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Universal Hyperbolic Geometry IV: Sydpoints and Twin Circumcircles

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ABSTRACT

We introduce the new notion of *sydpoints* into projective triangle geometry with respect to a general bilinear form. These are analogs of midpoints, and allow us to extend hyperbolic triangle geometry to non-classical triangles with points inside and outside of the null conic. Surprising analogs of circumcircles may be defined, involving the appearance of pairs of *twin circles*, yielding in general eight circles with interesting intersection properties.

Key words: universal hyperbolic geometry, triangle geometry, projective geometry, bilinear form, sydpoints, twin circumcircles

MSC 2000: 51M10, 14N99, 51E99

Univerzalna hiperbolička geometrija IV: sidtočke i kružnice blizanke

SAŽETAK

Uvodimo novi pojam sidtočaka u projektivnu geometriju trokuta s obzirom na opću bilinearnu formu. One su analogoni polovišta i dopuštaju nam proširiti hiperboličku geometriju trokuta ka neklasičnim trokutima s točkama unutar i van apsolutne konike. Mogu se definirati neočekivani analogoni opisanih kružnica koji uključuju pojavljivanje kružnica blizanki što vodi ka osam kružnica sa zanimljivim svojstvima presjeka.

Ključne riječi: univerzalna hiperbolička geometrija, geometrija trokuta, projektivna geometrija, bilinearna forma, sidtočka, kružnice blizanke

1 Introduction

In this paper we continue a study of hyperbolic triangle geometry, parallel to, but with different features to the Euclidean case laid out in [5] and [6], and in a related but different direction from [9], [10] and [11], using the framework of Universal hyperbolic geometry (UHG), developed by Wildberger in [13], [14], [15] and [16]. We study the new notion of *sydpoints* s of a side $\overline{ab}$—this is analogous and somewhat complementary to the more familiar notion of *midpoints* m; the related idea of *twin circumcircles* of a triangle; and introduce *circumlinear coordinates* to build up the Circumcenter hierarchy of a triangle, treating midpoints and sydpoints uniformly.

In [16] we saw that if each of the three sides of a triangle (in UHG) has midpoints m, then these six points lie three at a time on four circumlines C, whose duals are the four *circumcenters* c. These are the centers of the four *circumcircles* which pass through the three points of the triangle.

This is shown for a classical triangle in Figure 1, where the larger blue circle is the *null circle* defining the metrical structure, together with the *midlines* M—traditionally called perpendicular bisectors. While the red circumcircle is a classical circle in the Cayley Beltrami Klein model of hyperbolic geometry, the other three are usually described as *curves of constant width*, but for us they are *all just circles*. This is the start of the Circumcenter hierarchy in UHG.

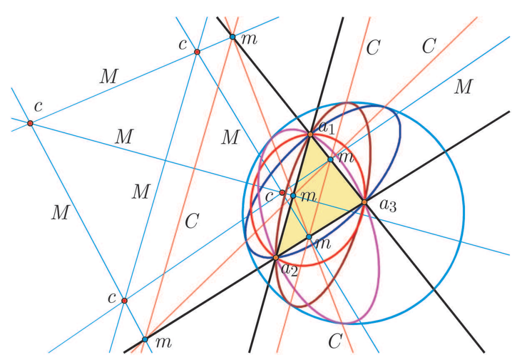

Figure 1: *Midpoints, Midlines, Circumlines, Circumcenters and Circumcircles*

Remarkably, much of this extends also to triangles with points both interior and exterior to the null circle, but we also find new phenomenon relating to circumcircles, that suggest a reconsideration of the classical case above.

The fundamental metrical notion between points in UHG is the *quadrance* q, and a midpoint of $\overline{ab}$ is a point m on ab satisfying $q(a,m) = q(b,m)$. Our key new concept is the following: a **sydpoint** of $\overline{ab}$ is a point s on ab satisfying

$$q(a,s) = -q(b,s).$$

While the existence of midpoints is equivalent to $1 - q(a,b)$ being a square in the field, the existence of sydpoints is equivalent to $q(a,b) - 1$ being a square. As with midpoints, if sydpoints exist there are generally two of them.

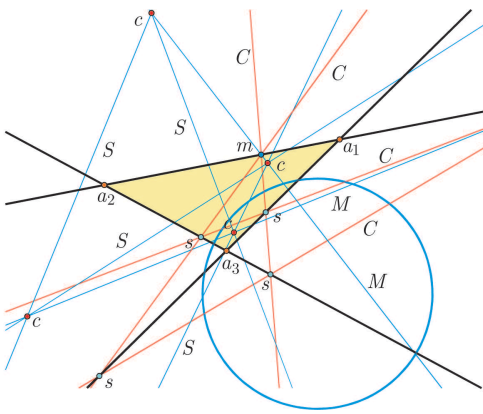

Figure 2: *A non-classical triangle with both midpoints and sydpoints*

In Figure 2, the non-classical triangle $\overline{a_1a_2a_3}$ has one side $\overline{a_1a_2}$ with midpoints m whose duals are *midlines* M, and two sides $\overline{a_1a_3}$ and $\overline{a_2a_3}$ with sydpoints s whose duals are *sydlines* S. The six midpoints and sydpoints lie three at a time on four *circumlines* C, whose duals are the four *circumcenters* c. The connection between these new circumcenters and the idea of circumcircles is particularly interesting, since in this case it is impossible to find *any* circles which pass through all three points of the triangle $\overline{a_1a_2a_3}$.

In UHG circles can often be paired: two circles are **twins** if they share the same center and their quadrances sum to 2. The circumcenters c are the centers of *twin circumcircles* passing through collectively the three points of the triangle. This notion extends our understanding even in the classical case. The four pairs of twin circumcircles give eight *generalized circumcircles* (even for the classical case), and these meet in a surprising way in the *CircumMeet points*, some of which pleasantly depend only the side of the triangle on which they lie.

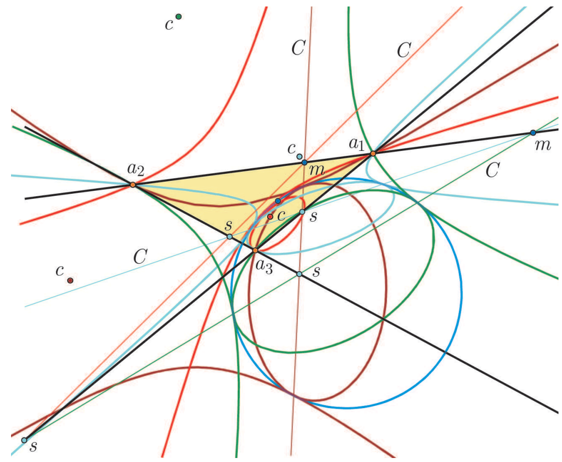

Figure 3: *Four twin circumcircles of a non-classical triangle*

In Figure 3 we see the twin circumcircles of the triangle of the previous Figure; some of these appear in this model as hyperbolas tangent to the null circle—these are invisible in classical hyperbolic geometry, but have a natural interpretation in terms of hyperboloids of one sheet in three-dimensional space (DeSitter space).

The other main contribution of this paper is in setting up *circumlinear coordinates*. UHG is more algebraic than the classical theory ([2], [1], [3], [4], [8]), emphasizing a projective metrical formulation without transcendental functions for Cayley-Klein geometries, valid both inside and outside the usual null circle (or absolute), and working over a general field, generally not of characteristic two. In [16], triangle geometry was studied in the more general setting of a projective plane over a field, with a metrical structure induced by a symmetric bilinear form on the associated three-dimensional vector space, or equivalently a general conic playing the role of the null circle or absolute. That paper focussed on *ortholinear coordinates*, and gave derivations for many initial constructions in the Incenter hierarchy, and only dual statements for the corresponding results for the Circumcenter hierarchy.

In this paper we introduce the complementary *circumlinear coordinates*, which are well suited for studying midpoints and sydpoints simultaneously. Finding formulas for key points and lines is, as always, a main aim. If the triangle $\overline{a_1a_2a_3}$ has either midpoints or sydpoints for each of its sides, a change of coordinates allows us to write $a_1 = [1:0:0]$, $a_2 = [0:1:0]$ and $a_3 = [0:0:1]$, with the bilinear form given by a matrix

$$C = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & \varepsilon \end{bmatrix} \tag{1}$$

where $\varepsilon^2 = \pm 1$. We reformulate formulas of the Orthocenter hierarchy of ([16]) using circumlinear coordinates, including the *Orthoaxis A* with the five important points h, s, b, x and z, and then turn to the Circumcenter hierarchy, studying *Medians*, *Centroids*, *CircumCentroids*, *CircumDual points*, *Tangent lines*, *Jay lines*, *Wren lines*, *CircumMeet points* and some new associated points and lines, and finish with a nice correspondence between the Circumcenters and four *Sound conics* passing two at a time through the twelve *Sound points*. Note that when we study a particular triangle, we adopt the convention of Capitalizing major points and lines of that Triangle. Although the paper is one of a series, we have tried to make it largely self-contained.

1.1 Projective duality and midpoint constructions

One can approach Universal Hyperbolic Geometry from either a synthetic projective geometry or an analytic linear algebra point of view; both are useful, and they shed light on each other. In this section we give a synthetic introduction useful for dynamic geometry packages such as GSP, C.a.R., Cabri, GeoGebra and Cinderella. We work in the projective plane over a field, which in our pictures will be the rational numbers, with a distinguished conic, called the **null circle**, but elsewhere also the *absolute*. In our pictures, this will be the familiar unit circle, always in blue, with points lying on it called **null points**.

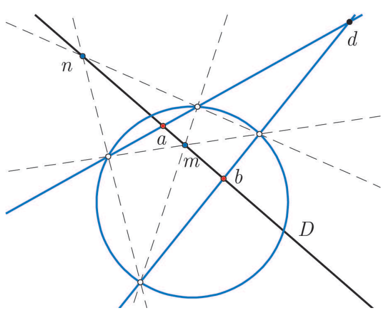

Figure 4: *Duals and perpendicularity*

The key duality, or polarity, between points and lines induced by the null circle allows a notion of perpendicularity: two points a and b are **perpendicular**, written $a \perp b$, precisely when b lies on the dual of a, or conversely a lies on the dual of b (these are equivalent), and similarly two lines L and M are **perpendicular**, written $L \perp M$, precisely when L passes through the dual of M, or conversely M passes through the dual of L.

In Figure 4 we see a construction for the *dual* of a point d; this is the line D formed by the other two diagonals n and m of any null quadrangle for which d is a diagonal point. Then d is perpendicular to any point on $D \equiv nm$, and any line through d is perpendicular to D. To construct the dual of a line L, take the meet of the duals of any two points on it.

The basic isometries in such a geometry are reflections in points (or reflections in lines—these two notions turn out to be the same). If m is not a null point, the reflection r_m in m interchanges the two null points on any line through m, should there be such. In Figure 5 for example, r_m interchanges x and w, and interchanges y and z. It is then a remarkable and fundamental fact that r_m extends to a projective transformation: to find the image of a point a, construct any line through a which meets the null circle at two points, say x and y, then find the images of x and y under r_m, namely w and z, and then define $r_m(a) = b \equiv (am)(wz)$ as shown. Perpendicularity of both points and lines is preserved by r_m.

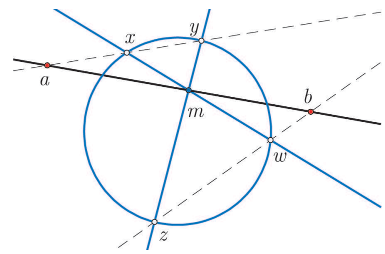

Figure 5: *Reflection r_m in m sends a to b*

The notion of reflection allows us to define midpoints without metrical measurements: if $r_m(a) = b$ then we may say that m is a **midpoint** of the side $\overline{ab}$. To construct the midpoints of a side $\overline{ab}$, when they exist (this is essentially a quadratic condition), we essentially invert the above construction.

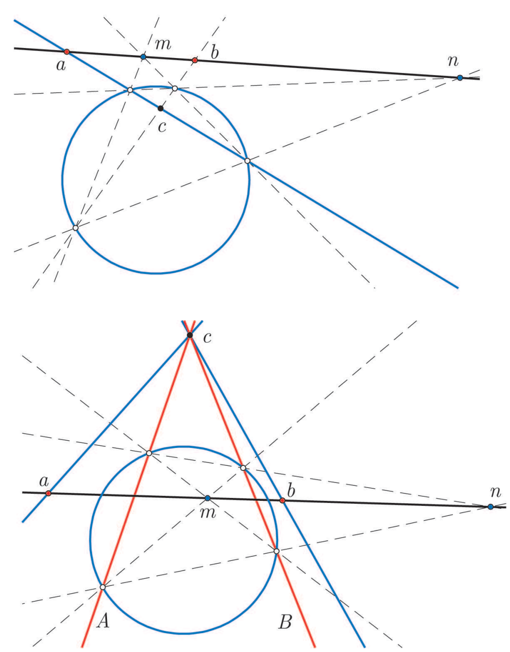

Figure 6: *Constructing midpoints m and n of the side $\overline{ab}$*

Figure 6 shows two situations where we can construct midpoints m and n of the side $\overline{ab}$, at least approximately over the rational numbers, which is the orientation of Geometer's Sketchpad and other dynamic geometry packages. In the top diagram, we take the dual c of the line ab, and if the lines ac and bc meet the null circle we take the other two diagonal points of this null quadrangle. This is also the case in Figure 4. In the bottom diagram, the lines ac and bc do not meet the null circle, but the dual lines A and B of a and b, which necessarily pass through c, do meet the null circle in a quadrangle, whose other diagonal points are the required midpoints m and n.

To define a circle $\mathcal{C}$ in this projective setting, suppose that c and p are points; then the locus of the reflections $r_x(p)$ as x runs along the dual line of c is the **circle** with center c through p. This projective definition immediately gives a correspondence between a circle and a line. Of course there is also a metrical definition, once we have set up quadrance and spread.

2 Metrical projective linear algebra

While the synthetic framework is attractive, for explicit computations and formulas it is useful to work with analytic geometry in the context of (projective) linear algebra. Our strategy, as in [16], will be to set up coordinates so that our basic triangle is as simple as possible, and all the complexity resides in the bilinear form. We begin with establishing some notation and basic results in the affine setting, although the projective setting is the main interest.

The three-dimensional vector space V over a field $\mathbb{F}$, of characteristic not two, consists of row vectors $v=(x,y,z)$ or equivalently 1×3 matrices $\begin{pmatrix} x & y & z\end{pmatrix}$. A metrical structure is determined by a *symmetric bilinear form*

$$v\cdot u = vu \equiv vCu^T$$

where C is an invertible symmetric 3×3 matrix. Note in particular our use of the algebraic notation vu. The dual vector space V^* may be viewed as column vectors $f=(l,m,n)^T$ or equivalently 3×1 matrices.

Vectors v, u are **perpendicular** precisely when $v\cdot u = vu = 0$. The **quadrance** of a vector v is the number $Q_v \equiv v\cdot v = v^2$. A vector v is **null** precisely when $Q_v = v^2 = 0$.

A variant of the following also appears in [7].

Theorem 1 (Parallel vectors) *If vectors v and u are parallel then*

$$Q_vQ_u = (vu)^2. \tag{2}$$

Conversely if (2) holds then either v and u are parallel, or the bilinear form restricted to the span of u and v is degenerate.

Proof. Consider a two-dimensional space containing v and u and the bilinear form restricted to it, given by a matrix $\widetilde{C} = \begin{pmatrix} a & b \\ b & c\end{pmatrix}$ with respect to some basis. If in this basis $v=(x,y)$ and $u=(u,v)$, then we may calculate that

$$Q_vQ_u-(vu)^2 = -\frac{(xv-yu)^4\left(ac-b^2\right)^2}{(au^2+2buv+cv^2)^2\,(ax^2+2bxy+cy^2)^2}.$$

So if v and u are parallel, the left hand side is zero, and conversely if the left hand side is zero, then either $ac-b^2\neq 0$ in which case the bilinear form restricted to the span of v and u is degenerate, or $xv-yu=0$, meaning that the vectors v and u are parallel. □

The previous result motivates the following measure of the non-parallelism of two vectors. The **(affine) spread** between non-null vectors v and u is the number

$$s(v,u)\equiv 1-\frac{(vu)^2}{Q_vQ_u}.$$

The spread is unchanged if either v or u are multiplied by a non-zero number.

2.1 Basic notation and definitions

One-dimensional and two-dimensional subspaces of $V=\mathbb{F}^3$ may be viewed as the basic objects forming the projective plane, with metrical notions coming from the affine notions of quadrance and spread in the associated vector space, but we prefer to give independent definitions so that logically neither the affine nor projective settings have priority. In general our notation in the projective setting is *opposite* to that in the affine setting, in the sense that the roles of small and capital letters are reversed throughout.

A **(projective) point** is a proportion $a=[x:y:z]$ in square brackets, or equivalently a projective row vector $a=\begin{bmatrix} x & y & z\end{bmatrix}$ where the square brackets in the latter are interpreted projectively: unchanged if multiplied by a non-zero number. A **(projective) line** is a proportion $L=\langle l:m:n\rangle$ in pointed brackets, or equivalently a projective column vector

$$L=\begin{bmatrix} l \\ m \\ n\end{bmatrix}.$$

When the context is clear, we refer to projective points and projective lines simply as **points** and **lines**. The **incidence** between the point $a=[x:y:z]$ and the line $L=\langle l:m:n\rangle$ is given by the relation

$$aL=\begin{bmatrix} x & y & z\end{bmatrix}\begin{bmatrix} l \\ m \\ n\end{bmatrix} = lx+my+nz=0.$$

In such a case we say a **lies on** L, or L **passes through** a.

The **join** a_1a_2 of distinct points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is the line

$$\begin{aligned} a_1a_2 &\equiv [x_1 : y_1 : z_1] \times [x_2 : y_2 : z_2] \\ &\equiv \langle y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1 \rangle . \end{aligned} \tag{3}$$

This is the unique line passing through a_1 and a_2. The **meet** L_1L_2 of distinct lines $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ and $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ is the point

$$\begin{aligned} L_1L_2 &\equiv \langle l_1 : m_1 : n_1 \rangle \times \langle l_2 : m_2 : n_2 \rangle \\ &\equiv [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1] . \end{aligned} \tag{4}$$

This is the unique point lying on L_1 and L_2.
Three points a_1, a_2, a_3 are **collinear** precisely when they lie on a line L; in this case we will sometimes write $L = a_1a_2a_3$. Similarly three lines L_1,L_2,L_3 are **concurrent** precisely when they pass through a point a; in this case we will sometimes write $a = L_1L_2L_3$.
It will be convenient to connect the affine and projective frameworks by the following conventions. If $v = (x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix}$ is a vector, then $a = [v] = [x : y : z] = \begin{bmatrix} x & y & z \end{bmatrix}$ is the **associated projective point**, and v is a **representative vector** for a. If $f = (l,m,n)^T$ is a dual vector, then $L = [f] = \langle l : m : n \rangle = \begin{bmatrix} l & m & n \end{bmatrix}^T$ is the **associated projective line**, and f is a **representative dual vector** for L.

2.2 Projective quadrance and spread

If C is a symmetric invertible 3×3 matrix, with entries in $\mathbb{F}$, and D is its adjugate matrix (the inverse, up to a multiple), then we denote by $\mathbf{C}$ and $\mathbf{D}$ the corresponding projective matrices, each defined up to a non-zero multiple. This pair of projective matrices determine a metrical structure on projective points and lines, as follows.
The (projective) points a_1 and a_2 are **perpendicular** precisely when $a_1\mathbf{C}a_2^T = 0$, written $a_1 \perp a_2$. This is a symmetric relation, and is well-defined. Similarly (projective) lines L_1 and L_2 are **perpendicular** precisely when $L_1^T\mathbf{D}L_2 = 0$, written $L_1 \perp L_2$. The point a and the line L are **dual** precisely when

$$L = a^\perp \equiv \mathbf{C}a^T \quad \text{or equivalently} \quad a = L^\perp \equiv L^T\mathbf{D}. \tag{5}$$

Then two points are perpendicular precisely when *one is incident with the dual of the other*, and similarly for two lines. So $a_1 \perp a_2$ precisely when $a_1^\perp \perp a_2^\perp$, because of the projective relation

$$\begin{aligned} \left(\mathbf{C}a_1^T\right)^T \mathbf{D}\left(\mathbf{C}a_2^T\right) &= \left(a_1\mathbf{C}^T\right)\mathbf{D}\left(\mathbf{C}a_2^T\right) = a_1\left(\mathbf{CD}\right)\left(\mathbf{C}a_2^T\right) \\ &= a_1\mathbf{C}a_2^T . \end{aligned}$$

A point a is **null** precisely when it is perpendicular to itself, that is, when $a\mathbf{C}a^T = 0$, and a line L is **null** precisely when it is perpendicular to itself, that is, when $L^T\mathbf{D}L = 0$. The null points determine the **null conic**, sometimes also called the *absolute*.
Hyperbolic and *elliptic geometries* arise respectively from the special cases

$$C = J \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D \qquad \text{and}$$

$$C = I \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D. \tag{6}$$

In the hyperbolic case, which forms the basis for almost all examples in this paper, the point $a = [x : y : z]$ is null precisely when $x^2 + y^2 - z^2 = 0$, and dually the line $L = (l : m : n)$ is null precisely when $l^2 + m^2 - n^2 = 0$. This is the reason we can picture the null circle in affine coordinates $X \equiv x/z$ and $Y \equiv y/z$ as the (blue) circle $X^2 + Y^2 = 1$. Note that in the elliptic case the null circle, over the rational numbers, has no points lying on it. This is why visualizing hyperbolic geometry is often easier than elliptic geometry.
The bilinear forms determined by C and D can be used to define the metrical structure in the associated projective setting. The dual notions of (**projective**) **quadrance** $q(a_1,a_2)$ between points a_1 and a_2, and (**projective**) **spread** $S(L_1,L_2)$ between lines L_1 and L_2, are

$$q(a_1,a_2) \equiv 1 - \frac{\left(a_1\mathbf{C}a_2^T\right)^2}{\left(a_1\mathbf{C}a_1^T\right)\left(a_2\mathbf{C}a_2^T\right)} \qquad \text{and}$$

$$S(L_1,L_2) \equiv 1 - \frac{\left(L_1^T\mathbf{D}L_2\right)^2}{\left(L_1^T\mathbf{D}L_1\right)\left(L_2^T\mathbf{D}L_2\right)}. \tag{7}$$

While the numerators and denominators of these expressions depend on choices of representative vectors and matrices for $a_1, a_2, \mathbf{C}, L_1, L_2$ and $\mathbf{D}$, the *quotients are independent of scaling*, so the overall expressions are indeed well-defined projectively.If $a_1 = [v_1]$, $a_2 = [v_2]$, and $L_1 = [f_1]$, $L_2 = [f_2]$, then we may write

$$q(a_1,a_2) = 1 - \frac{(v_1 \cdot v_2)^2}{(v_1 \cdot v_1)(v_2 \cdot v_2)} \qquad \text{and}$$

$$S(L_1,L_2) = 1 - \frac{(f_1 \odot f_2)^2}{(f_1 \odot f_1)(f_2 \odot f_2)}$$

where we introduce the dual bilinear form on column vectors by $f_1 \odot f_2 \equiv f_1^T D f_2$.
Clearly $q(a,a) = 0$ and $S(L,L) = 0$, while $q(a_1,a_2) = 1$ precisely when $a_1 \perp a_2$, and dually $S(L_1,L_2) = 1$ precisely when $L_1 \perp L_2$. Then using (5)

$$S\left(a_1^\perp, a_2^\perp\right) = q(a_1,a_2).$$

In [14], we showed that both these metrical notions can also be reformulated projectively and rationally using suitable cross ratios (and no transcendental functions!)
The following formula, introduced in [12], is is given in a more general setting in [13].

Theorem 2 (Hyperbolic Triple quad formula) *Suppose that a_1,a_2,a_3 are collinear points, with quadrances $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$ and $q_3 \equiv q(a_1,a_2)$. Then*

$$(q_1+q_2+q_3)^2 = 2\left(q_1^2+q_2^2+q_3^2\right)+4q_1q_2q_3. \tag{8}$$

Proof. We may assume at least two of the points distinct, as otherwise the relation is trivial. Suppose that representative vectors are then v_1,v_2 and $v_3 \equiv kv_1+lv_2$, with v_1 and v_2 linearly independent. Consider just the two-dimensional subspace spanned by v_1 and v_2. The bilinear form restricted to the subspace spanned by the ordered basis v_1,v_2 is given by some symmetric matrix $\widetilde{C} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then in this basis $v_1 = (1,0)$, $v_2 = (0,1)$ and $v_3 = (k,l)$, and we may compute that

$$q_3 = s(v_1,v_2) = \frac{ac-b^2}{ac}$$

$$q_2 = s(v_1,v_3) = \frac{l^2\left(ac-b^2\right)}{a\left(ak^2+2bkl+cl^2\right)}$$

$$q_1 = s(v_2,v_3) = \frac{k^2\left(ac-b^2\right)}{c\left(ak^2+2bkl+cl^2\right)}.$$

Then (8) is an identity. □

Here are a few useful consequences of the Triple quad formula. If one of the quadrances is $q_3 = 1$, then $q_1+q_2 = 1$; this is a consequence of the identity

$$(q_1+q_2+1)^2-2q_1^2-2q_2^2-2-4q_1q_2 = -(q_1+q_2-1)^2.$$

Also if two of the quadrances are equal, say $q_1 = q_2 = r$, then $q_3 = 0$ or $q_3 = 4r(1-r)$; this follows from the identity

$$(2r+q_3)^2-4r^2-2q_3^2-4r^2q_3 = -q_3\left(q_3-4r+4r^2\right).$$

2.3 Midpoints of a side

Midpoints are defined very simply using the metrical structure.

Definition 1 *A **midpoint** of a non-null side $\overline{ab}$ is a point m lying on ab which satisfies*

$$q(a,m) = q(b,m).$$

We exclude null sides because every two points on such a side have quadrance 0.

Theorem 3 (Side midpoints) *Suppose that a and b are distinct non-null points and $\overline{ab}$ is a non-null side. Then $\overline{ab}$ has a midpoint precisely when the quantity $1-q(a,b)$ is a square number. In this case, we may find representative vectors v and u for a and b respectively satisfying $v^2 = u^2$, and then there are exactly two midpoints of $\overline{ab}$, namely $m = [u+v]$ and $n = [u-v]$. These two midpoints are perpendicular. Furthermore a,m,b,n form a harmonic range.*

Proof. Suppose that $a = [v]$ and $b = [u]$ so that

$$1-q(a,b) = \frac{(vu)^2}{Q_vQ_u}.$$

A general point m on ab has representative non-zero vector $w = kv+lu$. The condition $q(a,m) = q(b,m)$ amounts to

$$\begin{aligned}\frac{(vw)^2}{Q_vQ_w} = \frac{(uw)^2}{Q_uQ_w} &\Leftrightarrow u^2\left(kv^2+l(vu)\right)^2 = v^2\left(k(vu)+lu^2\right)^2 \\ &\Leftrightarrow k^2u^2\left(v^2\right)^2+l^2(vu)^2u^2 = k^2v^2(vu)^2+l^2v^2\left(u^2\right)^2 \\ &\Leftrightarrow \left(v^2u^2-(vu)^2\right)\left(k^2v^2-l^2u^2\right) = 0.\end{aligned}$$

If $v^2u^2 = (vu)^2$ then by the Parallel vectors theorem either v and u are parallel, which is impossible since a and b are distinct, or the bilinear form restricted to $[v,u]$ is degenerate, which implies that the side $\overline{ab}$ is null. So a midpoint m exists precisely when $k^2v^2 = l^2u^2$.
In this case since a and b are non-null, v^2 and u^2 are non-zero, so k and l are also, since by assumption $w = kv+lu$ is non-zero, and we may renormalize v and u so that $v^2 = u^2$ (by for example setting $\widetilde{v} = kv$ and $\widetilde{u} = lu$, and then replacing $\widetilde{v}$,$\widetilde{u}$ by v,u again).
After this renormalization $1-q(a,b) = (vu)^2/\left(v^2\right)^2$ is then a square, and there are two midpoints $[v+u]$ and $[v-u]$. Since $(v+u)(v-u) = v^2-u^2 = 0$, the two midpoints are perpendicular. It is well known that for any two vectors v and u, the four lines $[v]$,$[v+u]$,$[u]$,$[v-u]$ form a harmonic range.
Conversely suppose that $1-q(a,b) = (vu)^2/\left(v^2u^2\right)$ is a square, say r^2. Then the ratio of v^2 to u^2 is a square, so v and u can be renormalized so that $v^2 = u^2$, at which point the above calculations show that $[v+u]$ and $[v-u]$ are both midpoints. □

We can also relate this to hyperbolic trigonometry as in [14]. If $q(a,b) = r \neq 0$, and m is a midpoint of the side $\overline{ab}$ with $q(a,m) = q(b,m) = q$, then $\{r,q,q\}$ satisfies the Triple quad formula. So as we observed earlier, $r = 4q(1-q)$, and in particular $1-r = 1-4q(1-q) = (2q-1)^2$ is a square number.
The dual lines M and N of the midpoints m and n of a side are called the **midlines** of the side. Since m and n are perpendicular, these each pass through the other midpoint, and

so might also be called the *perpendicular bisectors* of the side.
The dual concept of a midpoint of a side is the following.

Definition 2 *A **biline** of a non-null vertex $\overline{AB}$ is a line L passing through AB which satisfies*

$$S(A,L)=S(B,L).$$

From duality the vertex $\overline{AB}$ has a biline precisely when the quantity $1-S(A,B)$ is a square number, and in this case we have exactly two bilines which are perpendicular. The symmetry between midpoints and bilines is reflected in the duality between the Incenter and Circumcenter hierarchies in UHG. This notion of symmetry is absent in classical hyperbolic geometry, since there we always have only one midpoint of a side and two bilines (usually called angle bisectors); the number-theoretic considerations with the existence of these are generally invisible—the price of working over the "real numbers"!

2.4 Sydpoints of a side

Definition 3 *A **sydpoint** of a non-null side $\overline{ab}$ is a point s lying on ab which satisfies*

$$q(a,s)=-q(b,s).$$

Note both the similarities and differences between the following theorem and the Side midpoints theorem.

Theorem 4 (Side sydpoints) *Suppose that a and b are distinct non-null points and $\overline{ab}$ is a non-null side. Then $\overline{ab}$ has a sydpoint precisely when $q(a,b)-1$ is a square number. In this case we can find representative vectors v and u for a and b respectively satisfying $v^2=-u^2$, and then there are exactly two sydpoints of $\overline{ab}$, namely $s=[v+u]$ and $r=[v-u]$. In such a case, a and b are also sydpoints of the side $\overline{sr}$, and while s and r are not in general perpendicular, we do have*

$$q(a,s)=q(b,r) \qquad \text{and} \qquad q(a,r)=q(b,s).$$

Furthermore a,s,b,r form a harmonic range.

Proof. Suppose that $a=[v]$ and $b=[u]$ so that a general point $s=[w]$ on ab has representative vector $w=kv+lu$. Then the relation $q(a,s)=-q(b,s)$ amounts to

$$\begin{aligned}
&1-\frac{(vw)^2}{Q_vQ_w}=-1+\frac{(uw)^2}{Q_uQ_w}\\
\Leftrightarrow\ & 2u^2v^2(kv+lu)^2-u^2\left(kv^2+l(vu)\right)^2=v^2\left(k(vu)+lu^2\right)^2\\
\Leftrightarrow\ & k^2u^2\left(v^2\right)^2+l^2\left(u^2\right)^2v^2-\left(k^2v^2+l^2u^2\right)(vu)^2=0\\
\Leftrightarrow\ & \left(v^2u^2-(vu)^2\right)\left(k^2v^2+l^2u^2\right)=0.
\end{aligned}$$

If $v^2u^2=(vu)^2$ then by the Parallel vectors theorem either v and u are parallel, which is impossible since a and b are distinct, or the bilinear form restricted to $[v,u]$ is degenerate, which implies that the side $\overline{ab}$ is null. So a sydpoint s exists precisely when $k^2v^2=-l^2u^2$. In this case we may renormalize v and u so that $v^2=-u^2$, so that $s\equiv[v+u]$ and $r\equiv[v-u]$ are sydpoints. If $q(a,s)=-q(b,s)=d$, $q(a,r)=-q(b,r)=e$ and also $q(r,s)=f$, then the Triple quad formula applied to $\{a,r,s\}$ and $\{b,r,s\}$ implies that both

$$(f+d+e)^2=2\left(f^2+d^2+e^2\right)+4fde \qquad \text{and}$$

$$(f-d-e)^2=2\left(f^2+d^2+e^2\right)+4fde$$

which implies that $f+d+e=\pm(f-d-e)$. Since $f\neq 0$, we conclude that $d=-e$, which shows that

$$q(a,s)=q(b,r) \qquad \text{and} \qquad q(a,r)=q(b,s).$$

Now $(v+u)(v-u)=v^2-u^2=2v^2$ so the two sydpoints s and r are *not* in general perpendicular. However

$$(v+u)^2=v^2+2uv+u^2=2uv \qquad \text{and}$$

$$(v-u)^2=v^2-2uv+u^2=-2uv$$

so that $(v+u)^2=-(v-u)^2$. By symmetry this implies that $[(v+u)+(v-u)]=[2v]=a$ and $[(v+u)-(v-u)]=[-2u]=b$ are sydpoints of $\overline{rs}$. $\square$

For a fixed q there is at most one sydpoint s of $\overline{ab}$ for which $q(a,s)=q$; the other sydpoint r then satisfies $q(a,r)=-q\neq q$ since q is non-zero.

Example 1 *In the hyperbolic case, suppose that $a=[x:0:1]$ and $b=[y:0:1]$. Then from [14], Ex. 6*

$$q(a,b)=-\frac{(x-y)^2}{(1-x^2)(1-y^2)}$$

and so midpoints $m=[w:0:1]$ and sydpoints $s=[z:0:1]$ of $\overline{ab}$ exist precisely when $\left(x^2-1\right)\left(y^2-1\right)=r^2$ and $\left(x^2-1\right)\left(y^2-1\right)=-t^2$ respectively, in which cases

$$w=\frac{xy+1\pm r}{x+y} \qquad \text{and} \qquad z=\frac{(1-xy)(x+y)\pm t(x-y)}{x^2+y^2-2}.$$

So we see that algebraically sydpoints are somewhat more complicated than midpoints in general.

Over the rational numbers, any non-null side either approximately has midpoints or sydpoints, since being a square is approximately the same as being positive.
There are a few related notions which are useful to define. The duals S and R of the sydpoints s and r of a side $\overline{ab}$ are the **sydlines** of the side $\overline{ab}$. They do not in general pass through the sydpoints themselves. There is also a dual notion to that of sydpoints of a side which applies to vertices.

Definition 4 *A **siline** of a vertex $\overline{AB}$ is a line L which passes through AB and satisfies $S(A,L) = -S(B,L)$.*

Again by duality we deduce that a vertex $\overline{AB}$ has a siline precisely when the quantity $S(A,B)-1$ is a square number, and in this case there are exactly two silines L and K of the vertex $\overline{AB}$. Then also A, B, L and K are a harmonic pencil of lines. The duals of the silines are the **sipoints** of a vertex $\overline{AB}$.

2.5 The construction of Sydpoints

The following theorem is helpful in constructing sydpoints using a dynamic geometry package.

Theorem 5 (Sydpoints null points) *Suppose that the non-null side $\overline{ab}$ has sydpoints s and r, and that $\overline{ac}$ has midpoints m and n, where $c = (ab)^{\perp}$. Then $x \equiv (mr)(bc) = (ns)(bc)$ and $y \equiv (ms)(bc) = (nr)(bc)$ are null points.*

Proof. Suppose that $a = [v]$, $b = [u]$ and $c = [w]$. Then $vw = uw = 0$, since $c = (ab)^{\perp}$, and also since ab is not null v, u and w are independent. If $\overline{ac}$ has midpoints, in which case we may assume that $v^2 = w^2$, these are $m \equiv [v+w]$ and $n \equiv [v-w]$. If also $\overline{ab}$ has sydpoints, in which case we may assume that $v^2 = -u^2$, these are $s = [v+u]$ and $r = [v-u]$. Note that this renormalization can be made independent of the previous one.
Now consider $x \equiv (mr)(bc)$. This is a point with a representative vector of the form $k(v+w)+l(v-u)$ for some numbers k and l. Since x has a representative vector which is also in the span of u and w, it must be a multiple of $(v+w)-(v-u) = u+w$. But then

$$(u+w)^2 = u^2 + 2uw + w^2 = 0$$

since $uw = 0$ and $u^2 = -w^2$. So x is a null point, and similarly for y. $\square$

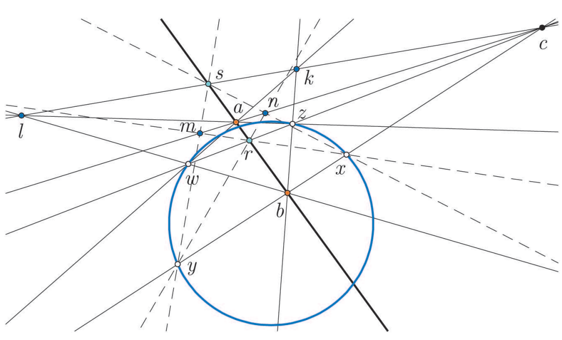

Figure 7: *Construction of sydpoints of $\overline{ab}$*

We make some remarks that are useful for practical constructions involving Geometer's Sketchpad, C.a.R., Cabri, GeoGebra or Cinderella etc. To approximately construct the sydpoints r and s of $\overline{ab}$ as in Figure 7, first construct the dual $c = (ab)^{\perp}$, then the midpoints m and n of $\overline{ac}$, and then use the null points x and y lying on bc as shown (we are assuming these exist—for a dynamic geometry package, approximately is sufficient!
The required points are $s = (nx)(ab) = (my)(ab)$ and $r = (ny)(ab) = (mx)(ab)$. Similarly, given the sydpoints r and s of $\overline{ab}$, a and b can be constructed as the sydpoints of $\overline{rs}$ using the null points w and z lying on rc and the midpoints k and l of $\overline{cs}$, the required points are $a = (lz)(rs) = (kw)(rs)$ and $b = (lw)(rs) = (kz)(rs)$. So the construction of sydpoints can be reduced, at least in this kind of situation, to computations of midpoints.
Once we establish the Circumlines theorem, it is interesting that Figure 7 can be viewed as a limiting case applied to the triangle $\overline{abc}$—the null points x and y act as midpoints of $\overline{bc}$, so mrx acts as a circumline.
Another useful construction is to find, given the point b and one of the sydpoints s, the other point a and the other sydpoint r as in Figure 8. First construct the dual $c = (bs)^{\perp}$, then find the midpoints k and l of $\overline{cs}$. Use the null points u,t lying on bk and the null points v,w lying on bl to construct $r = (cuv)(bs)$ and $a = (lu)(bs) = (kv)(bs)$.
However by symmetry there is a second solution: $\overline{r} = (cwt)(bs)$ and $\overline{a} = (lt)(bs) = (kw)(bs)$. Thus, we can think of s and r as being the sydpoints of the side $\overline{ab}$, and s and $\overline{r}$ as the sydpoints of the side $\overline{\overline{a}b}$. Notice also that b is a midpoint of the side $r\overline{r}$ and similarly s is a midpoint of the side $a\overline{a}$, and in fact $q(b,r) = q(b,\overline{r}) = q(s,a) = q(s,\overline{a})$.

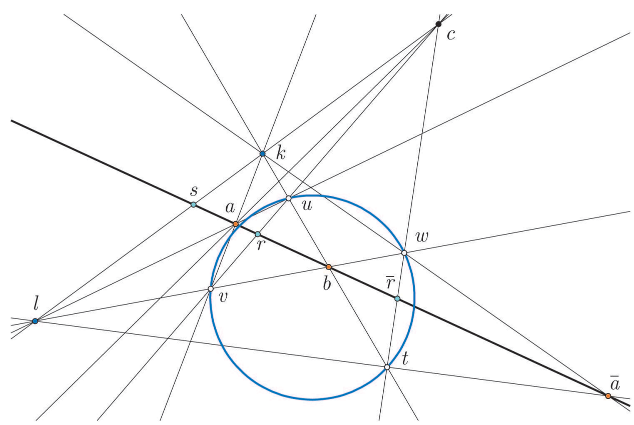

Figure 8: *Constructing r and a (or $\overline{r}$ and $\overline{a}$) from s and b*

2.6 Twin circles

In the geometry we are studying, a circle $\mathcal{C}$ may be defined as an equation of the form $q(c,x) = k$, for a fixed point c called the **center**, and a fixed number k called the **quadrance** of the circle. We also write $\mathcal{C}_c^k$ for this circle, and say that a point a **lies on** the circle precisely when $q(c,a) = k$. Since in this case the circle is also determined by c and a, we write $\mathcal{C}_c^k = \mathcal{C}_c^{(a)}$. The bracket reminds us that a is not unique.

Definition 5 *Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with the same center c and quadrances q_1 and q_2 are* ***twins*** *precisely when*

$$q_1 + q_2 = 2.$$

We now show that twin circles are naturally connected with sydpoints.

Theorem 6 (Sydpoint twin circle) *If s is a sydpoint of $\overline{ab}$, and c lies on $S \equiv s^\perp$, then the circles $\mathcal{C}_c^{(a)}$ and $\mathcal{C}_c^{(b)}$ are twins. Conversely if $\mathcal{C}_c^{(a)}$ and $\mathcal{C}_c^{(b)}$ are twins, then $s \equiv c^\perp(ab)$ is a sydpoint of $\overline{ab}$.*

Proof. If s is a sydpoint of $\overline{ab}$ then $q(a,s) = q = -q(b,s)$ for some q. Then since c and s are perpendicular, $q(c,s) = 1$. Let $d = s^\perp(ab)$. Then since d and s are perpendicular, $q(d,s) = 1$, and then $q(a,d) = 1 - q(a,s) = 1 - q$ and $q(b,d) = 1 - q(b,s) = 1 + q$. So $q(a,d) + q(b,d) = 2$. Now suppose that $q(c,d) = r$. Then by Pythagoras' theorem (see [13], [14]) in the right triangle $\overline{cda}$ we have

$$q(c,a) = r + (1-q) - r(1-q)$$

while in the right triangle $\overline{cdb}$ we have

$$q(c,b) = r + (1+q) - r(1+q).$$

Then

$$q(c,a) + q(c,b) =$$
$$= r + (1-q) - r(1-q) + r + (1+q) - r(1+q) = 2.$$

The argument can be reversed to show the converse. □

We note that the theorem has another possible interpretation: the locus of a point c such that $q(a,c) + q(b,c) = 2$ is a line.

2.7 Constructions of twin circles

The Sydpoint twin circle theorem assists us to construct twin circles; we generally expect this to reduce to finding midpoints, but there are also some simpler scenarios. Suppose we are given a circle $\mathcal{C}$ (in brown) with center c as in Figure 9. Choose an arbitrary point a on the circle $\mathcal{C}$ and construct $C \equiv c^\perp$, then let s be the meet of ac and C, and t the meet of $A \equiv a^\perp$ and C.

Now, we can apply the construction of Figure 8; suppose that the side $\overline{st}$ has midpoints m and n, and that x and y are null points on am, and z and w are null points on an. Then $b \equiv (mz)(ac) = (ny)(ac)$ and $e \equiv (mw)(ac) = (nx)(ac)$ lie on the twin circle $\mathcal{D}$ to $\mathcal{C}$. Symmetry implies that we could also use $d \equiv (mw)(ct) = (ny)(ct)$ and $f \equiv (mz)(ct) = (nx)(ct)$.

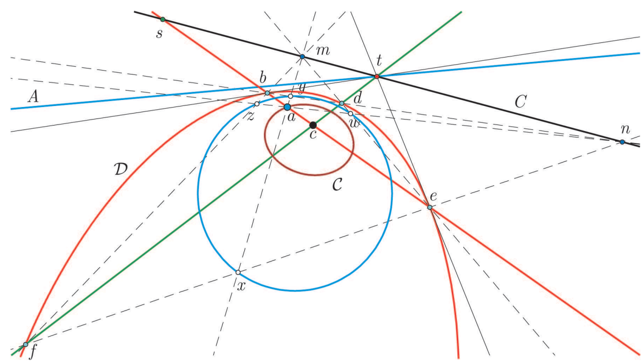

Figure 9: *Constructing the twin circle $\mathcal{D}$ of $\mathcal{C}$*

Figure 10 shows another example of constructing the twin $\mathcal{D}$ of a given circle $\mathcal{C}$ (in brown) with center c. In this case c is outside the null circle, so its dual line C passes through null points x and y (approximately—remember that a dynamic geometry package usually only deals with decimal approximations, so the number-theoretical subtlety is diminished). Choose a point a on $\mathcal{C}$ with dual line $A = a^\perp$. Then the twin circle $\mathcal{D}$ (in red) is the locus of the point $b = (ax)A$ or the point $d = (ay)A$ as a moves along $\mathcal{C}$.

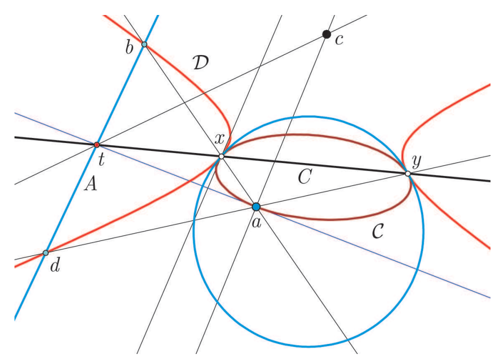

Figure 10: *Another construction of a twin circle*

The fact that $q(a,c) + q(b,c) = 2$ follows by applying either the Nil Cross law ([14, Thm 80]) or the Null subtended quadrance theorem ([14, Thm 90]) to the triangle $\overline{abc}$. Similarly, given the red circle $\mathcal{D}$, its twin circle $\mathcal{C}$ (in brown) can be constructed as the locus of the point $a = (bx)b^\perp$ when moving the point b on $\mathcal{D}$.

It should also be noted that we have *not at all established that the twin of any circle necessarily exists*. In fact over the rational numbers, the twin circle of a given circle does not always exist. For example over the rational numbers, if c is inside the null circle, then $q(c,a)$ never takes on values in the range $(0,1)$, but it can take on values in the range $(1,2)$.

3 Circumlinear coordinates and the Orthocenter hierarchy

In the paper ([16]) we focussed on ortholinear coordinates, as the Orthocenter is arguably the most important point in hyperbolic triangle geometry, and secondly on the Incenter hierarchy. In this paper we are primarily interested in the Circumcenter hierarchy, and we introduce *circumlinear coordinates* to work efficiently with both *midpoints* and *sydpoints* simultaneously. While triangle geometry involving sydpoints will be new and somewhat unfamiliar, the natural beauty and elegance of this theory is very compelling indeed.

Suppose the bilinear form $v\cdot u = vAu^T$ in the associated three-dimensional vector space $V=\mathbb{F}^3$ is given by a symmetric matrix A, and that $T:V\to V$ is a linear transformation given by an invertible 3×3 matrix M, so that $T(v)=vM=w$, with inverse matrix N, so that $wN=v$. The new bilinear form $\circ$ defined by

$$\begin{aligned} w_1\circ w_2 &\equiv (w_1N)\cdot(w_2N)=(w_1N)A(w_2N)^T \\ &= w_1(NAN^T)w_2^T \end{aligned} \qquad (9)$$

has matrix $C=NAN^T$.

So let us start with three (projective) points a_1, a_2 and a_3 such that *each of the three sides of the triangle* $\overline{a_1a_2a_3}$ *has either midpoints or sydpoints*. That means we can find representative vectors v_1, v_2 and v_3 in V so that for any i and j, $v_i^2=\pm v_j^2$. There are two possibilities up to relabelling and re-scaling: 1) $v_1^2=v_2^2=v_3^2=1$ (this corresponds to three midsides) and 2) $v_1^2=v_2^2=-v_3^2=1$ (this corresponds to one midside and two sydsides). We can incorporate both situations at once by supposing that

$$v_1^2=v_2^2=\varepsilon v_3^2=1 \qquad \text{where} \qquad \varepsilon=\pm 1.$$

Now we can find a linear transformation to map v_1, v_2 and v_3 to the basis vectors $e_1=(1,0,0)$, $e_2=(0,1,0)$ and $e_3=(0,0,1)$ respectively. With respect to this new basis, the bilinear form is then given by a new matrix of the form

$$C=\begin{pmatrix} 1 & a & b\\ a & 1 & c\\ b & c & \varepsilon\end{pmatrix} \qquad \text{with adjugate}$$

$$D=\begin{pmatrix} c^2-\varepsilon & a\varepsilon-bc & b-ac\\ a\varepsilon-bc & b^2-\varepsilon & c-ab\\ b-ac & c-ab & a^2-1\end{pmatrix} \qquad (10)$$

where the diagonal entries of C ensure that $e_1^2=e_2^2=1$ and $e_3^2=\varepsilon$, and otherwise $e_1e_2=a$, $e_1e_3=b$ and $e_2e_3=c$ are arbitrary. So the metrical structure depends on the numbers a, b and c and (the sign of) ε. Note that

$$\det\begin{pmatrix} 1 & a & b\\ a & 1 & c\\ b & c & \varepsilon\end{pmatrix}=-a^2\varepsilon-b^2-c^2+\varepsilon+2abc.$$

This quantity appears as a common factor in several of the derivations of proportions in the paper, and since it is by assumption non-zero, we simply cancel it without mention.

We now reformulate some of the formulas of the Orthocenter hierarchy of ([16]) using circumlinear coordinates, maintaining the convention of using capital letters for various constructions associated to a base triangle. The projective matrices corresponding to C and D are denoted $\mathbf{C}$ and $\mathbf{D}$ respectively.

Our starting point is that the basic Triangle $\overline{a_1a_2a_3}$ has been projectively transformed so that its **Points** are

$$a_1=[1:0:0] \quad a_2=[0:1:0] \quad a_3=[0:0:1]. \qquad (11)$$

The **Lines** of the Triangle are then

$$L_1=\langle 1:0:0\rangle \quad L_2=\langle 0:1:0\rangle \quad L_3=\langle 0:0:1\rangle.$$

The main assumption is that each of the three sides is either a midside or a sydside, or possibly both, which we have seen allows us to write the bilinear form using the projective matrices (10). The Triangle will have three midsides if $\varepsilon=1$, and two sydsides and one midside if $\varepsilon=-1$. The computations are based on two basic operations: *finding joins and meets*, which essentially amounts to taking cross products as in (3) and (4); and *finding duals*, either by multiplying transposes of points by C on the left, or transposes of lines by D on the right as in (5).

Our goal is to establish formulas for important points and lines to facilitate determining relationships between them: the reader is encouraged to follow along and check our computations, which are mostly elementary. Occasionally we simplify a proportion by cancelling a common factor: naturally this factor should not be zero, so we state this as a condition.

3.1 Change of coordinates and the main example

Most of the diagrams in this paper deal with the particular triangle in Figure 11 created with GSP, with affine points $a_1\approx[-0.03959,0.15272]$, $a_2\approx[-0.20363,0.78056]$ and $a_3\approx[-1.75344,0.19797]$, and corresponding representative vectors $v_1\approx(-0.237,0.914,5.985)$, $v_2\approx(-2.036,7.806,10)$ and $v_3\approx(-7.128,0.805,4.065)$. These have been normalized so that

$$Q_{v_1}=Q_{v_2}=-Q_{v_3}$$

with respect to the bilinear form $v\cdot u\equiv vJu^T$, where J is defined in (6).

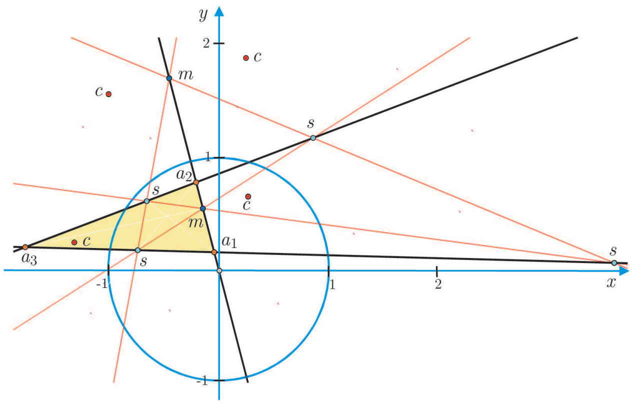

Figure 11: *Basic example triangle with coordinates*

We now show how to explicitly change coordinates, following Section 1.5 of [16]. The linear transformation $T(v) = vN$, where N is

$$N = \begin{pmatrix} -0.237 & 0.914 & 5.985 \\ -2.036 & 7.806 & 10 \\ -7.128 & 0.805 & 4.065 \end{pmatrix},$$

sends $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ to v_1, v_2 and v_3 respectively. The inverse matrix $M = N^{-1}$ sends the vectors v_1, v_2 and v_3 to e_1, e_2 and e_3. Following (9), after we apply the linear transformation T, J is replaced by the matrix

$$C = NJN^T \approx \begin{pmatrix} 1.0 & 1.495 & 0.627 \\ 1.495 & 1 & 0.568 \\ 0.627 & 0.568 & -1 \end{pmatrix} \quad \text{with adjugate}$$

$$D = \begin{pmatrix} 1.327 & -1.851 & -0.222 \\ -1.851 & 1.393 & -0.369 \\ -0.222 & -0.369 & 1.235 \end{pmatrix}.$$

We get the constants

$$a = 1.495 \qquad b = 0.627 \qquad c = 0.568 \qquad \varepsilon = -1.$$

As an example of how to explicitly apply the theorems of this paper to our specific triangle, consider the midpoints of the side $\overline{a_1a_2}$ in standard coordinates which are $m = n_{1+} = [1:1:0]$ and $m = n_{1-} = [1:-1:0]$. Multiply by N and then renormalize so that $z = 1$, to find these midpoints in the original triangle to be

$$\begin{aligned} n_{1+} &= [1:1:0]\,N = \begin{bmatrix} -2.273 & 8.72 & 15.985 \end{bmatrix} \\ &= \begin{bmatrix} -0.142 & 0.546 & 1.0 \end{bmatrix} \\ n_{1-} &= [1:-1:0]\,N = \begin{bmatrix} 1.799 & -6.892 & -4.015 \end{bmatrix} \\ &= \begin{bmatrix} -0.448 & 1.72 & 1.0 \end{bmatrix}. \end{aligned}$$

As another example, using the formulas from the Circumlines/Circumcenter theorem, we may similarly compute that the circumcenters c, in agreement with Figure 11, are

$$c_0 = \begin{bmatrix} 0.268 & 0.653 & 1.0 \end{bmatrix} \quad c_1 = \begin{bmatrix} -0.997 & 1.573 & 1.0 \end{bmatrix}$$
$$c_2 = \begin{bmatrix} 0.249 & 1.898 & 1.0 \end{bmatrix} \quad c_3 = \begin{bmatrix} -1.308 & 0.241 & 1.0 \end{bmatrix}.$$

3.2 Altitudes, Orthocenter and Orthic triangle

The **Dual lines** are

$$\begin{aligned} A_1 &\equiv a_1^\perp = \mathbf{C}a_1^T = \langle 1:a:b\rangle \\ A_2 &\equiv a_2^\perp = \mathbf{C}a_2^T = \langle a:1:c\rangle \\ A_3 &\equiv a_3^\perp = \mathbf{C}a_3^T = \langle b:c:\varepsilon\rangle. \end{aligned}$$

The **Dual points** are

$$\begin{aligned} l_1 &\equiv L_1^T D = \left[c^2-\varepsilon : \varepsilon a - bc : b - ac\right] \\ l_2 &= \left[\varepsilon a - bc : b^2 - \varepsilon : c - ab\right] \\ l_3 &= \left[b - ac : c - ab : a^2 - 1\right]. \end{aligned}$$

The **Altitudes** are

$$\begin{aligned} N_1 &\equiv a_1 l_1 = \langle 0 : ac - b : \varepsilon a - bc\rangle \\ N_2 &\equiv a_2 l_2 = \langle c - ab : 0 : bc - \varepsilon a\rangle \\ N_3 &\equiv a_3 l_3 = \langle ab - c : b - ac : 0\rangle \end{aligned}$$

and the **Altitude dual points** are

$$\begin{aligned} n_1 &\equiv A_1 L_1 = [0:-b:a] \\ n_2 &\equiv A_2 L_2 = [c:0:-a] \\ n_3 &\equiv A_3 L_3 = [-c:b:0]. \end{aligned}$$

The **Base points** are

$$\begin{aligned} b_1 &\equiv N_1 L_1 = [0 : \varepsilon a - bc : b - ac] \\ b_2 &\equiv N_2 L_2 = [\varepsilon a - bc : 0 : c - ab] \\ b_3 &\equiv N_3 L_3 = [b - ac : c - ab : 0] \end{aligned}$$

and the **Base lines** are

$$\begin{aligned} B_1 &\equiv n_1 l_1 = \left\langle b^2 - 2abc + a^2\varepsilon : a\left(\varepsilon - c^2\right) : b\left(\varepsilon - c^2\right)\right\rangle \\ B_2 &\equiv n_2 l_2 = \left\langle a\left(\varepsilon - b^2\right) : c^2 - 2abc + a^2\varepsilon : c\left(\varepsilon - b^2\right)\right\rangle \\ B_3 &\equiv n_3 l_3 = \left\langle b\left(1 - a^2\right) : c\left(1 - a^2\right) : b^2 - 2abc + c^2\right\rangle. \end{aligned}$$

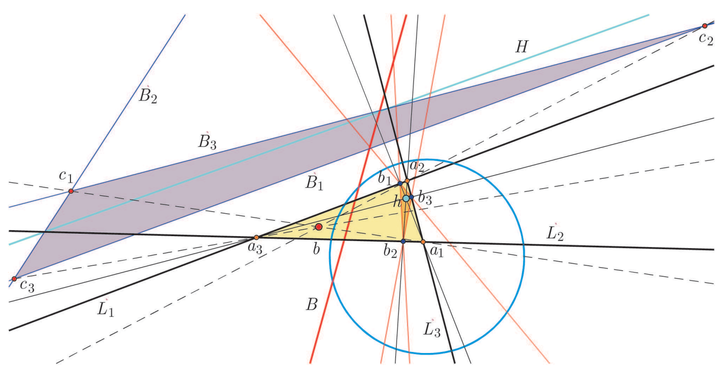

Figure 12: *Altitudes, Orthocenter, Orthic triangle and Base center b*

Assuming $a\varepsilon - bc \neq 0$, $b - ac \neq 0$ and $c - ab \neq 0$, the **Orthic lines** are

$$\begin{aligned} C_1 &\equiv b_2 b_3 = \langle ab - c : b - ac : \varepsilon a - bc\rangle \\ C_2 &\equiv b_1 b_3 = \langle c - ab : ac - b : \varepsilon a - bc\rangle \\ C_3 &\equiv b_1 b_2 = \langle c - ab : b - ac : bc - a\varepsilon\rangle. \end{aligned}$$

The **Orthic points** are

$$\begin{aligned} c_1 \equiv B_2B_3 =& \left[\left(2ca^2-ba-c\right)\varepsilon+c\left(2b^2+c^2-3abc\right):\right.\\ &\left.:\left(ac-b\right)\left(b^2-\varepsilon\right):\left(bc-a\varepsilon\right)\left(a^2-1\right)\right]\\ c_2 \equiv B_1B_3 =& \left[\left(ab-c\right)\left(c^2-\varepsilon\right):\left(2ba^2-ca-b\right)\varepsilon+\right.\\ &\left.+b\left(b^2+2c^2-3abc\right):\left(bc-a\varepsilon\right)\left(a^2-1\right)\right]\\ c_3 \equiv B_1B_2 =& \left[\left(ab-c\right)\left(c^2-\varepsilon\right):\left(ac-b\right)\left(b^2-\varepsilon\right):\right.\\ &\left.:a\left(a^2-1\right)\varepsilon+\left(2ab^2-3a^2bc+2ac^2-bc\right)\right]. \end{aligned}$$

The **Orthocenter** is arguably the most important point in triangle geometry, it is

$$\begin{aligned} h &\equiv N_1N_2 = N_2N_3 = N_1N_3 \\ &= \left[(b-ac)(a\varepsilon-bc):(c-ab)(a\varepsilon-bc):(ac-b)(ab-c)\right]. \end{aligned}$$

The dual line is the **Ortholine**

$$H \equiv n_1n_2 = n_1n_3 = n_2n_3 = \langle ab:ac:bc\rangle.$$

The **Orthic triangle** $\overline{b_1b_2b_3}$ is perspective with the Triangle $\overline{a_1a_2a_3}$ with center of perspectivity the Orthocenter h.

The *Triangle Base center theorem* states that the **Orthic dual triangle** $\overline{c_1c_2c_3}$ is perspective with the Triangle $\overline{a_1a_2a_3}$. The center of perspectivity is the **Base center**

$$b = \left[(ab-c)\left(c^2-\varepsilon\right):(ac-b)\left(b^2-\varepsilon\right):(bc-\varepsilon a)\left(a^2-1\right)\right]$$

with dual line the **Base axis**

$$B = \langle c+ab:b+ac:\varepsilon a+bc\rangle.$$

In Figure 12 we see the Altitudes, Orthocenter h and the dual Ortholine H, the Orthic triangle $\overline{b_1b_2b_3}$, Orthic dual triangle $\overline{c_1c_2c_3}$, base center b and Base axis B.

3.3 Desargues points and the Orthoaxis

The **Desargues points** are the meets of corresponding Orthic lines and Lines:

$$\begin{aligned} g_1 &\equiv C_1L_1 = [0:bc-\varepsilon a:b-ac]\\ g_2 &\equiv C_2L_2 = [bc-\varepsilon a:0:c-ab]\\ g_3 &\equiv C_3L_3 = [b-ac:ab-c:0] \end{aligned}$$

and the dual **Desargues lines** are

$$\begin{aligned} G_1 &= \left\langle b^2-a^2\varepsilon:2bc-ac^2-a\varepsilon:bc^2+b\varepsilon-2ac\varepsilon\right\rangle\\ G_2 &= \left\langle 2bc-ab^2-a\varepsilon:c^2-a^2\varepsilon:b^2c+c\varepsilon-2ab\varepsilon\right\rangle\\ G_3 &= \left\langle b+a^2b-2ac:2ab-c-a^2c:b^2-c^2\right\rangle. \end{aligned}$$

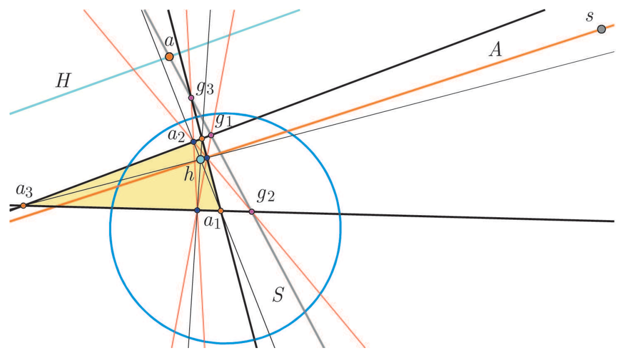

Figure 13: *Desargues points, Orthic axis S and Orthoaxis A*

Desargues' theorem implies that the Desargues points g_1, g_2, g_3 are collinear. They lie on the **Orthic axis**

$$S = \langle ab-c:ac-b:bc-a\varepsilon\rangle. \qquad (12)$$

Dually the Desargues lines G_1, G_2, G_3 are concurrent, passing through the **Orthostar**

$$s = \begin{bmatrix} \left(2ca^2-3ba+c\right)\varepsilon+c\left(2b^2-c^2-abc\right): \\ \left(2ba^2-3ca+b\right)\varepsilon-b\left(b^2-2c^2+abc\right): \\ a\left(1-a^2\right)\varepsilon+\left(2ab^2-a^2bc+2ac^2-3bc\right) \end{bmatrix}.$$

The **Orthoaxis** A**,** introduced in [16], is arguably the most important line in hyperbolic triangle geometry; it and its dual the **Orthoaxis point** a are

$$\begin{aligned} A \equiv sh =& \left\langle(ab-c)\left(a^2\varepsilon-b^2\right):(b-ac)\left(a^2\varepsilon-c^2\right):\right.\\ &\left.:(bc-a\varepsilon)\left(b^2-c^2\right)\right\rangle\\ a \equiv SH =& \left[c\left(a^2\varepsilon-b^2\right):b\left(c^2-\varepsilon a^2\right):a\left(b^2-c^2\right)\right]. \end{aligned}$$

The *Base center on Orthoaxis theorem* asserts that the Orthoaxis A passes through the Base center b.

3.4 Parallels and the Double triangle

Recall from [14] that the **parallel line** P **through a point** a **to a line** L is the line through a perpendicular to the altitude from a to L. This motivates the definition of the Double triangle of a Triangle. The **Parallel lines**

$$\begin{aligned} P_1 &\equiv a_1n_1 = \langle 0:a:b\rangle\\ P_2 &\equiv a_2n_2 = \langle a:0:c\rangle\\ P_3 &\equiv a_3n_3 = \langle b:c:0\rangle \end{aligned}$$

are the joins of corresponding Points a and Altitude points n, and their duals are the **Parallel points**

$$\begin{aligned} p_1 &= \left[b^2-2abc+a^2\varepsilon:bc-a\varepsilon:ac-b\right]\\ p_2 &= \left[bc-a\varepsilon:c^2-2abc+a^2\varepsilon:ab-c\right]\\ p_3 &= \left[\varepsilon(ac-b):\varepsilon(ab-c):b^2-2abc+c^2\right]. \end{aligned}$$

Assuming $a \neq 0$, $b \neq 0$ and $c \neq 0$, the meets of Parallel lines are the **Double points**

$$d_1 \equiv P_2P_3 = [-c : b : a]$$
$$d_2 \equiv P_1P_3 = [c : -b : a]$$
$$d_3 \equiv P_1P_2 = [c : b : -a]$$

and their duals are the **Double lines**

$$D_1 \equiv p_2p_3 = \langle 2ab - c : b : \varepsilon a \rangle$$
$$D_2 \equiv p_1p_3 = \langle c : 2ac - b : \varepsilon a \rangle$$
$$D_3 \equiv p_1p_2 = \langle c : b : 2bc - \varepsilon a \rangle .$$

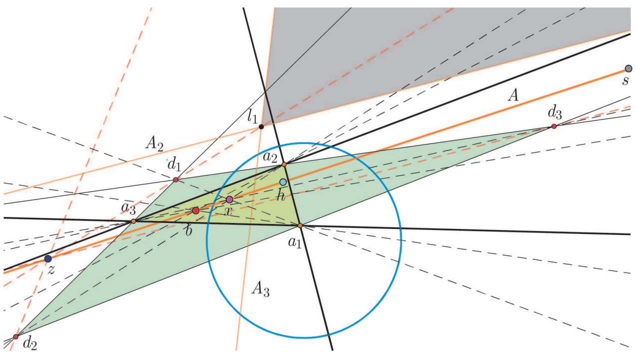

Figure 14: *The Double triangle, Orthoaxis A, and the points z, b, x, h and s*

We give here another proof of the following result, involving a simpler computation than in [16].

Theorem 7 (Double triangle midpoint) *The Points a_1, a_2, a_3 are midpoints of the Double triangle $\overline{d_1d_2d_3}$.*

Proof. We compute

$$q(d_1, a_3) = \frac{-b^2 - c^2 + 2abc}{a^2 - b^2 - c^2 + 2abc} = q(d_2, a_3).$$

Similarly, a_1 is a midpoint of $\overline{d_2d_3}$, and a_2 is a midpoint of $\overline{d_1d_3}$. □

The *Double triangle perspectivity theorem* states that the Double triangle $\overline{d_1d_2d_3}$ and the Triangle $\overline{a_1a_2a_3}$ are perspective from a point, the **Double point**, or x **point**

$$x = [c : b : a]$$

which lies on the Orthoaxis A. The proof is very simple in these coordinates: we compute that

$$a_1d_1 = \langle 0 : -a : b \rangle$$
$$a_2d_2 = \langle a : 0 : -c \rangle$$
$$a_3d_3 = \langle -b : c : 0 \rangle$$

and then observe that these lines meet at x.

The dual of the x point is the X **line**

$$X = \langle 2ab + c : 2ac + b : 2bc + a\varepsilon \rangle .$$

The *Double dual triangle perspectivity theorem* asserts that the Double triangle $\overline{d_1d_2d_3}$ and the Dual triangle $\overline{l_1l_2l_3}$ are perspective from a point, the **Double dual point**, or z **point**

$$z = \begin{bmatrix} \left(ca^2 - 2ba + c\right)\varepsilon + c\left(b^2 - c^2\right) : \\ \left(ba^2 - 2ca + b\right)\varepsilon - b\left(b^2 - c^2\right) : \\ a\left(1 - a^2\right)\varepsilon + ab^2 - 2bc + ac^2 \end{bmatrix}.$$

Its dual is the Z **line**

$$Z = \langle c : b : \varepsilon a \rangle .$$

The z point lies on the Orthoaxis A, or equivalently the Orthoaxis point a lies on the Z line.

4 The Circumcenter hierarchy

We now begin the study of the Circumcenter hierarchy. The basic assumption that we used to set up circumlinear coordinates was that each side of the triangle was either a midside or a sydside. We wish to treat both cases symmetrically, hence we introduce the notion that a **smydpoint** n of the side $\overline{ab}$ is either a midpoint or a sydpoint (or possibly both). Smydpoints exists precisely when $1 - q(a, b)$ is either a square or the negative of a square (or possibly both). Our diagrams will illustrate the situation when one side has midpoints and the other two sides have sydpoints. We introduce consistent labelling to bring out the four-fold symmetry in this situation.

4.1 Circumcenters, medians and centroids

By the Side midpoints and Side sydpoints theorems, in Circumlinear coordinates the smydpoints are

$$n_{1+} = [0 : 1 : 1] \text{ and } n_{1-} = [0 : -1 : 1] \quad \text{on } \overline{a_2a_3}$$
$$n_{2+} = [1 : 0 : 1] \text{ and } n_{2-} = [1 : 0 : -1] \quad \text{on } \overline{a_1a_3}$$
$$n_{3+} = [1 : 1 : 0] \text{ and } n_{3-} = [1 : -1 : 0] \quad \text{on } \overline{a_1a_2}.$$

Note that the indices of our labelling reflect the positions and relative signs of the non-zero entries.

Theorem 8 (Circumlines/Circumcenters) *The six Smydpoints lie three at a time on four* ***Circumlines***

$$C_0 \equiv n_{1-}n_{2-}n_{3-} = \langle 1 : 1 : 1 \rangle$$
$$C_1 \equiv n_{1-}n_{2+}n_{3+} = \langle -1 : 1 : 1 \rangle$$
$$C_2 \equiv n_{2-}n_{1+}n_{3+} = \langle 1 : -1 : 1 \rangle$$
$$C_3 \equiv n_{3-}n_{1+}n_{2+} = \langle 1 : 1 : -1 \rangle .$$

The duals are the ***Circumcenters***

$$c_0 = C_0^{\perp} = \begin{bmatrix} (a-1)\varepsilon - c(a+b-c)+b : \\ (a-1)\varepsilon - b(a-b+c)+c : \\ (a-1)(a-b-c+1) \end{bmatrix}$$

$$c_1 = C_1^{\perp} = \begin{bmatrix} (a+1)\varepsilon - c(a+b+c)+b : \\ c-(a+1)\varepsilon - b(a-b-c) : \\ (a+1)(a-b+c-1) \end{bmatrix}$$

$$c_2 = C_2^{\perp} = \begin{bmatrix} b-(a+1)\varepsilon - c(a-b-c) : \\ (a+1)\varepsilon - b(a+b+c)+c : \\ (a+1)(a+b-c-1) \end{bmatrix}$$

$$c_3 = C_3^{\perp} = \begin{bmatrix} b-(a-1)\varepsilon - c(a-b+c) : \\ c-(a-1)\varepsilon - b(a+b-c) : \\ (a-1)(a+b+c+1) \end{bmatrix}.$$

Proof. The formulas for the Circumlines can be checked immediately, the Circumcenter formulas are computations using duality. □

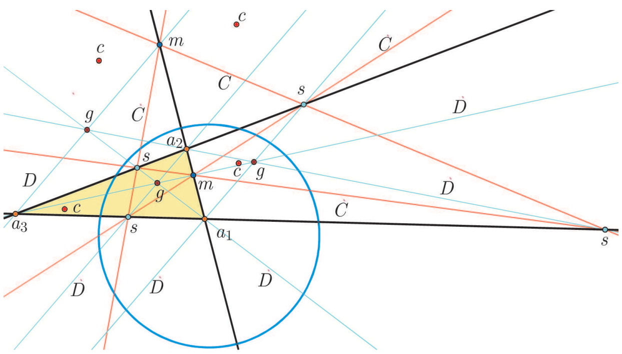

Figure 15: *Circumlines, Circumcenters, Medians and Centroids*

Median lines (or just **medians**) are joins of Points a and Smydpoints n which lie on the opposite lines:

$$D_{1-} \equiv a_1 n_{1-} = \langle 0:1:1\rangle \qquad D_{1+} \equiv a_1 n_{1+} = \langle 0:-1:1\rangle$$
$$D_{2+} \equiv a_2 n_{2+} = \langle 1:0:-1\rangle \qquad D_{2-} \equiv a_2 n_{2-} = \langle 1:0:1\rangle$$
$$D_{3-} \equiv a_3 n_{3-} = \langle 1:1:0\rangle \qquad D_{3+} \equiv a_3 n_{3+} = \langle -1:1:0\rangle.$$

Figure 15 shows the six Medians and their meets.

Theorem 9 (Centroids) *The Median lines D are concurrent in threes, meeting at four **Centroid points***

$$g_0 \equiv D_{1+}D_{2+}D_{3+} = [1:1:1]$$
$$g_1 \equiv D_{1+}D_{2-}D_{3-} = [-1:1:1]$$
$$g_2 \equiv D_{1-}D_{2+}D_{3-} = [1:-1:1]$$
$$g_3 \equiv D_{1-}D_{2-}D_{3+} = [1:1:-1].$$

*The dual **Centroid lines** are*

$$G_0 = \langle a+b+1 : a+c+1 : b+c+\varepsilon\rangle$$
$$G_1 = \langle a+b-1 : c-a+1 : c-b+\varepsilon\rangle$$
$$G_2 = \langle b-a+1 : a+c-1 : b-c+\varepsilon\rangle$$
$$G_3 = \langle a-b+1 : a-c+1 : b+c-\varepsilon\rangle.$$

Proof. Straightforward. □

4.2 CircumCentroids

While many aspects of the Circumcenter hierarchy are independent of ε, there are some that are not. The following is an extension of the similarly named result in [16].

Theorem 10 (CircumCentroid axis) *The meets of corresponding Circumlines and Centroid lines are collinear precisely when either $b = \pm c$ or $\varepsilon = 1$. If $\varepsilon = 1$, the common line is the Z axis $\langle c : b : \varepsilon a\rangle$, and the joins of corresponding Circumcenters and Centroid points meet at the z point. If $b = c$, then the common line is $\langle b : b : a+\varepsilon-1\rangle$, while if $b = -c$, then the common line is $\langle -b : b : a-\varepsilon+1\rangle$.*

Proof. The meets of Circumlines C_0, C_1, C_2, C_3 and corresponding Centroid lines G_0, G_1, G_2, G_3 are the four **CircumCentroid points**

$$z_0 \equiv C_0G_0 = [a-b-\varepsilon+1 : c-a+\varepsilon-1 : b-c]$$
$$z_1 \equiv C_1G_1 = [b-a-\varepsilon+1 : 1-a-c-\varepsilon : b+c]$$
$$z_2 \equiv C_2G_2 = [1-a-b-\varepsilon : c-a-\varepsilon+1 : b+c]$$
$$z_3 \equiv C_3G_3 = [a+b-\varepsilon+1 : \varepsilon-a-c-1 : b-c].$$

The determinants

$$\det \begin{bmatrix} a-b-\varepsilon+1 & c-a+\varepsilon-1 & b-c \\ b-a-\varepsilon+1 & 1-a-c-\varepsilon & b+c \\ 1-a-b-\varepsilon & c-a-\varepsilon+1 & b+c \end{bmatrix} = -4\left(b^2-c^2\right)(\varepsilon-1)$$

$$\det \begin{bmatrix} a-b-\varepsilon+1 & c-a+\varepsilon-1 & b-c \\ 1-a-b-\varepsilon & c-a-\varepsilon+1 & b+c \\ a+b-\varepsilon+1 & \varepsilon-a-c-1 & b-c \end{bmatrix} = 4\left(b^2-c^2\right)(\varepsilon-1)$$

show that the CircumCentroid points are collinear precisely when $\varepsilon = 1$ or $b = \pm c$. If $\varepsilon = 1$ the common line is $\langle c : b : a\rangle$ which in this case agrees with $Z = \langle c : b : \varepsilon a\rangle$. If $b = c$ we can check that the common line is $\langle b : b : a+\varepsilon-1\rangle$, and if $b = -c$ the common line is $\langle -b : b : a-\varepsilon+1\rangle$. □

4.3 Twin Circumcircles of a Triangle

If a triangle has three midsides, then corresponding Circumcenters will be centers of circles which pass through all three points, as in the classical triangle in Figure 1. This situation also holds for a triangle such as $\overline{a_1a_2a_3}$ in Figure 16, lying outside the null circle (still in blue) shown with three of its Midpoints m, (the other three are off the page), six Midlines M, three of the four Circumlines C, the four Circumcenters c, and the corresponding Circumcircles.

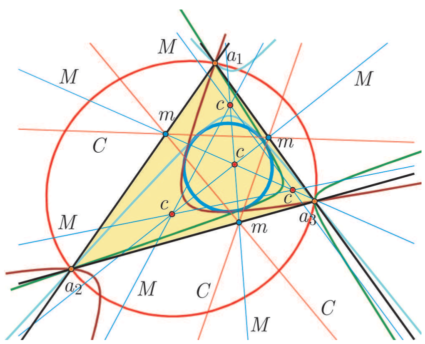

Figure 16: *Circumcenters of a triangle outside the null circle*

But what happens if a triangle has some points inside and some outside the null circle? In that case it turns out that we need to consider special *pairs* of circles, which collectively play the role of circumcircles. We do not know of any classical precedents for this phenomenon.

Definition 6 *Twin circles $\mathcal{C}$ and $\overline{\mathcal{C}}$ are* ***twin circumcircles*** *for a triangle $\overline{a_1a_2a_3}$ precisely when each of a_1, a_2, a_3 lie on either $\mathcal{C}$ or $\overline{\mathcal{C}}$.*

Theorem 11 (Twin circumcircles) *If a triangle $\overline{a_1a_2a_3}$ has smydpoints on all three sides, then the four circumcenters c_0, c_1, c_2, c_3 are each the center of twin circumcircles for $\overline{a_1a_2a_3}$.*

Proof. If n is a smydpoint of the side $\overline{a_ka_l}$ then its dual $n^{\perp}$ passes through two circumcenters, say c_i and c_j. Let's consider just c_i. If n is a sydpoint of $\overline{a_ka_l}$ then the Sydpoint twin circle theorem shows that the circles $\mathcal{C}_{c_i}^{(a_k)}$ and $\mathcal{C}_{c_i}^{(a_l)}$ are twin circles. If n is a midpoint of $\overline{a_ka_l}$ then the reflection r_n interchanges a_k and a_l and fixes both c_i and c_j, so that $\mathcal{C}_{c_i}^{(a_k)}$ and $\mathcal{C}_{c_i}^{(a_l)}$ coincide.

Since c_i is perpendicular to two smydpoints on different lines of the triangle $\overline{a_1a_2a_3}$, the argument can be repeated, so that either there is one circle with center at c_i that passes through all three points, or one of the twin circles $\mathcal{C}_{c_i}^{(a_k)}$ and $\mathcal{C}_{c_i}^{(a_l)}$ also passes through the third point of the triangle, in which case these are twin circumcircles. $\square$

Now let's introduce some labelling and explicit formulas. Consider the circles $\mathcal{C}_i = \mathcal{C}_i^{(a_3)}$ centered at c_i and passing through a_3, for $i = 0,1,2,3$. Their equations $q(p,c_i) = q(c_i,a_3)$ in a variable point $p = [x:y:z]$, can be written, after factoring a common term $-\varepsilon + a^2\varepsilon + b^2 + c^2 - 2abc$, as

$$\mathcal{C}_0\colon (1-\varepsilon)\left(x^2+y^2\right) + 2(a-\varepsilon)xy + 2(b-\varepsilon)xz + 2(c-\varepsilon)yz = 0$$
$$\mathcal{C}_1\colon (1-\varepsilon)\left(x^2+y^2\right) + 2(a+\varepsilon)xy + 2(b+\varepsilon)xz + 2(c-\varepsilon)yz = 0$$
$$\mathcal{C}_2\colon (1-\varepsilon)\left(x^2+y^2\right) + 2(a+\varepsilon)xy + 2(b-\varepsilon)xz + 2(c+\varepsilon)yz = 0$$
$$\mathcal{C}_3\colon (1-\varepsilon)\left(x^2+y^2\right) + 2(a-\varepsilon)xy + 2(b+\varepsilon)xz + 2(c+\varepsilon)yz = 0.$$

The respective twin circles $\overline{\mathcal{C}_i}$ with equations $q(p,c_i) = 2 - q(c_i,a_3)$ can be written as

$$\begin{aligned}\overline{\mathcal{C}}_0 : (1+\varepsilon)\left(x^2+y^2\right) + 2\varepsilon z^2 + 2(a+\varepsilon)xy \\ + 2(b+\varepsilon)xz + 2(c+\varepsilon)yz = 0\end{aligned}$$
$$\begin{aligned}\overline{\mathcal{C}}_1 : (1+\varepsilon)\left(x^2+y^2\right) + 2\varepsilon z^2 + 2(a-\varepsilon)xy \\ + 2(b-\varepsilon)xz + 2(c+\varepsilon)yz = 0\end{aligned}$$
$$\begin{aligned}\overline{\mathcal{C}}_2 : (1+\varepsilon)\left(x^2+y^2\right) + 2\varepsilon z^2 + 2(a-\varepsilon)xy \\ + 2(b+\varepsilon)xz + 2(c-\varepsilon)yz = 0\end{aligned}$$
$$\begin{aligned}\overline{\mathcal{C}}_3 : (1+\varepsilon)\left(x^2+y^2\right) + 2\varepsilon z^2 + 2(a+\varepsilon)xy \\ + 2(b-\varepsilon)xz + 2(c-\varepsilon)yz = 0.\end{aligned}$$

If $\varepsilon = 1$, then each of the four circumcircles $\mathcal{C}_i$ passes through all three points of the triangle, while their twins $\overline{\mathcal{C}}_i$ pass through none of the points of the triangle; even so, their presence is felt.

In Figure 17 we see a triangle $\overline{a_1a_2a_3}$ with all three points inside the null circle, together with its four pairs of twin circumcircles, each pair with the same colour. The reader might enjoy looking for interesting relations between these circles.

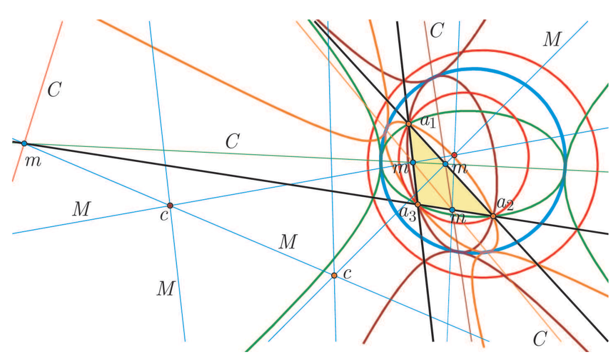

Figure 17: *Twin circumcircles for a classical triangle*

4.4 CircumDual points, Tangent lines and Sound points

If $\varepsilon = -1$, then the circumcircles $\mathcal{C}_i$ pass only through c_3, while the twins $\overline{\mathcal{C}}_i$ pass through c_1 and c_2. In each case we have four twin circumcircle pairs of the Triangle**.** These eight circles are shown for our standard example Triangle in Figure 18, along with the Tangent lines, which we now introduce.

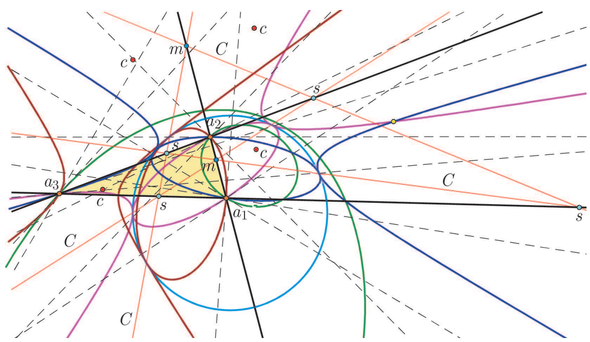

Figure 18: *Twin Circumcircles and Tangent lines*

The **CircumDual point** p_{ij} is the meet of the Dual line A_i and the Circumline C_j, for $i = 1,2,3$ and $j = 0,1,2,3$. Then

$$p_{10}=[a-b:b-1:-a+1] \qquad p_{20}=[c-1:a-c:-a+1]$$
$$p_{11}=[a-b:-b-1:a+1] \qquad p_{21}=[1-c:-a-c:a+1]$$
$$p_{12}=[a+b:b-1:-a-1] \qquad p_{22}=[c+1:c-a:-a-1]$$
$$p_{13}=[-a-b:b+1:1-a] \qquad p_{23}=[-c-1:a+c:a-1]$$

$$p_{30} = [\varepsilon-c:b-\varepsilon:-b+c]$$
$$p_{31} = [c-\varepsilon:-b-\varepsilon:b+c]$$
$$p_{32} = [-c-\varepsilon:b-\varepsilon:b+c]$$
$$p_{33} = [-c-\varepsilon:b+\varepsilon:b-c].$$

The **Tangent line** T_{ij} is the join of the CircumDual point p_{ij} and the point a_i. This line is indeed tangent to the circumcircle $\mathcal{C}_i$ at the point a_i if this circle passes through a_i. The twelve Tangent lines are:

$$T_{10} = \langle 0:a-1:b-1\rangle \qquad T_{20} = \langle a-1:0:c-1\rangle$$
$$T_{11} = \langle 0:a+1:b+1\rangle \qquad T_{21} = \langle a+1:0:c-1\rangle$$
$$T_{12} = \langle 0:a+1:b-1\rangle \qquad T_{22} = \langle a+1:0:c+1\rangle$$
$$T_{13} = \langle 0:a-1:b+1\rangle \qquad T_{23} = \langle a-1:0:c+1\rangle$$

$$T_{30} = \langle b-\varepsilon:c-\varepsilon:0\rangle$$
$$T_{31} = \langle b+\varepsilon:c-\varepsilon:0\rangle$$
$$T_{32} = \langle b-\varepsilon:c+\varepsilon:0\rangle$$
$$T_{33} = \langle b+\varepsilon:c+\varepsilon:0\rangle.$$

The **Sound point** s_{ij} is the meet of the Tangent line T_{ij} with the opposite Line L_i. The twelve Sound points are:

$$s_{10} = [0:1-b:a-1] \qquad s_{20} = [1-c:0:a-1]$$
$$s_{11} = [0:-b-1:a+1] \qquad s_{21} = [1-c:0:a+1]$$
$$s_{12} = [0:1-b:a+1] \qquad s_{22} = [-1-c:0:a+1]$$
$$s_{13} = [0:b+1:1-a] \qquad s_{23} = [1+c:0:1-a]$$

$$s_{30} = [\varepsilon-c:b-\varepsilon:0]$$
$$s_{31} = [\varepsilon-c:b+\varepsilon:0]$$
$$s_{32} = [c+\varepsilon:\varepsilon-b:0]$$
$$s_{33} = [-c-\varepsilon:b+\varepsilon:0].$$

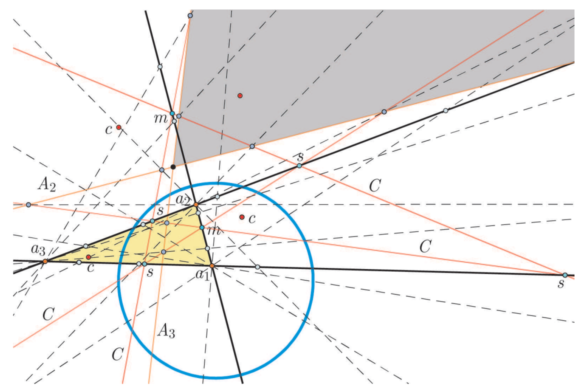

Figure 19: *CircumDual points and Sound points*

4.5 Jay and Wren lines

In this section we begin to see more divergence between the $\varepsilon = 1$ and $\varepsilon = -1$ cases. In the latter case a symmetry emerges between the Circumcenters c_0 and c_3, and between c_1 and c_2.

Theorem 12 (Jay lines) *If $\varepsilon = 1$ then the sets of Sound points $\{s_{10}, s_{20}, s_{30}\}$, $\{s_{11}, s_{21}, s_{31}\}$, $\{s_{12}, s_{22}, s_{32}\}$ and $\{s_{13}, s_{23}, s_{33}\}$ are each collinear, while if $\varepsilon = -1$ then the sets of Sound points $\{s_{10}, s_{20}, s_{33}\}$, $\{s_{11}, s_{21}, s_{32}\}$, $\{s_{12}, s_{22}, s_{31}\}$ and $\{s_{13}, s_{23}, s_{30}\}$ are each collinear. In both cases the common lines are respectively the four **Jay lines***

$$J_0 = \langle (a-1)(b-1):(a-1)(c-1):(c-1)(b-1)\rangle$$
$$J_1 = \langle (a+1)(b+1):(a+1)(c-1):(c-1)(b+1)\rangle$$
$$J_2 = \langle (a+1)(b-1):(a+1)(c+1):(c+1)(b-1)\rangle$$
$$J_3 = \langle (a-1)(b+1):(a-1)(c+1):(c+1)(b+1)\rangle.$$

Proof. The forms of the Sound points and Jay lines make verifying these incidences almost trivial. Note that changing the sign of ε interchanges s_{30} with s_{33}, and s_{31} with s_{32}. This explains why the two lists appear different in these two cases. □

In the case of $\varepsilon = 1$ we associate each triple of Sound points to the Circumline which is involved in each term. In the case of $\varepsilon = -1$ we associate each triple to the Circumline which is involved in *two* of the three elements of the triple.

There are four meets of Circumlines and associated Jay lines called **CircumJay points**, namely

$$t_0 \equiv C_0J_0 = [(c-1)(a-b) : (-b+1)(a-c) : (a-1)(b-c)]$$
$$t_1 \equiv C_1J_1 = [(c-1)(a-b) : -(b+1)(a+c) : (a+1)(b+c)]$$
$$t_2 \equiv C_2J_2 = [(c+1)(a+b) : (1-b)(a-c) : -(a+1)(b+c)]$$
$$t_3 \equiv C_3J_3 = [-(c+1)(a+b) : (b+1)(a+c) : (a-1)(b-c)].$$

Note that these formulas are independent of ε.

Theorem 13 (CircumJay) *The four CircumJay points t_0, t_1, t_2, t_3 are collinear and lie on the line*

$$T = \langle c+ab : b+ac : a+bc \rangle.$$

*When $\varepsilon = 1$ this coincides with the Base axis B. When $\varepsilon = -1$, this is a new line which we call the T **axis**. In the case of $\varepsilon = -1$, T, B and L_3 are concurrent at a new point*

$$\overline{t} = [-(b+ac) : c+ab : 0].$$

Proof. The CircumJay point t_0 lies on T since

$$(c-1)(a-b)(c+ab) + (-b+1)(a-c)(b+ac) + (a-1)(b-c)(a+bc) = 0$$

and similarly for the other points. The T axis agrees with the Base axis $B = \langle c+ab : b+ac : \varepsilon a+bc \rangle$ if $\varepsilon = 1$. For $\varepsilon = -1$, the verification of $\overline{t} = TB$ is also straightforward, and clearly it lies on L_3. □

Theorem 14 (Wren lines) *If $\varepsilon = 1$ then the sets of Sound points $\{s_{11}, s_{22}, s_{33}\}$, $\{s_{10}, s_{32}, s_{23}\}$, $\{s_{31}, s_{20}, s_{13}\}$ and $\{s_{21}, s_{12}, s_{30}\}$ are each collinear, while if $\varepsilon = -1$ then the sets of Sound points $\{s_{11}, s_{22}, s_{30}\}$, $\{s_{10}, s_{23}, s_{31}\}$, $\{s_{13}, s_{20}, s_{32}\}$ and $\{s_{12}, s_{21}, s_{33}\}$ are each collinear. In both cases the common lines are respectively the four **Wren lines***

$$W_0 = \langle (a+1)(b+1) : (a+1)(c+1) : (b+1)(c+1) \rangle$$
$$W_1 = \langle (a-1)(b-1) : (c+1)(a-1) : (c+1)(b-1) \rangle$$
$$W_2 = \langle (b+1)(a-1) : (a-1)(c-1) : (b+1)(c-1) \rangle$$
$$W_3 = \langle (a+1)(b-1) : (a+1)(c-1) : (b-1)(c-1) \rangle.$$

Proof. Again, with the formulas for Sound points and Wren lines, it is straightforward to check incidences. As with the Jay lines, changing the sign of ε interchanges s_{03} with s_{33}, and s_{13} with s_{23}. □

Notice that each set of collinear Sound points is associated to the Circumcenter which is not involved in the indices of that group. **CircumWren points** are the meets of Circumlines and associated Wren lines. These points are

$$u_0 \equiv C_0W_0 = [(c+1)(a-b) : -(b+1)(a-c) : (a+1)(b-c)]$$
$$u_1 \equiv C_1W_1 = [(c+1)(a-b) : (-b+1)(a+c) : (a-1)(b+c)]$$
$$u_2 \equiv C_2W_2 = [(c-1)(a+b) : -(b+1)(a-c) : (-a+1)(b+c)]$$
$$u_3 \equiv C_3W_3 = [(-c+1)(a+b) : (b-1)(a+c) : (a+1)(b-c)].$$

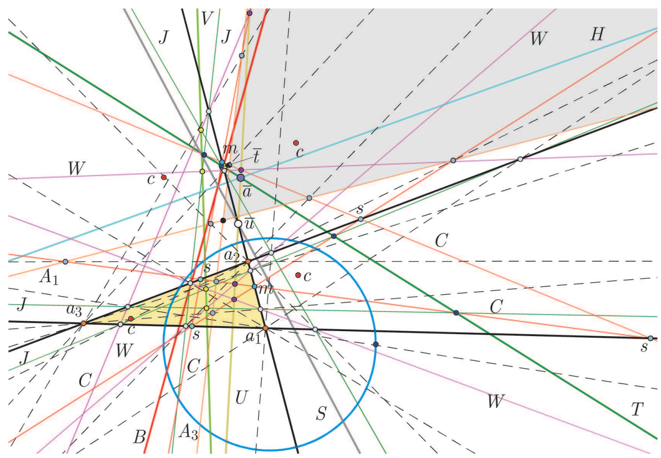

Figure 20: *Jay lines J, Wren lines W, T, U, V axes and new points $\overline{a}, \overline{u}, \overline{t}$*

Theorem 15 (CircumWren) *The four CircumWren points u_0, u_1, u_2, u_3 are collinear and lie on the line*

$$U \equiv \langle ab-c : ac-b : bc-a \rangle.$$

*When $\varepsilon = 1$ this coincides with the Orthic axis S. When $\varepsilon = -1$, this is a new line which we call the U **axis**. In case $\varepsilon = -1$, S, U and L_3 are concurrent in a new point*

$$\overline{u} = [ac-b : c-ab : 0].$$

Proof. We may compute that v_0 lies on U since

$$(c+1)(a-b)(ab-c) - (b+1)(a-c)(ac-b) + (a+1)(b-c)(bc-a) = 0.$$

The other incidences are similar. From (12) we recall that the Orthic axis has equation $S = \langle ab-c : ac-b : bc-a\varepsilon \rangle$ which agrees with U precisely when $\varepsilon = 1$. Again the formula for $\overline{u}$ is easy. □

In Figure 20 we see the CircumJay points t_j (dark blue) on T, the CircumWren points u_j (purple) on U, and the JayWren points v_j (yellow) on V.

Theorem 16 (CircumJayWren) *The lines U, T and H are concurrent, and pass through*

$$\overline{a} \equiv \left[c\left(a^2-b^2\right):b\left(c^2-a^2\right):a\left(b^2-c^2\right)\right]. \qquad (13)$$

If $\varepsilon = 1$ then $\overline{a}$ agrees with the Orthoaxis point $a = \left[c\left(a^2\varepsilon-b^2\right):b\left(c^2-\varepsilon a^2\right):a\left(b^2-c^2\right)\right]$.

Proof. The concurrence of these lines follows from

$$\det\begin{bmatrix} ab-c & ac-b & bc-a \\ c+ab & b+ac & a+bc \\ ab & ac & bc \end{bmatrix} = 0.$$

The common incidence with (13) is also readily checked. The last statement is self-evident. □

There are four **JayWren points** which are the meets of associated Jay lines and Wren lines:

$$\begin{aligned}
v_0 &= J_0W_0 = \left[\left(c^2-1\right)(a-b):\left(b^2-1\right)(c-a):\left(a^2-1\right)(b-c)\right] \\
v_1 &= J_1W_1 = \left[\left(c^2-1\right)(a-b):\left(b^2-1\right)(a+c):\left(1-a^2\right)(b+c)\right] \\
v_2 &= J_2W_2 = \left[\left(c^2-1\right)(a+b):\left(b^2-1\right)(a-c):\left(1-a^2\right)(b+c)\right] \\
v_3 &= J_3W_3 = \left[\left(c^2-1\right)(a+b):\left(1-b^2\right)(a+c):\left(a^2-1\right)(b-c)\right].
\end{aligned}$$

Theorem 17 (JayWren) *The four JayWren points v_0, v_1, v_2, v_3 are collinear and lie on the **JayWren axis**, or the V line*

$$V = \left\langle c\left(b^2-1\right)\left(a^2-1\right):b\left(c^2-1\right)\left(a^2-1\right):a\left(c^2-1\right)\left(b^2-1\right)\right\rangle.$$

Proof. The JayWren point v_0 lies on V since

$$\begin{aligned}
&\left(c^2-1\right)(a-b)c\left(b^2-1\right)\left(a^2-1\right) \\
&\quad -\left(b^2-1\right)(a-c)b\left(c^2-1\right)\left(a^2-1\right) \\
&\quad +\left(a^2-1\right)(b-c)a\left(c^2-1\right)\left(b^2-1\right) = 0.
\end{aligned}$$

Checking the other incidences is similar. □

4.6 CircumMeets and reflections

One of the interesting features of this situation concerns the meets of the eight **generalized circumcircles** forming the four twin circumcircles of a triangle with six smydpoints. We establish easily a basic fact.

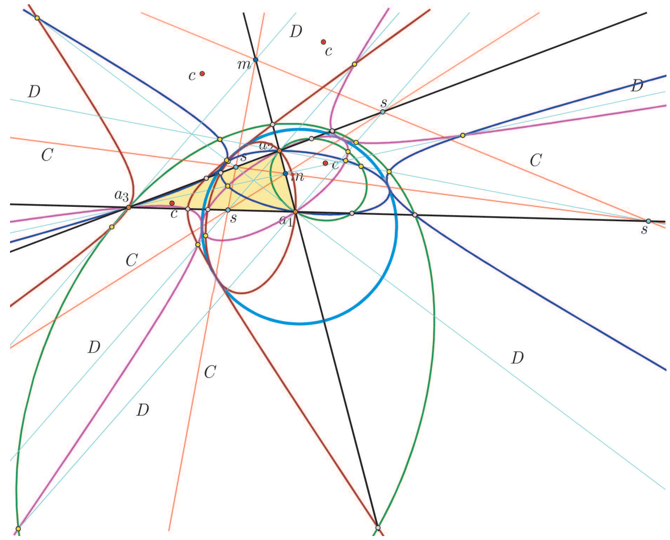

Figure 21: *Circumcircles and CircumMeet points*

Theorem 18 (Smydpoint reflection) *Suppose that a generalized circumcircle $\mathcal{C}$ has center c_j perpendicular to a smydpoint n. If $\mathcal{C}$ passes through a point a_k of the Triangle, then it also passes through the reflection $r_n(a_k)$.*

Proof. If n is perpendicular to c_j, then the reflection r_n in n fixes the center c_j of $\mathcal{C}$, and so fixes $\mathcal{C}$. Thus if $\mathcal{C}$ passes through a_k, it also passes through $r_n(a_k)$. □

This theorem helps explain why in Figure 21 the meets of the generalized circumcircles lie either on the lines of the Triangle, or on the Medians. We see that reflections of Points in Sydpoints are also interesting points of the Triangle—in fact somewhat surprisingly these CircumMeet points are independent of the third Point of the Triangle, and depend only on the particular side on which they lie. The reader can verify with a dynamic geometry package that as we vary one point of the Triangle, the generalized circumcircles move, but their meets on the opposite Line do not.

In general meets of circles are complicated by number-theoretical issues (circles do not have to meet, after all). We conjecture that whenever generalized Circumcircles meet, they do so either on Lines or Medians. We hope to explain the more detailed structure of these CircumMeet points in a future paper.

4.7 Sound conics

The twelve sound points are quite interesting, supporting the linear structures of Jay and Wren lines. They also are connected with four special conics in an interesting way, each conic naturally also associated with a circumcenter.

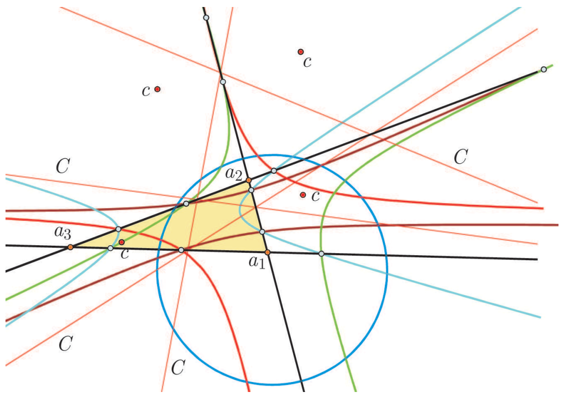

Figure 22: *Sound conics*

Theorem 19 *The sextuples* $\{s_{12},s_{13},s_{21},s_{23},s_{31},s_{32}\}$, $\{s_{12},s_{13},s_{20},s_{22},s_{30},s_{33}\}$, $\{s_{10},s_{11},s_{21},s_{23},s_{30},s_{33}\}$ *and* $\{s_{10},s_{11},s_{20},s_{22},s_{31},s_{32}\}$ *of sound points all lie on conics. Each of these four* ***Sound conics*** $\mathcal{K}_j$ *is associated to a Circumcenter* c_j.

Proof. We compute the coefficients of the equation of the (blue) conic

$$\mathcal{K}_0 : a_1x^2+a_2y^2+a_3z^2+a_4xy+a_5xz+a_6yz=0$$

passing through points $s_{12},s_{13},s_{21},s_{23},s_{31}$ by solving the linear system

$$(1-b)^2a_2+(a+1)^2a_3+(1-b)(a+1)a_6=0$$
$$(1+b)^2a_2+(1-a)^2a_3+(1+b)(1-a)a_6=0$$
$$(1-c)^2a_1+(a+1)^2a_3+(1-c)(a+1)a_5=0$$
$$(1+c)^2a_1+(1-a)^2a_3+(1+c)(1-a)a_5=0$$
$$(\varepsilon-c)^2a_1+(b+\varepsilon)^2a_2+(\varepsilon-c)(b+\varepsilon)a_4=0$$

This results in the values

$$a_1=(c+\varepsilon)(b-\varepsilon)\left(b^2-1\right)\left(a^2-1\right)$$
$$a_2=(c+\varepsilon)(b-\varepsilon)\left(c^2-1\right)\left(a^2-1\right)$$
$$a_3=(c+\varepsilon)(b-\varepsilon)\left(c^2-1\right)\left(b^2-1\right)$$
$$a_4=2\left(a^2-1\right)(bc+1)(b\varepsilon-c\varepsilon+bc-1)$$
$$a_5=2(c+\varepsilon)(b-\varepsilon)\left(b^2-1\right)(ac+1)$$
$$a_6=2(c+\varepsilon)(b-\varepsilon)(ab+1)\left(c^2-1\right).$$

When substituting the coordinates of s_{32} in the above equation with these coefficients, we obtain equality precisely when

$$\left(\varepsilon^2-1\right)\left(a^2-1\right)\left((b-c)\left(4bc+b^2+c^2+2\right)\varepsilon + (bc-1)\left(b^2+c^2-2\right)\right)=0$$

which is true since $\varepsilon^2=1$.

By following the same argument, we can obtain the equations of the (red) conic

$$\mathcal{K}_1 : b_1x^2+b_2y^2+b_3z^2+b_4xy+b_5xz+b_6yz=0$$

through $s_{12},s_{13},s_{20},s_{22},s_{30},s_{33}$ with coefficients

$$b_1=(b+\varepsilon)(c+\varepsilon)\left(b^2-1\right)\left(a^2-1\right)$$
$$b_2=(b+\varepsilon)(c+\varepsilon)\left(c^2-1\right)\left(a^2-1\right)$$
$$b_3=(b+\varepsilon)(c+\varepsilon)\left(c^2-1\right)\left(b^2-1\right)$$
$$b_4=2(bc-1)\left(a^2-1\right)(b\varepsilon+c\varepsilon+bc+1)$$
$$b_5=2(b+\varepsilon)(c+\varepsilon)(ac-1)\left(b^2-1\right)$$
$$b_6=2(b+\varepsilon)(c+\varepsilon)(ab+1)\left(c^2-1\right),$$

the (green) conic

$$\mathcal{K}_2 : c_1x^2+c_2y^2+c_3z^2+c_4xy+c_5xz+c_6yz=0$$

through $s_{10},s_{11},s_{21},s_{23},s_{30},s_{33}$ with coefficients

$$c_1=(c+\varepsilon)(b+\varepsilon)\left(b^2-1\right)\left(a^2-1\right)$$
$$c_2=(c+\varepsilon)(b+\varepsilon)\left(c^2-1\right)\left(a^2-1\right)$$
$$c_3=(c+\varepsilon)(b+\varepsilon)\left(c^2-1\right)\left(b^2-1\right)$$
$$c_4=2(bc-1)(b\varepsilon+c\varepsilon+bc+1)\left(a^2-1\right)$$
$$c_5=2(c+\varepsilon)(b+\varepsilon)(ac+1)\left(b^2-1\right)$$
$$c_6=2(c+\varepsilon)(b+\varepsilon)(ab-1)\left(c^2-1\right),$$

and the (brown) conic

$$\mathcal{K}_3 : d_1x^2+d_2y^2+d_3z^2+d_4xy+d_5xz+d_6yz=0$$

through $s_{10},s_{11},s_{20},s_{22},s_{31},s_{32}$ with coefficients

$$d_1=(c+\varepsilon)(b-\varepsilon)\left(b^2-1\right)\left(a^2-1\right)$$
$$d_2=(c+\varepsilon)(b-\varepsilon)\left(c^2-1\right)\left(a^2-1\right)$$
$$d_3=(c+\varepsilon)(b-\varepsilon)\left(c^2-1\right)\left(b^2-1\right)$$
$$d_4=2(bc+1)(b\varepsilon-c\varepsilon+bc-1)\left(a^2-1\right)$$
$$d_5=2(c+\varepsilon)(b-\varepsilon)(ac-1)\left(b^2-1\right)$$
$$d_6=2(c+\varepsilon)(b-\varepsilon)(ab-1)\left(c^2-1\right).$$

We associate each Sound conic $\mathcal{K}_j$ to the Circumcenter c_j not involved in any of the six Sound points lying on it. □

5 Further directions

We can now extend hyperbolic triangle geometry from classical triangles to more general ones. Taking duals we get also analogous results for the Incenter hierarchy, and it is worthwhile to elaborate these and then investigate further the links between Incenter and Circumcenter hierarchies.

The close relations between twin circles ought to have consequences for relativistic physics, as points inside the null circle correspond to time-like lines and points outside to space-like lines. The geometry we are investigating suggests these two aspects of relativistic geometry ought to be much more closely linked.

Another direction is that over certain finite fields, we can expect some sides to have both midpoints and sydpoints! This is an interesting aspect for those with a number theoretical or combinatorial bend. It turns out that sydpoints play a big role in the theory of conics in UHG as well, as we will explain in a future paper.

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