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ABSTRACT

In the present paper, we study a set $\mathbf{T} = \{\mathcal{T}_{(r,d)} : d \in \mathbb{R}\}$ of the certain one-parameter families of triangles. The traces of some triangle points within the set are analyzed and described.

Key words: tangential triangle, hyperosculating circle, pencil of conics

MSC 2000: 51N20, 51N15

O nekim familijama trokuta

SAŽETAK

U ovom radu proučava se skup $\mathbf{T} = \{\mathcal{T}_{(r,d)} : d \in \mathbb{R}\}$ specijalnih jednoparametarskih familija trokuta. Analizirat će se i opisati krivulje mjesta nekih točaka trokuta unutar danog skupa.

Ključne riječi: tangencijalni trokut, hiperoskulacijska kružnica, pramen konika

1 Introduction

The study of triangles and their families even nowadays attracts many geometers. Various problems in connection to the triangles and their families are studied in [1], [2], [3]. Nowadays, the use of modern geometry softwares (*GeoGebra*, *Cinderella*, *The Geometer's Sketchpad* ...) enables the dynamic geometric constructions which, in general, facilitate the analysis of the movement of the triangles, or some triangle points, within the specified system.

When it comes to the families of triangles, there are many ways to associate triangles with each other. One such is defined in this paper generalizing the concept of the tangential triangles.

Generally, given a triangle $\Delta A_1A_2A_3$, the triangle $\Delta T_1T_2T_3$ is said to be the tangential triangle if it is formed by the lines tangent to the circumcircle of $\Delta A_1A_2A_3$ at its vertices. Hereafter, we will use the term a tangential triangle in connection to a circle. Hence, a triangle will be called a tangential triangle to a given circle $\mathcal{C}$ iff it is formed by the lines tangent to $\mathcal{C}$. Naturally, given the circle, there are ∞^3 such triangles. By adding some more elements into the specified family, a one–parameter family of triangles is defined in this paper. Furthermore, the connection between the added elements and the given circle–tangent configuration is studied.

Denoting by $PG(2,\mathbb{R})$ the projective closure of $\mathbb{R}^2$, we always assume that $PG(2,\mathbb{R})$ is embedded into its complexification $PG(2,\mathbb{R} \subset \mathbb{C})$. Choosing the line at infinity f as $x_3 = 0$, the interchange between homogeneous and Cartesian coordinates in $\mathbb{R}^2$ is realized.

1.1 The family of triangles $\mathcal{T}_{(r,d)}$

Let a circle $\Phi(S,r)$ with radius r and one of its tangents t be given. For $d \in \mathbb{R}$ a one–parameter family of triangles $\mathcal{T}_{(r,d)}$ is defined such that a triangle $\Delta ABC \in \mathcal{T}_{(r,d)}$ iff it satisfies the following two properties:

F1) a triangle ΔABC is tangential to the given circle $\Phi(S,r)$,

F2) $A,B \in t$ and $d = \pm|\overrightarrow{AB}|$.

Hence, as a segment of the fixed length d moves along the tangent t, a triangle ΔABC traversers a one–parameter family $\mathcal{T}_{(r,d)}$. This motion is continuous, but not rigid for the remaining two triangle sides which are therefore continuously changing.

Furthermore, by varying d a set $\mathbf{T} = \{\mathcal{T}_{(r,d)} : d \in \mathbb{R}\}$ of the triangle families is obtained in connection to the given circle and its fixed tangent.

Fig. 1 shows two triangles ΔABC and ΔBDE of the family $\mathcal{T}_{(r,d)}$ obtained for some d. The circle Φ is its ex– and incircle, and they share the side and one vertex lying on it. Altohough not necessary, it is convenient to introduce an orientation onto the tangent t to ensure that the position of only one vertex uniquely determines the remaining two vertices.

Obviously, given a configuration of a circle Φ and tangent t, the loci of many triangle points within two families $\mathcal{T}_{(r,d)}, \mathcal{T}_{(r,-d)} \in \mathbf{T}$ will coincide. This follows directly from the geometric construction, since the loci of the triangle centers within $\mathcal{T}_{(r,d)}$ are symmetric with respect to the circle diameter perpendicular to the given tangent t, as it will be shown later.

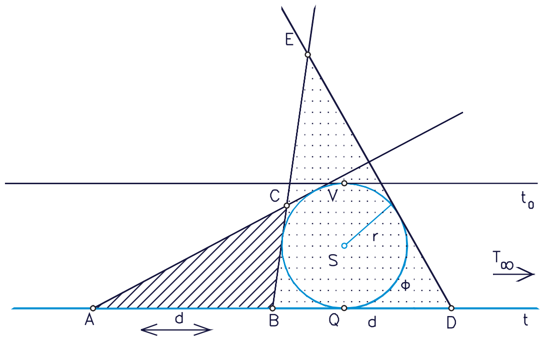


Figure 1

Before we continue with the traces of some points within triangle, let us focus onto some special triangles within a family $\mathcal{T}_{(r,d)} \in \mathbf{T}$. Following the similar approach as in [1], the position of those triangles with respect to Φ and t will play an important role in the determination of the traces of the triangle points within the family. The special triangles are degenerated triangle with one of the vertices lying on the given circle Φ or at the infinity. Hereafter, let Q be the contact point of Φ and t, and let t_1 be the tangent of Φ parallel to t. We distinguish the following three types of the special triangles within each family $\mathcal{T}_{(r,d)}$:

S1) If one of the vertices lying on t coincides with Q, the triangle degenerates into the segment of the length d. In each family there are two such triangles.

S2) Furthermore, it is possible that two triangle sides, having only one of the vertices on t, are parallel. Then their intersection point lies at infinity and determines the third vertex. The number of such triangles within each family depends on the relation between the given length d and the circle diameter $2r$.
For if $|d| > 2r$, we have two such triangles, if $|d| = 2r$ only one such triangle is possible, and for $|d| < 2r$ there are no real triangles satisfying this condition.

S3) If one of the points lying on t converges to the point at infinity T_∞ of the line t, then the intersection points of the tangents drawn to Φ converges to the point V, the contact point of the circle Φ and its tangent t_1 parallel to t.

Hence, in the first two cases we have the classes of special triangles obtained by varying d within the set $\mathbf{T}$. Interestingly, in the case S3 only one such triangle remains fixed within all families $\mathcal{T}_{(r,d)}$.

The aim of this paper is to examine the traces of some triangle points within the specified tangential families of triangles. The results will be presented analytically and their analysis will be provided by the use of the three classes of degenerated triangles. Furthermore, the connection between the given elements and obtained curves is studied.

The constructions in this work are done by *The Geometer's Sketchpad* and the computations with *Mathematica*.

In the section 2 it will be shown that the specified tangential families, which are subject to the present paper, belong to the special poristic families of triangles, [2]. The third triangle vertex lies on the conic which hyperosculates the given cirlce Φ at the point V. Since the triangle vertices are running on a singular cubic curve while the three lines spanned by the respective vertices envelope a circle, the triangles within a family $\mathcal{T}_{(r,d)}$ are triangles with a certain circumscribed degenerated cubic curve (a conic section and line t) tangential to the given circle Φ.

2 The locus of C

Naturally, we will start with the locus of the third triangle vertex, not lying on the tangent t. For $d \in \mathbb{R}$, let a familiy $\mathcal{T}_{(r,d)} \in \mathbf{T}$ be given.

Without loss of generality we can assume that the circle Φ and the tangent t are given by the equations

$$\Phi: \quad x^2 + y^2 = r^2, \qquad t: \quad y = -r. \tag{1}$$

Let $\Delta ABC \in \mathcal{T}_{(r,d)}$. Aiming at parametrization of the third vertex C, not lying on the given tangent t, let us denote the vertex A of ΔABC by A_λ given by $A_\lambda(\lambda, -r)$, $\lambda \in \mathbb{R}$.

The third vertex C_λ is uniquely determined as the intersection point of the tangents drawn from A_λ and $B_\lambda = (\lambda + d, -r)$ to the given circle Φ. Its homogeneous coordinates depend on the the parameter λ and reads

$$C_\lambda = \Big(r^2(2\lambda + d) : r\big(\lambda(\lambda + d) - r^2\big) : \lambda(\lambda + d) + r^2\Big). \tag{2}$$

Thus, the one–parameter family of triangles $\mathcal{T}_{(r,d)}$ is described with λ as well, i.e. $\mathcal{T}_{(r,d)} = \{\Delta_\lambda ABC : \lambda \in \mathbb{R}\}$.

Our first goal is to describe the locus curve Γ_d of the vertex C which obviously lies on some conic. Note that Γ_d is symmetric with respect to the circle diameter perpendicular to t. For verifying that, let (T_∞) be the pencil of lines with vertex T_∞, $T_\infty \in t$. Every line line $q_i \in (T_\infty)$ carries at most two triangle vertices of the family $\mathcal{T}_{(r,d)}$. For, if $C_\lambda \in q_i$, such that $\Delta_\lambda ABC \in \mathcal{T}_{(r,d)}$, then the triangle $\Delta_{-\lambda-d}ABC$ also lies in $\mathcal{T}_{(r,d)}$ having $C_{-\lambda-d} \in q_i$. Namely, if $\alpha_i := \angle A_\lambda C_\lambda B_\lambda$, then the locus of points where the circle Φ is seen under the same angle α_i is the concentric circle $\Phi_i(S, |SC|)$. The intersection points of q_i and Φ_i are the vertices C_λ and $C_{-\lambda-d}$. Furthermore, $\Delta_\lambda ABC \cong \Delta_{-\lambda-d}BCA$ and they are symmetric with respect to the axis $o_t \ni S$, $o_t \perp t$ (see Fig. 2).

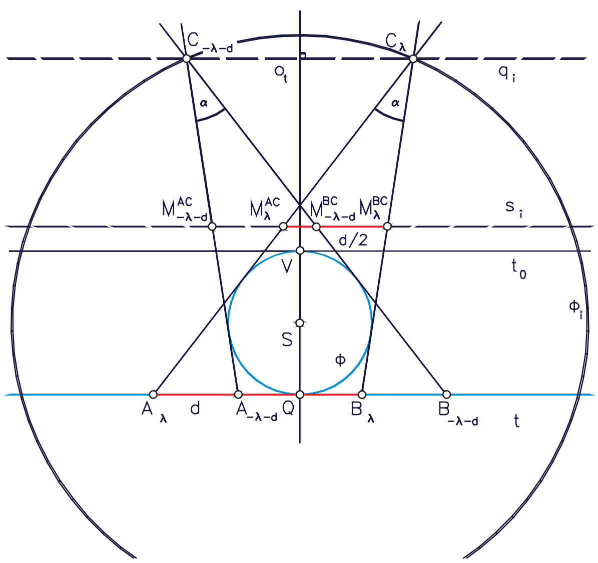


Figure 2

For $\lambda = -\frac{d}{2}$ the vertex C_λ lies on o_t, the both intersection points of the line $q_i \in (T_\infty)$ and Φ_i coincide and the line q_i is the tangent to the conic Γ_d with the vertex $C_{-\frac{d}{2}}$. The associated triangle $\Delta_{-\frac{d}{2}}ABC \in \mathcal{T}_{(r,d)}$ is an isosceles triangle. Especially, for $d = 2r$ such an isosceles triangle degenerates and one vertex coincides with the ideal point of the axis of symmetry o_t.

Before we derive an implicit equation of this curve let us determine the coordinates of the vertices of the special triangles given with S1–S3. Hence, we get for $\lambda \in \{0, -d\}$ the vertices $C_0 = (d, -r)$ and $C_{-d} = (-d, -r)$ lying on the tangent t. From (2) the coordinates of the vertices C_λ lying at infinity are given with $\lambda = \lambda_1$ or $\lambda = \lambda_2$, where

$$\lambda_{1,2} := \frac{-d \mp \tau}{2}, \quad \tau := \sqrt{d^2 - 4r^2}. \tag{3}$$

Thus, one distinguishes three cases depending on the number of triangles $\Delta ABC \in \mathcal{T}_{(r,d)}$ with the vertex C at infinity. They all depend on the relation between the circle diameter and given length d. Therefore, for the vertex C of the triangle $\Delta ABC \in \mathcal{T}_{(r,d)}$, we have:

i) if $d \geq 2r$, the two vertices $C_{\lambda_{1,2}} = (\tau : \mp 2r : 0)$ are lying on f and Γ_d is a hyperbola;

ii) if $d = 2r$, only one such vertex $C_{\lambda_1} = C_{\lambda_2} = (0:1:0)$ lies on f and Γ_d is a parabola;

iii) if $d < 2r$ there are no real vertices on f and Γ_d is an ellipse.

When $\lambda \to \pm\infty$, as a limiting point of (2) we get $C \to C_\infty = V = (0, r) \in \Phi \cap t_1$, and the circle tangent t_1 is given with

$$t_1: \quad y = r, \quad t_1 || t. \tag{4}$$

The third case S3 determines one of the vertices V of the conic Γ_d lying on the axis o_t. The line t_1 given by (4) is then the common tangent of the conic Γ_d and given circle Φ. It remains fixed for all tangential families of triangles within the set $\mathbf{T}$. For $d \neq 0$, the point V is the only common point of the conics Φ and Γ_d. A one–parameter family of conics $\mathbf{P} = \{\Gamma_d : d \in \mathbb{R}\}$, obtained by varying d, belongs to the pencil of hyperosculating conics. We can see that this pencil is uniquely determined with two of its conics, the given circle Φ and the only degenerated conic within the pencil, two coinciding lines t_1.

Similar observations can be obtained by deriving the implicit equation of the required locus of the vertex $C(x,y)$ of the triangle $\Delta ABC \in \mathcal{T}_{(r,d)}$ from (2). It turns out to be a conic Γ_d given by

$$\Gamma_d : d^2(y-r)^2 - 4r^2(x^2+y^2-r^2) = 0. \tag{5}$$

For a given circle Φ and tangent t, all three types of hyperosculating conics Γ_d within the one–parameter family $\mathbf{P}$ obtained by varying d are shown in Fig. 3.

Thus we have:

Theorem 1 *Assume we are given a circle $\Phi(S,r)$, one of its tangents t, and a segment AB of length $d \in \mathbb{R}$ lying on t.*

The locus Γ_d of the vertex C such that $\Delta ABC \in \mathcal{T}_{(r,d)}$ is contained in the pencil of conics hyperosculating Φ at V, where $V \notin t$ and $o_t := VS \perp t$ is the focal axis of Γ_d. The length d serves as a parameter within the pencil.
The conic Γ_d is an ellipse, a parabola, or a hyperbola iff $|d| < 2r$, $|d| = 2r$, or $|d| > 2r$.

Let us conclude this section with another formulation of Theorem 1 which shows an interesting loci property of conic:

Proposition 1 *For given circle Φ the set of all points X such that the tangents drawn to Φ cut at one of its fixed tangent segments of equal length is a conic $\mathcal{C}$ that hyperosculates Φ.*

3 Some locus curves

As a result of the similarity of the triangles $\Delta_\lambda ABC$ and $\Delta_{-\lambda-d}ABC$ within the family $\mathcal{T}_{(r,d)} = \{\Delta ABC : \lambda \in \mathbb{R}\} \in \mathbf{T}$, the traces of the triangle centers lie on the symmetric curves with respect to the axis of symmetry o_t perpendicular to t. Many triangle points lie on the symmetric curves as well but their axis of symmetry may not coincide with o_t.

In what follows the traces of one such triangle point (the side midpoint) is analyzed, as well as the trace of one triangle center, the triangle circumcenter.

3.1 The midpoint M_{AC}

Let $d \in \mathbb{R}$ and a tangential family $\mathcal{T}_{(r,d)} \in \mathbf{T}$ be given. For $\Delta ABC \in \mathcal{T}_{(r,d)}$, let the vertices A and B lie onto t. The midpoints of the variable sides AC and BC trace the corresponding curves Ψ_d^{AC} and Ψ_d^{BC}. Since $\Gamma_d \equiv \Gamma_{-d}$ and $\Psi_d^{AC} \equiv \Psi_{-d}^{BC}$, the curves Ψ_d^{AC} and Ψ_d^{BC} are symmetric with respect to the axis o_t. Thus, in what follows only the the locus of the midpoints of the side variable side AC of the triangle ΔABC is given.

If the circle Φ and tangent t are given with (1), starting with the special triangles within the family $\mathcal{T}_{(r,d)}$ we can easily calculate the midpoints $M_{-d}^{AC} = (-d,-r)$ and $M_0^{AC} = (d/2,-r)$ in the case S1, the midpoints $M_{\lambda_{1,2}} = C_{\lambda_{1,2}} = (\tau : \mp 2r : 0)$ in the case S2, and the midpoint $M_\infty = T_\infty = (1:0:0)$ lying at infinity and obtained as the limiting point in the case S3.

Obviously, Ψ_d^{AC} is a symmetric cubic. For each line $q_i \in (T_\infty)$ let an involution in the pencil of lines (T_∞) having the lines t and q_i for its double lines be given. Then there is the line $s_i \in (T_\infty)$ associated to the line f at infinity such that the lines $(t,q_i;f,s_i)$ are harmonically related (see Fig. 2). Furthermore, in the previous section to the line q_i of the pencil (T_∞) two triangles $\Delta_\lambda ABC$ and $\Delta_{-\lambda-d}ABC$ are associated, if the vertices $C_\lambda, C_{-\lambda-d}$ are lying on it. Since the midpoints M_λ^{AC} and $M_{-\lambda-d}^{AC}$ are also symmetric with respect to the axis o_t, the midpoints $M_{-\lambda-d}^{AC}$ and $M_{-\lambda-d}^{BC}$ lying on $s_i \in (T_\infty)$ are at the distance $\frac{d}{2}$, the midsegment length of all tangential triangles within $\mathcal{T}_{(r,d)}$. Thus, the midpoints M_λ^{AC} and $M_{-\lambda-d}^{AC}$ are symmetric with respect to the axis o_M parallel to o_t and $d(o_t,o_M) = \frac{d}{4}$.

The obtained curve has a vertex lying on axis o_M associated to the isosceles triangle when $\lambda = -\frac{d}{2}$ and it coordiantes are given with $M_{-\frac{d}{2}} = \left(-\frac{d}{4}, \frac{4r^3}{\tau^2}\right) \in o_M$. The other intersection point with the axis o_M determines the double point of the midpoint trace and reads $M_{\lambda_{3,4}} = \left(-\frac{d}{4}, \frac{r}{2}\right)$ for $\lambda_{3,4} = \frac{-d\pm\sqrt{d^2-12r^2}}{2}$. Therefore, the cubic Ψ_{AC} has a cusp at $M_{\lambda_{3,4}}$ exactly if $d^2 = 12r^2$. If $d^2 < 12r^2$, $M_{\lambda_{3,4}}$ is an isolated double point.

Furthermore, since the midpoint M_∞ is the limiting point in S3 for all $d \in \mathbb{R}$ as $\lambda \to \pm\infty$, the line $t_0 \in (T_\infty)$ passing through M_∞ is the common asymptote for the curves Ψ_d^{AC} of the one–parameter family $\mathbf{G}^{AC} = \{\Psi_d^{AC} : d \in \mathbb{R}\}$ obtained by varying d. Since $(t,t_1;f,t_0)$ are harmonically related, it follows that t_0 passes through the circle center S.

Thus, we have shown:

Theorem 2 *The midpoint of the variable triangle side AC such that $\Delta ABC \in \mathcal{T}_{(r,d)}$ lies on a rational symmetric cubic Ψ_d^{AC} asymptotic to a line t_o which is parallel to the given tangent t and passes through the circle center S.*
It has a cusp at the double point if $d^2 = 12r^2$, a node if $d^2 > 12r^2$ and an isolated double point if $d^2 < 12r^2$.

An elementary computation using the equations of the triangle sides yields the homogenous coordinates of the triangle midpoints M_λ^{AC} as

$$M_\lambda^{AC}\Big(\lambda^2(d+\lambda)+(d+3\lambda)r^2 : -2r^3 : 2\big(\lambda(\lambda+d)+r^2\big)\Big) \tag{6}$$

if the circle Φ and the tangent t are given by (1). The equation of the cubic parameterized by (6) in terms of Cartesian coordinates reads

$$\Psi_d^{AC} : y^3(d^2-4r^2) = r\left(d^2y^2 + r\Big(2x(d+2x)-3r^2\Big)y + r^4\right). \tag{7}$$

The triangle family can be used for the parametrization of the locus and also for solving some complex problems whose computation cannot be done in an acceptable amount of time using computers. For example, the determination of the intersection points of the cubic Ψ^d and a circle Φ follows easily using the properties of the isosceles triangles. The midpoint M_λ^{AC} lies on the given circle Φ precisely when it coincides with one of the point of tangency of the inscribed (or escribed) circle Φ of the triangle ΔABC lying on the line AC. This is the case when $\det(S, M_\lambda^{AC}, B_\lambda) = 0$ where S is the center of Φ, i.e. when λ satisfies the following equality $\lambda^3 \cdot r + \lambda^2 \cdot dr + \lambda \cdot r^3 - dr^3 = 0$. Hence, the cubic Ψ_d^{AC} touches the given circle Φ once, or three times (see Fig. 3).

3.2 The circumcenter O

Again, for $d \in \mathbb{R}$, let a family $\mathcal{T}_{(r,d)} \in \mathbf{T}$ be given. Furthermore, let Υ_d be the locus of the circumcenter O_λ^d of a triangle $\Delta_\lambda ABC \in \mathcal{T}_{(r,d)}$. Since the circumcenter O_λ^d can be calculated as the intersection point of the perpendicular bisectors of the sides AC and AB, if the circle Φ and tangent t are given with (1), it yields

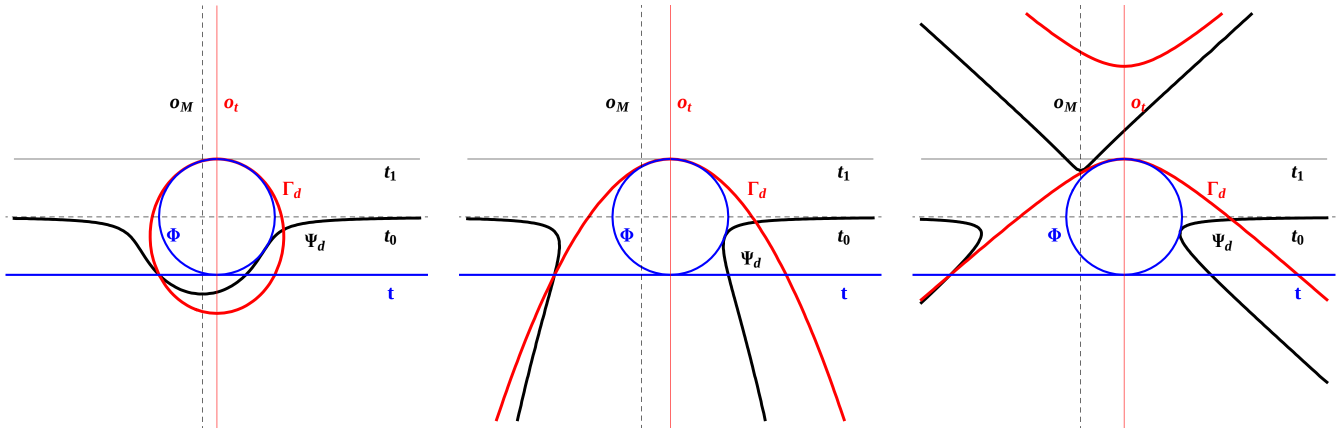
$$\begin{aligned} O_\lambda \;=\; & \Big(2r(d+2\lambda)\big(\lambda(\lambda+d)+r^2\big) : \lambda^2(\lambda+d)^2 - \\ & - r^2\Big((\lambda+d)^2+\lambda^2\Big) - 3r^4 : 4r\lambda\big((\lambda+d)+r^2\big) \end{aligned} \tag{8}$$

which parameterizes the rational symmetric quartic Υ_d with equation

$$\Upsilon_d : \big(4x^2 - 8r\cdot y - (d^2+4r^2)\big)^2 = 16r^2\big(d^2+4(r+y)^2\big) \tag{9}$$

Similar observations can be provided by the use of the tangential family $\mathcal{T}_{(r,d)}$ as well as the further analysis of the obtained curve.

Using the special triangles within the family we get the following. To the degenerated triangles $\Delta_{-d}ABC$ and $\Delta_0 ABC$ in S1 the circumcenters $O_0 = \left(\frac{d}{2}, -\frac{1}{4r}(d^2+3r^2)\right)$ and $O_{-d} = \left(-\frac{d}{2}, -\frac{1}{4r}(d^2+3r^2)\right)$ are associated lying at the perpendicular bisectors of the segment AB. They are symmetric with respect to the axis of symmetry o_t.



$d < 2r$ $\quad$ $d = 2r$ $\quad$ $d > 2r$

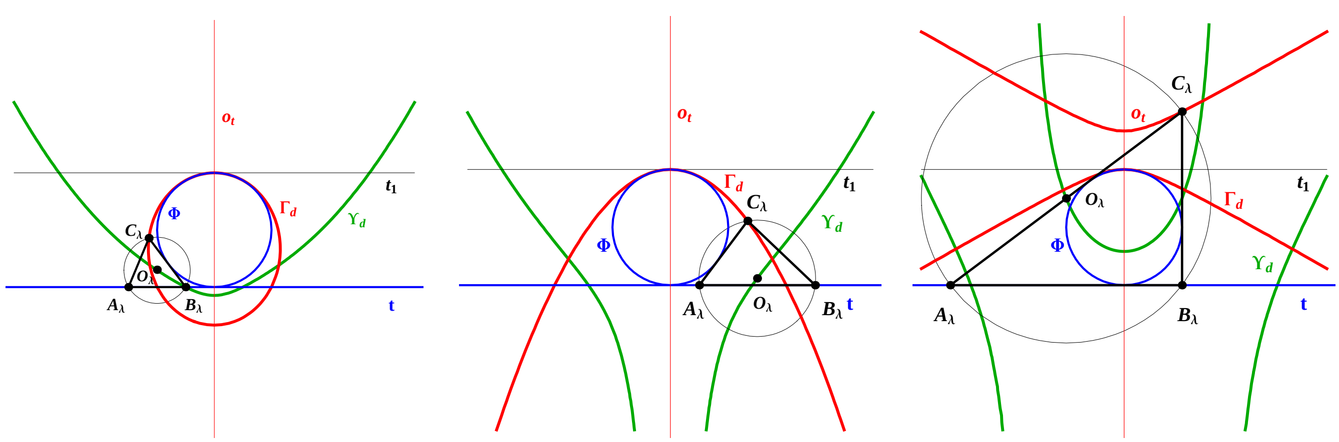
Figure 3

In S2, we get the circumcenters $O_{\lambda_1} = O_{\lambda_2} = O_\infty = (0:1:0)$, where λ_1 and λ_2 are given with (3), coinciding with the ideal point O_∞ of o_t. It is the cuspidal point of a quartic if $d^2 = 4r^2$, the nodal point if $d^2 > 4r^2$ and the isolated point if $d^2 < 4r^2$. In the case S3, as λ converges to the infinity, the circumcenter converges to the point O_∞ as well. Thus, it is actually the triple point of Υ_d belonging also to the circumcenter of the special triangle $\Delta_\infty ABC$ at which the line f touches the obtained symmetric quartic.

We can state:

Theorem 3 *The circumcenter O^d of the triangle $\Delta ABC \in \mathcal{T}_{(r,d)}$ lies on a rational symmetric quartic Υ_d with a triple point at infinity. It is the cuspidal point if $d^2 = 4r^2$, the nodal point if $d^2 > 4r^2$ and the isolated point if $d^2 < 4r^2$. One of the tangents at the quartic triple point is the infinity line, while the other two are perpendicular to the given tangent t.*

Fig. 4 displays some conics Γ_d of the one–parameter family $\mathbf{P} = \{\Gamma_d : d \in \mathbb{R}\}$ and associated quartics Υ_d belonging to the one–parameter family $\mathbf{O} = \{\Upsilon_d : d \in \mathbb{R}\}$. Those curve appear as traces of a circumcenter and vertex of a tangential triangle ΔABC within the family $\mathcal{T}_{(r,d)}$ associated to the given circle Φ and its tangent t for some real number d.



$d < 2r$ $\quad$ $d = 2r$ $\quad$ $d > 2r$

Figure 4

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