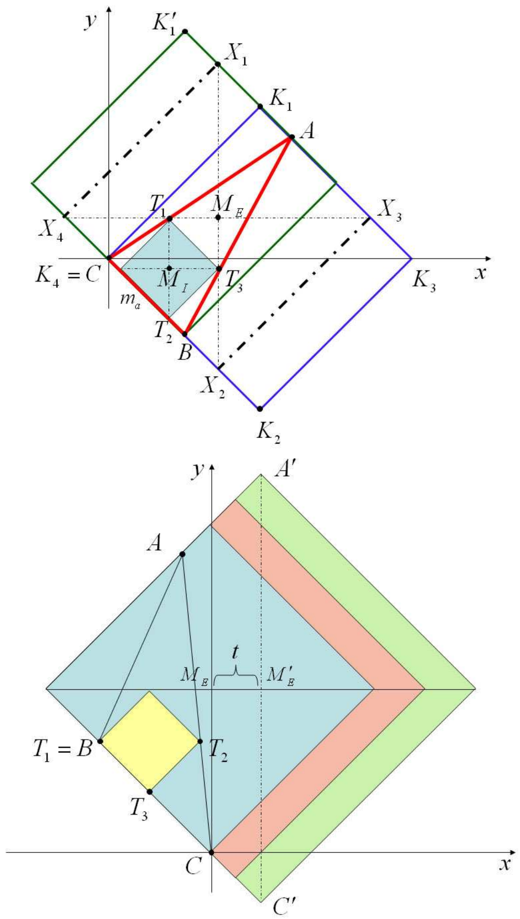


Figure 7

**Case 2:** The proof can be given as in Case 1.

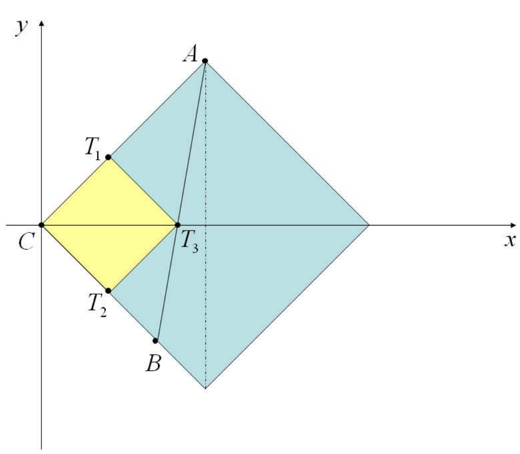


Figure 8

□

The following theorem gives the relations between radii of the incircle and circumcircle of a given triangle with one or two sides are on a separator line.

**Theorem 6** *Let $\overset{\triangle}{ABC}$ be a triangle in the taxicab plane, $m_a, m_b$ and $m_c$ denote slopes of sides $BC, AC$ and $AB$, respectively. If the triangle $\overset{\triangle}{ABC}$ has a taxicab incircle with radius* **r** *and circumcircle with radius* **R**, *and* **t** *is quantity of change, then*

$$\frac{\mathbf{r}}{2\mathbf{R}} = \begin{cases} \rho(m_a, m_b, m_c), & |m_a| \leq 1, |m_b| < 1 \text{ and } |m_c| > 1 \\ \rho\left(-m_a^{-1}, -m_b^{-1}, -m_c^{-1}\right), & |m_a| \geq 1, |m_b| > 1 \text{ and } |m_c| < 1. \end{cases}$$

*and*

$$\frac{\mathbf{r}}{2(\mathbf{R}-\mathbf{t})} = \begin{cases} \rho(m_c, m_b, m_a), & |m_a| = 1, |m_b| < 1 \text{ and } |m_c| < 1 \\ \rho\left(-m_c^{-1}, -m_b^{-1}, -m_a^{-1}\right), & |m_a| = 1, |m_b| > 1 \text{ and } |m_c| > 1 \\ \rho(m_a, m_b, m_c), & |m_a| = 1, |m_b| = 1 \text{ and } |m_c| > 1 \\ \rho\left(-m_a^{-1}, -m_b^{-1}, -m_c^{-1}\right), & |m_a| = 1, |m_b| = 1 \text{ and } |m_c| < 1 \end{cases}$$

*where* $\rho(m_a, m_b, m_c) =$

$$\frac{|m_b - m_a|\,|\delta(m_a, m_b) - m_c|}{\max\{|1+\delta(m_a,m_b)|, |1-\delta(m_a,m_b)|\}\,|\max\{m_a,m_b\}(1-m_c) + \min\{m_a,m_b\}(1+m_c) - 2m_c|}$$

*with*

$$\delta(m_a, m_b) = \begin{cases} m_a, & a > b \\ m_b, & a < b\,. \end{cases}$$

**Proof.** Let $\overset{\triangle}{ABC}$ be a triangle which at least one side of it on a separator line. One can easily see possible positions of the triangle from Theorem 5. Let $A = (x_a, y_a), B = (x_b, y_b)$ be two points such that $x_a > x_b > 0$, $y_a > 0 > y_b$, $m_a = -1$, $m_b < 1$ and $m_c > 1$.

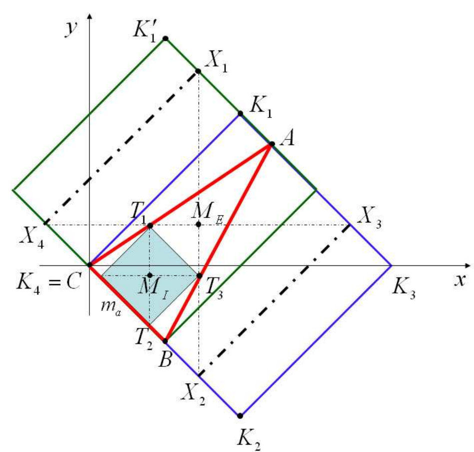


Figure 9

In this case a side of the circumcircle coincide with the side $BC$ of the triangle, and vertex $A$ is on opposite side of the circle to the side containing $BC$. So there are infinite taxicab circumcircles which of all have the same radius. $K_1 = \left(\frac{x_a+y_a}{2}, \frac{x_a+y_a}{2}\right)$ is the top vertex of the circumcircle that a vertex of it is $C$. Similarly, $K_1^{'} = (x_b, -x_b+x_a+y_a)$ is the top vertex of the circumcircle such that a vertex of it is $B$. $X_1$ denotes the top vertex of all circumcircles. So $X_1 = \lambda K_1 + (1-\lambda)K_1^{'}$ for all $\lambda \in [0,1]$ (see Figure 9). Therefore the vertices of circumcircles are

$$\begin{aligned} X_1 &= (\lambda p + x_b, -\lambda p + (-x_b + x_a + y_a)),\\ X_2 &= (\lambda p + x_b, -\lambda p - x_b),\\ X_3 &= \left(\lambda p + \frac{x_a+y_a}{2} + x_b, (1-\lambda)p\right),\\ X_4 &= ((1-\lambda)p, (1-\lambda)p), \end{aligned}$$

where $p = \frac{x_a+y_a}{2} - x_b$. Thus, center and radius of the taxicab circumcircle having vertex $X_i$ of the triangle $\overset{\triangle}{ABC}$ are

$$M_E = (\lambda p + x_b, (1-\lambda)p) \text{ and } \mathbf{R} = \frac{x_a+y_a}{2}.$$

Three vertices of the taxicab incircle are on lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ with the equations $y = m_c(x-x_a)+y_a$, $y=-x$ and $y = m_b x$, respectively. Let $T_1$ and $T_2$ be on lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ respectively. That is $T_1=(\gamma, m_b\gamma)$ and $T_2=(\gamma,-\gamma)$ for $\gamma \in \mathbb{R}^+$. Thus center and radius of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$ are

$$M_I = \left(\gamma, \frac{(m_b-1)\gamma}{2}\right) \text{ and } \mathbf{r} = \frac{(m_b+1)\gamma}{2}.$$

Since $T_3=\left(\frac{(m_b+3)\gamma}{2}, \frac{(m_b-1)\gamma}{2}\right)$ and point $T_3$ is on $y = m_c(x-x_a)+y_a$, one can find as $\gamma = \frac{2x_a(m_b-m_c)}{m_b-3m_c-1-m_bm_c}$. Thus the relation between $\mathbf{R}$ and $\mathbf{r}$ is obtained as

$$\mathbf{r} = \frac{2\mathbf{R}(m_b-m_c)}{m_b-3m_c+m_a(1+m_bm_c)}.$$

$m$ stand for slope of a line $l$. If $l$ reflects with $y-$axis, then $m$ changed to $-m$. So the above relation is valid for $m_a$=1, $m_b > -1$ and $m_c < -1$. The relation about other case for $|m_a| = 1, |m_b| > 1$ and $|m_c| < 1$ can be found when $m_a$, $m_b$, $m_c$ replace with $-m_a^{-1}, -m_b^{-1}$ and $-m_c^{-1}$ in the relations for $|m_a|$=1, $|m_b| < 1$ and $|m_c| > 1$. It is easily seen that all relations satisfy $\rho(m_a,m_b,m_c)$ for $|m_a|$=1, $|m_b| < 1, |m_c| > 1$ and $\rho\left(-m_a^{-1}, -m_b^{-1}, -m_c^{-1}\right)$ for $|m_a|$=1, $|m_b| > 1$ and $|m_c| < 1$.

Let $\overset{\triangle}{ABC}$ be a triangle such that $|m_a| = 1, |m_b| > 1$ and $|m_c| > 1$ (see Figure 10). $A = (x_a,y_a), B = (x_b,y_b)$ and $C = (0,0)$ are three points such that $x_b < x_a < 0$ and $0 < y_b < y_a$.

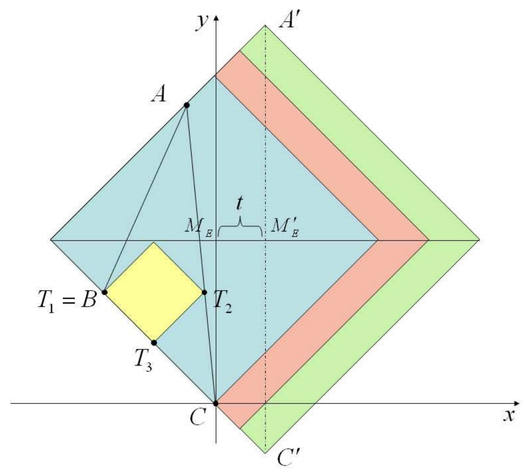


Figure 10

Therefore, center and radius of the taxicab circumcircle changing depend on parameter $\mathbf{t}$ of the triangle $\overset{\triangle}{ABC}$ are

$$M_E = \left(\mathbf{t}, \frac{-x_a+y_a}{2}\right) \text{ and } \mathbf{R} = \mathbf{t} + \frac{-x_a+y_a}{2}$$

where $\mathbf{t}$ is quantity of change. From last equation of $\mathbf{R}$, one can find that $x_a = \frac{2(\mathbf{R}-\mathbf{t})}{m_b-1}$. Three vertices of the taxicab incircle are on lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ with the equations $y = m_c(x-x_a)+y_a$, $y=-x$ and $y = m_b x$, respectively. Let $T_1$ and $T_2$ be on lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, respectively. So $T_1$=$(x_b,y_b)$ and $T_2$=$\left(\frac{y_b}{m_b}, y_b\right)$. Thus center and radius of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$ are

$$M_I = \left(x_b + \frac{x_b(m_a-m_b)}{2m_b}, y_b\right) \text{ and } \mathbf{r} = \frac{x_b(m_a-m_b)}{2m_b}.$$

Since $m_c = \frac{y_a-y_b}{x_a-x_b}$, one can find as $x_b = \frac{x_a(m_c-m_b)}{m_c-m_a}$. Thus the relation between $\mathbf{R}$ and $\mathbf{r}$ is

$$\mathbf{r} = \frac{2(\mathbf{R}-\mathbf{t})(m_b-m_c)(m_b-m_a)}{2m_b(m_b+m_a)(m_c-m_a)}.$$

This relation is also valid if one can take $-m_a, -m_b, -m_c$ instead of $m_a$, $m_b$, $m_c$, respectively. The relation about other case for $|m_a|$=1, $|m_b| < 1$ and $|m_c| < 1$ can be found if $m_a, m_b, m_c$ replace with $-m_a^{-1}, -m_b^{-1}, -m_c^{-1}$ in the relations for $|m_a| = 1, |m_b| > 1$ and $|m_c| > 1$. It is easily seen that all relations satisfy $\rho(m_a,m_b,m_c)$ for

$|m_a| = 1, |m_b| < 1, |m_c| < 1$ and $\rho\left(-m_a^{-1}, -m_b^{-1}, -m_c^{-1}\right)$ for $|m_a| = 1, |m_b| > 1$ and $|m_c| > 1$.

Let $\overset{\triangle}{ABC}$ be a triangle with $|m_a|$=1, $|m_b|$=1 and $|m_c| > 1$ (see Figure 11). $A = (x_a, y_a)$, $B = (x_b, y_b)$ and $C = (0,0)$ are three points such that $x_a > x_b > 0$ and $y_a > 0 > y_b$.

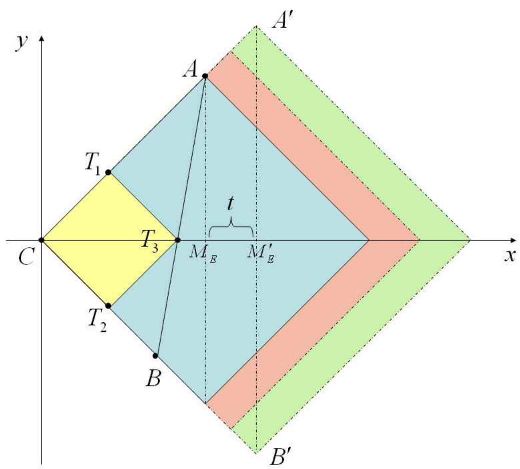


Figure 11

In this case, there are infinite taxicab circumcircles, and their centers move along a vertical or a horizontal line according to slope of side $AB$. So, the center and radius of the taxicab circumcircle changing depend on parameter $\mathbf{t}$ of the triangle $\overset{\triangle}{ABC}$ are

$$M_E = (x_a + \mathbf{t}, 0) \quad \text{and} \quad \mathbf{R} = x_a + \mathbf{t}$$

where $\mathbf{t}$ is quantity of change. From last equation about $\mathbf{R}$ is obviously $x_a$=$\mathbf{R}$−$\mathbf{t}$.
Three vertices of the taxicab incircle are on lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ with the equations $y = m_c(x - x_a) + y_a$, $y = -x$ and $y = x$, respectively. Let $T_1$ and $T_2$ be on lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, respectively. So $T_1$=$(\gamma,\gamma)$ and $T_2$=$(\gamma,-\gamma)$ for $\gamma \in \mathbb{R}^+$. Thus center and radius of the taxicab incircle of the triangle $\overset{\triangle}{ABC}$ are

$$M_I = (\gamma, 0) \quad \text{and} \quad \mathbf{r} = \gamma.$$

Since $T_3 = (2\gamma, 0)$ and point $T_3$ is on $y = m_c(x - x_a) + y_a$, one can find as $\gamma = \dfrac{x_a(m_c - m_b)}{2m_c}$. Therefore the relation among $\mathbf{R}$, $\mathbf{r}$ and $\mathbf{t}$ is

$$\mathbf{r} = \frac{(\mathbf{R} - \mathbf{t})(|m_c| - 1)}{2|m_c|}.$$

This equation is also valid if $m_c$ replace with $-m_c$.

The other case for $|m_a| = |m_b| = 1$ and $|m_c| < 1$ can be shown if $m_a$, $m_b$, $m_c$ replace with $-m_a^{-1}$, $-m_b^{-1}$, $-m_c^{-1}$ in the relations for $|m_a| = |m_b|$=1 and $|m_c| > 1$. Thus, the proof of the theorem is completed. □

The next theorem gives relation between centers of the incircle and circumcircle.

**Theorem 7** *Let $\overset{\triangle}{ABC}$ be a triangle which at least one side is on a separator line in the taxicab plane. If the triangle $\overset{\triangle}{ABC}$ has a taxicab incircle and circumcircle, then the distance between the center $M_E$ of the taxicab circumcircle with radius $\mathbf{R}$, and the center $M_I$ of the incircle with radius $\mathbf{r}$ is*

$$d_T(M_E, M_I) = \mathbf{R} - \mathbf{r}.$$

**Proof.** The proof is given by using the values of $M_E$ and $M_I$ in the Theorem 6. □

The following theorem shows that expression of the Theorem 7 is also valid for a triangle which at least one side is on a separator line.

**Theorem 8** *Let $\overset{\triangle}{ABC}$ be a triangle with at least one side is on a separator line. If $\overset{\triangle}{ABC}$ has a taxicab incircle, then the center of incircle is the intersection point of the taxicab inside angle bisectors of the triangle $\overset{\triangle}{ABC}$.*

**Proof.** Let $\overset{\triangle}{ABC}$ be a triangle such that $m_a$, $m_b$ and $m_c$ are slopes of sides $BC$,$AC$ and $AB$, respectively. If at least one side of a triangle $\overset{\triangle}{ABC}$ lies on a separator line, there are six different cases according to slopes of the sides of triangles;

**i**) $|m_a| = 1$, $|m_b| < 1$ and $|m_c| > 1$
**ii**) $|m_a| = 1$, $|m_b| > 1$ and $|m_c| < 1$
**iii**) $|m_a| = 1$, $|m_b| > 1$ and $|m_c| > 1$
**iv**) $|m_a| = 1$, $|m_b| < 1$ and $|m_c| < 1$
**v**) $|m_a| = 1$, $|m_b| = 1$ and $|m_c| > 1$
**vi**) $|m_a| = 1$, $|m_b| = 1$ and $|m_c| < 1$

**Case i)** Let $\overset{\triangle}{ABC}$ be a triangle with $|m_a|$=1, $|m_b| < 1$ and $|m_c| > 1$. Then $A$=$(x_a, y_a)$, $B = (x_b, y_b)$ and $C = (0,0)$ are three points such that $x_a > x_b > 0$ and $y_b > 0 > y_a$ (see Figure 12).

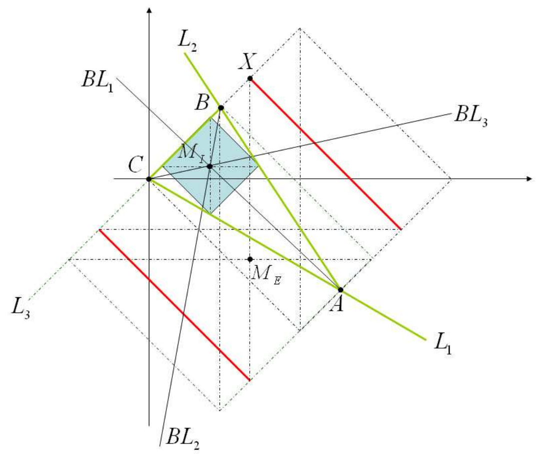


Figure 12

$L_1$, $L_2$ and $L_3$ denote the lines which lie on $AC$, $AB$ and $BC$, respectively. These lines $L_1$, $L_2$ and $L_3$ are $y = m_b x$, $y = m_c(x - x_a) + y_a$, and $y$=$x$, respectively. Using definition of distance from a point to a line in the taxicab plane, the taxicab inside angle bisector lines $BL_1, BL_2$ and $BL_3$ are found as follow:

Since $BL_1 = \{X = (x,y) \in \mathbb{R}^2 : d_T(X, L_1) = d_T(X, L_2)\}$; and

$$d_T(X, L_1) = \frac{|y - m_b x|}{\max\{|-m_b|, |1|\}} = y - m_b x$$

$$d_T(X, L_2) = \frac{|y + m_c(x_a - x) - y_a|}{\max\{|-m_c|, |1|\}} = \frac{y + m_c(x_a - x) - y_a}{m_c},$$

one can obtain the following equation:

$$y = \frac{m_c(m_b - 1)}{m_c - 1} x + \frac{x_a(m_c - m_b)}{m_c - 1}.$$

Similarly the equations of lines $BL_2$ and $BL_3$ are

$$y = \frac{2m_c}{1 + m_c} x + \frac{x_a(m_b - m_c)}{1 + m_c} \quad \text{and} \quad y = \frac{1 + m_b}{2} x$$

respectively. If the system of linear equations consisting of equations of $BL_1$, $BL_2$ and $BL_3$ is solved, then the solution is found as

$$\left( \frac{2x_a(m_b - m_c)}{m_b - 3m_c - 1 - m_b m_c}, \frac{x_a(m_b - m_c)(m_b - 1)}{m_b - 3m_c - 1 - m_b m_c} \right).$$

That is, lines $BL_1, BL_2$ and $BL_3$ pass through the same point. Also this point is the center of taxicab incircle of the triangle $\overset{\triangle}{ABC}$.

The proof of the other cases can be given by similar ways.

□

## References


[1] H. B. Çolakoğlu, R. Kaya, A synthetic approach to the taxicab circles, *Applied Sciences* **9** (2007), 67–77.

[2] H. B. Çolakoğlu, R. Kaya, Regular Polygons in the Taxicab Plane, *KoG* **12** (2008), 27–33.

[3] B. Divjak, Notes on Taxicab Geometry, *KoG* **5** (2000/01), 5–9.

[4] S. Ekmekçi, *Taksi Çemberleriyle İlgili Özellikler*, ESOGÜ, PhD, 2001.

[5] O. Gelisgen,R. Kaya, The Taxicab Space Group, *Acta Mathematica Hungarica* **122 (1–2)** (2009), 187–200.

[6] R. Kaya, Z. Akça, I. Gunaltili, M. Ozcan, General Equation for Taxicab Conics and Their Classification, *Mitt. Math. Ges. Hamburg* **19** (2000), 135–148.

[7] E. F. Krause, *Taxicab Geometry*, Addison-Wesley, Menlo Park, 1975.

[8] D. J. Schattschneider, The Taxicab Group, *Amer. Math. Monthly* **91** (1984), 423–428.

[9] S. S. So, Recent Developments in Taxicab Geometry, *Cubo Mat. Educ.* **4 (2)** (2002), 79–96.

[10] K. Thompson, Taxicab Triangle Incircles and Circumcircles, to appear in *The Pi Mu Epsilon Journal* (see author's home page).



**Temel Ermiş**
e-mail: termis@ogu.edu.tr

**Özcan Gelişgen**
e-mail: gelisgen@ogu.edu.tr

**Rustem Kaya**
e-mail: rkaya@ogu.edu.tr

Department of Mathematics and Computer Sciences
Faculty of Science and Arts
University of Eskişehir Osmangazi
26480 Eskişehir-Türkiye