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Nguyen Le
Norman John Wildberger

Universal Affine Triangle Geometry and Four-fold Incenter Symmetry

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ABSTRACT

We develop a generalized triangle geometry, using an arbitrary bilinear form in an affine plane over a general field. By introducing standardized coordinates we find canonical forms for some basic centers and lines. Strong concurrencies formed by quadruples of lines from the Incenter hierarchy are investigated, including joins of corresponding Incenters, Gergonne, Nagel, Spieker points, Mittenpunkts and the New points we introduce. The diagrams are taken from relativistic (green) geometry.

Key words: Triangle geometry, affine geometry, Rational trigonometry, bilinear form, incenter hierarchy, Euler line, Gergonne, Nagel, Mittenpunkt, chromogeometry

MSC 2000: 51M05, 51M10, 51N10

Univerzalna afina geometrija trokuta i četverostruka simetrija središta upisane kružnice

SAŽETAK

Razvijamo opću geometriju trokuta koristeći proizvoljnu bilinearnu formu u afinoj ravnini nad općim poljem. Uvodeći standardizirane koordinate pronalazimo kanonske oblike nekih osnovnih središta i pravaca. Proučavamo snažnu konkurentnost četvorki pravaca koji pripadaju "hijerarhiji središta upisane kružnice" uključujući i spojnice odgovarajućih sjecišta simetrala kutova trokuta, Georgonnovih točaka, Nagelovih točaka, Mittenpunkova (imenovano sa strane autora, op. ur.) te Novih točaka koje se uvode u članku. Slike su prikazane u tzv. zelenoj geometriji.

Ključne riječi: geometrija trokuta, afina geometrija, racionalna trigonometrija, bilinearna forma, hijerarhija središta upisane kružnice, Eulerov pravac, Georgonnova točka, Nagelova točka, Mittenpunkt, kromogeometrija

1 Introduction

This paper *repositions and extends triangle geometry* by developing it in the wider framework of Rational Trigonometry and Universal Geometry ([10], [11]), valid over arbitrary fields and with general quadratic forms. Our main focus is on strong concurrency results for quadruples of lines associated to the Incenter hierarchy.

Triangle geometry has a long and cyclical history ([1], [3], [16], [17]). The centroid $G = X_2$, circumcenter $C = X_3$, orthocenter $H = X_4$ and incenter $I = X_1$ were known to the ancient Greeks. Prominent mathematicians like Euler and Gauss contributed to the subject, but it took off mostly in the latter part of the 19th century and the first part of the 20th century, when many new centers, lines, conics, and cubics associated to a triangle were discovered and investigated. Then there was a period when the subject languished; and now it flourishes once more—spurred by the power of dynamic geometry packages like GSP, C.a.R., Cabri, GeoGebra, and Cinderella; by the heroic efforts of Clark Kimberling in organizing the massive amount of information on Triangle Centers in his Online Encyclopedia ([5], [6], [7]); and by the explorations and discussions of the Hyacinthos Yahoo group ([4]).

The increased interest in this rich and fascinating subject is to be applauded, but there are also mounting concerns about the consistency and accessibility of *proofs*, which have not kept up with the greater pace of *discoveries*. Another difficulty is that the current framework is modelled on the continuum as "real numbers", which often leads synthetic treatments to finesse number-theoretical issues.

One of our goals is to provide explicit algebraic formulas for points, lines and transformations of triangle geometry which hold in great generality, over the rational numbers, finite fields, and even the field of complex rational numbers, and with different bilinear forms determining the metrical structure without any recourse to transcendental quantities or "real numbers". Of course we proceed only a very small way down this road, but far enough to establish some analogs of results that have appeared first in Universal Hyperbolic Geometry ([14]); namely the concurrency of some *quadruples* of lines associated to the classical Incenters, Gergonne points, Nagel points, Mittenpunkts, Spieker points as well as the *New points* which we introduce here. We identify the resulting centers in Kimberling's list.

Our basic technology is simple but powerful: we propose to replace the affine study of a *general triangle under a particular bilinear form* with the study of a *particular triangle under a general bilinear form*—analogous to the projective situation as in ([14]), and using the framework of Rational Trigonometry ([10], [11]). By choosing a very elementary standard Triangle—with vertices the origin and the two standard basis vectors—we get reasonably pleasant and simple formulas for various points, lines and constructions. An affine change of coordinates changes any triangle under any bilinear form to the one we are studying, so our results are in fact very general.

Our principle results center around the classical four points, but a big difference with our treatment is that we acknowledge from the start that the very existence of the *Incenter hierarchy* is dependent on number-theoretical conditions which end up playing an intimate and ultimately rather interesting role in the theory. Algebraically it becomes difficult to separate the classical incenter from the three closely related excenters, and the quadratic relations that govern the existence of these carry a natural four-fold symmetry between them. This symmetry becomes crucial to simplifying formulas and establishing theorems. So in our framework, *there are four Incenters I_0, I_1, I_2 and I_3, not one*.

To showcase the generality of our results, we illustrate theorems not over the Euclidean plane, but in the *Minkowski plane* coming from *Einstein's special theory of relativity in null coordinates*, where the metrical structure is determined by the bilinear form

$$(x_1,y_1)\cdot(x_2,y_2) \equiv x_1y_2 + y_1x_2.$$

In the language of Chromogeometry ([12] , [13]), this is *green geometry*, with circles appearing as rectangular hyperbolas with asymptotes parallel to the coordinate axes. Green perpendicularity amounts to vectors being Euclidean reflections in these axes, while null vectors are parallel to the axes. It is eye-opening to see that triangle geometry is just as rich in such a relativistic setting as it is in the Euclidean one!

1.1 Summary of results

We summarize the main results of this paper using Figure 1 from green geometry. As established in ([13]), the triangle $\overline{A_1A_2A_3}$ has a *green Euler line CHG* just as in the Euclidean setting, where $C = X_3$ is the Circumcenter, $G = X_2$ is the Centroid, and $H = X_4$ is the Orthocenter, with the affine ratio $\overrightarrow{CG} : \overrightarrow{GH} = 1 : 2$, which we may express as $G = \frac{2}{3}C + \frac{1}{3}H$. The reader might like to check that using the green notation of perpendicularity, the green altitudes really do meet at H, and the green midlines/perpendicular bisectors really do meet at C.

In the general situation there are *four* Incenters/Excenters I_0, I_1, I_2 and I_3 which algebraically are naturally viewed symmetrically. Associated to any one Incenter I_j is a *Gergonne point $G_j = X_7$* (not to be confused with the centroid also labelled G), a *Nagel point $N_j = X_8$*, a *Mittenpunkt $D_j = X_9$*, a *Spieker point $S_j = X_{10}$* and most notably a *New point L_j*. It is not at all obvious that these various points can be defined for a general affine geometry, but this is the case, as we shall show. The New points L_0, L_1, L_2, L_3 are a particularly novel feature of this paper. They really do appear to be new, and it seems remarkable that these important points have not been intensively studied, as they fit naturally and simply into the Incenter hierarchy, as we shall see.

The four-fold symmetry between the four Incenters is maintained by all these points: so in fact there are *four* Gergonne, Nagel, Mittenpunkt, Spieker and New points, each associated to a particular Incenter, as also pointed out in ([8]). Figure 1 shows just one Incenter and its related hierarchy: as we proceed in this paper the reader will meet the other Incenters and hierarchies as well.

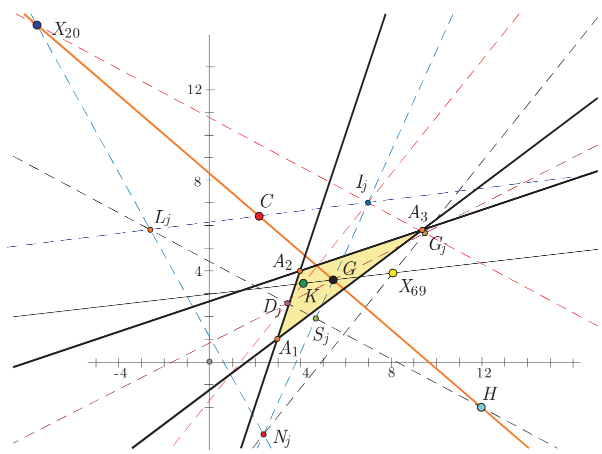

Figure 1: *Aspects of the Incenter hierarchy in green geometry*

The main aims of the paper are to set-up a coordinate system for triangle geometry that incorporates the number-theoretical aspects of the Incenter hierarchy, and respects the four-fold symmetry inherent in it, and then to use this to catalogue existing as well as new points and phenomenon. Kimberling's Triangle Center Encyclopedia ([6]) distinguishes the classical Incenter X_1 as the first and perhaps most important triangle center. Our embrace of the four-fold symmetry between incenters and excenters implies something of a re-evaluation of some aspects of classical triangle geometry; instead of certain distinguished centers we have rather distinguished quadruples of related points. Somewhat surprisingly, this point of view makes visible a number of remarkable *strong concurrences*—-where four symmetrically-defined lines meet in a center. The proofs of these relations are reasonably straight-forward but not automatic, as in general certain important quadratic relations are needed to simplify expressions for incidence. Here is a summary of our main results.

Main Results

i) The four lines $I_jG_j,\ j=0,1,2,3$, meet in the De Longchamps point X_{20} (orthocenter of the Double triangle) — these are the Soddy lines ([9]).

ii) The four lines I_jN_j meet in the Centroid $G=X_2$, and in fact $G=\frac{2}{3}I_j+\frac{1}{3}N_j$ — these are the Nagel lines. The Spieker points S_j also lie on the Nagel lines, and in fact $S_j=\frac{1}{2}I_j+\frac{1}{2}N_j$.

iii) The four lines I_jD_j meet in the Symmedian point $K=X_6$ (isogonal conjugate of the Centroid G) — the standard such line is labelled $L_{1,6}$ in [6].

iv) The four lines I_jL_j meet in the Circumcenter C, and in fact $C=\frac{1}{2}I_j+\frac{1}{2}L_j$ — the standard such line is labelled $L_{1,3}$.

v) The four lines G_jN_j meet in the point X_{69} (isotomic conjugate of the Orthocenter H) — these lines are labelled $L_{7,8}$.

vi) The four lines G_jD_j meet in the Centroid $G=X_2$, and in fact $G=\frac{2}{3}D_j+\frac{1}{3}G_j$ — the standard such line is labelled $L_{2,7}$.

vii) The four lines D_jS_j meet in the Orthocenter $H=X_4$ — the standard such line is labelled $L_{4,7}$.

viii) The four lines N_jL_j meet in the point X_{20} (orthocenter of the Double triangle), and in fact $L_j=\frac{1}{2}X_{20}+\frac{1}{2}N_j$ — the standard such line is labelled $L_{1,3}$.

ix) The New point L_j lies on the line D_jS_j which also passes through the Orthocenter H, and in fact $S_j=\frac{1}{2}H+\frac{1}{2}L_j$.

In particular the various points alluded to here have *consistent definitions over general fields and with arbitrary bilinear forms*! The New points are the meets of the lines $L_{1,3}$ and $L_{4,7}$, they are the reflections of the Incenters I_j in the Circumcenter C, and they are the reflections of the Orthocenter H in the Spieker points S_j.

It is also worth pointing out a few additional relations between the triangle centers that appear here: the point X_{69}, defined as the Isogonal conjugate of the Orthocenter H, is also the central dilation in the Centroid of the Symmedian point K; in our notation $X_{69}=\delta_{-1/2}(K)$. This implies that $G=\frac{2}{3}K+\frac{1}{3}X_{69}$. In addition the De Longchamps point X_{20}, defined as the orthocenter of the Double (or anti-medial) triangle is also the reflection of the Orthocenter H in the Circumcenter C. These relations continue to hold in the general situation.

Table 1 summarizes the various strong concurrences we have found. Note however that not all pairings yield concurrent quadruples: for example the lines joining corresponding Nagel points and Mittenpunkts are *not* in general concurrent.

In the final section of the paper, we give some further results and directions involving chromogeometry.

1.2 Affine structure and vectors

We begin with some terminology and concepts for elementary affine geometry in a linear algebra setting, following [10]. Fix a field F, of characteristic not two, whose elements will be called **numbers**. We work in a two-dimensional affine space $\mathbb{A}^2$ over F, with $\mathbb{V}^2$ the associated two-dimensional vector space. A **point** is then an ordered pair $A\equiv[x,y]$ of numbers enclosed in square brackets, typically denoted by capital letters, such as A,B,C etc. A **vector** of $\mathbb{V}^2$ is an ordered pair $v\equiv(x,y)$ of numbers enclosed in round brackets, typically u,v,w etc. Any pair of points A and B determines a vector $v=\overrightarrow{AB}$; so for example if $A\equiv[2,-1]$ and $B\equiv[5,1]$, then $v=\overrightarrow{AB}=(3,2)$, and this is the same vector $v=\overrightarrow{CD}$ determined by $C\equiv[4,1]$ and $D\equiv[7,3]$.

	Incenter I	Gergonne G	Nagel N	Mittenpunkt D	Spieker S	New L
Incenter I	–	X_{20}	$G=X_2$	$K=X_6$	$G=X_2$	$C=X_3$
Gergonne G	X_{20}	–	X_{69}	$G=X_2$	–	–
Nagel N	$G=X_2$	X_{69}	–	–	$G=X_2$	X_{20}
Mittenpunkt D	$K=X_6$	$G=X_2$	–	–	$H=X_4$	$H=X_4$
Spieker S	$G=X_2$	–	$G=X_2$	$H=X_4$	–	$H=X_4$
New L	$C=X_3$	–	X_{20}	$H=X_4$	$H=X_4$	–

Table 1

The non-zero vectors $v_1 \equiv (x_1, y_1)$ and $v_2 \equiv (x_2, y_2)$ are **parallel** precisely when one is a non-zero multiple of the other, this happens precisely when

$$x_1y_2 - x_2y_1 = 0.$$

Vectors may be scalar-multiplied and added component-wise, so that if v and w are vectors and α, β are numbers, the **linear combination** $\alpha v + \beta w$ is defined. For points A and B and a number λ, we may define the **affine combination** $C = (1-\lambda)A + \lambda B$ either by coordinates or by interpreting it as the sum $A + \lambda\overrightarrow{AB}$. An important special case is when $\lambda = 1/2$; in that case the point $C \equiv A/2 + B/2$ is the **midpoint** of $\overline{AB}$, a purely affine notion independent of any metrical framework.

Once we fix an origin $O \equiv [0,0]$, the affine space $\mathbb{A}^2$ and the associated vector space $\mathbb{V}^2$ are naturally identified: to every point $A \equiv [x,y]$ there is an associated position vector $a = \overrightarrow{OA} = (x,y)$. So points and vectors are almost the same thing, but not quite. The choice of distinguished point also allows us a useful notational shortcut: we agree that for a point $A \equiv [x,y]$ and a number λ we write

$$\lambda[x,y] \equiv (1-\lambda)O + \lambda A = [\lambda x, \lambda y]. \qquad (1)$$

A **line** is a proportion $l \equiv \langle a:b:c \rangle$ where a and b are not both zero. The point $A \equiv [x,y]$ **lies on** the line $l \equiv \langle a:b:c \rangle$, or equivalently the line l **passes through** the point A, precisely when

$$ax + by + c = 0.$$

For any two distinct points $A_1 \equiv [x_1,y_1]$ and $A_2 \equiv [x_2,y_2]$, there is a unique line $l \equiv A_1A_2$ which passes through them both; namely the **join**

$$A_1A_2 = \langle y_1 - y_2 : x_2 - x_1 : x_1y_2 - x_2y_1 \rangle. \qquad (2)$$

In vector form, this line has parametric equation $l : A_1 + \lambda v$, where $v = \overrightarrow{A_1A_2} = (x_2 - x_1, y_2 - y_1)$ is a **direction vector** for the line, and λ is a parameter. The direction vector of a line is unique up to a non-zero multiple. The line $l \equiv \langle a:b:c \rangle$ has a direction vector $v = (-b, a)$.

Two lines are **parallel** precisely when they have parallel direction vectors. For every point P and line l, there is then precisely one line m through P parallel to l, namely $m : P + \lambda v$, where v is any direction vector for l. For any two lines $l_1 \equiv \langle a_1 : b_1 : c_1 \rangle$ and $l_2 \equiv \langle a_2 : b_2 : c_2 \rangle$ which are not parallel, there is a unique point $A \equiv l_1l_2$ which lies on them both; using (1) we can write this **meet** as

$$A \equiv l_1l_2 = \left[\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right]$$
$$= (a_1b_2 - a_2b_1)^{-1}[b_1c_2 - b_2c_1, c_1a_2 - c_2a_1]. \qquad (3)$$

Three points $A_1 = [x_1,y_1], A_2 = [x_2,y_2], A_3 = [x_3,y_3]$ are **collinear** precisely when they lie on a common line, which amounts to the condition

$$x_1y_2 - x_1y_3 + x_2y_3 - x_3y_2 + x_3y_1 - x_2y_1 = 0.$$

Three lines $\langle a_1 : b_1 : c_1 \rangle$, $\langle a_2 : b_2 : c_2 \rangle$ and $\langle a_3 : b_3 : c_3 \rangle$ are **concurrent** precisely when they pass through the same point, which amounts to the condition

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_2b_1c_3 = 0.$$

1.3 Metrical structure: quadrance and spread

We now introduce a metrical structure, which is determined by a non-degenerate symmetric 2×2 matrix C, with entries in the fixed field $\mathbb{F}$ over which we work. This matrix defines a symmetric bilinear form on vectors, regarded as row matrices, by the formula

$$v \cdot u = vu = vCu^T.$$

Here non-degenerate means $\det C \neq 0$, and implies that if $v \cdot u = 0$ for all vectors u then $v = 0$.

Note our introduction of the simpler notation $v \cdot u = vu$, so that also $v \cdot v = v^2$. There should be no confusion with matrix multiplication, even if v and u are viewed as 1×2 matrices. Since C is symmetric, $v \cdot u = vu = uv = u \cdot v$.

Two vectors v and u are **perpendicular** precisely when $v \cdot u = 0$. Since the matrix C is non-degenerate, for any vector v there is, up to a scalar, exactly one vector u which is perpendicular to v.

The bilinear form determines the main metrical quantity: the **quadrance** of a vector v is the number

$$Q_v \equiv v \cdot v = v^2.$$

A vector v is **null** precisely when $Q_v = v \cdot v = v^2 = 0$, in other words precisely when v is perpendicular to itself.

The **quadrance** between the points A and B is

$$Q(A,B) \equiv Q_{\overrightarrow{AB}}.$$

In the Euclidean case, this is of course the square of the usual distance. But quadrance is a more elementary and fundamental notion than distance, and its algebraic nature makes it ideal for metrical geometry using other bilinear forms (as Einstein and Minkowski tried to teach us a century ago!)

Two lines l and m are **perpendicular** precisely when they have perpendicular direction vectors. A line is **null** precisely when it has a null direction vector (in which case all direction vectors are null).

We now make the important observation that the affine notion of parallelism may also be recaptured via the bilinear form. (This result also appears with the same title in [15].)

Theorem 1 (Parallel vectors) *Vectors v and u are parallel precisely when*

$$Q_v Q_u = (vu)^2.$$

Proof. If $C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $v = (x, y)$ and $u = (z, w)$, then an explicit computation shows that

$$Q_v Q_u - (vu)^2 = -\frac{(xw - yz)^4 (ac - b^2)^2}{(ax^2 + 2bxy + cy^2)^2 (az^2 + 2bzw + cw^2)^2}.$$

Since the quadratic form is non-degenerate, $ac - b^2 \neq 0$, so we see that the left hand side is zero precisely when $xw - yz = 0$, in other words precisely when v and u are parallel. □

This motivates the following measure of the non-parallelism of two vectors; the **spread** between non-null vectors v and u is the number

$$s(v, u) \equiv 1 - \frac{(vu)^2}{Q_v Q_u}.$$

This is the replacement in rational trigonometry for the transcendental notion of angle θ, and in the Euclidean case it has the value $\sin^2 \theta$. Spread is a more algebraic, logical, general and powerful notion than that of angle, and together quadrance and spread provide the foundation for *Rational Trigonometry*, a new approach to trigonometry developed in [10]. The current pre-occupation with distance and angle as the basis for Euclidean geometry is a historical aberration contrary to the explicit orientation of Euclid himself, and is a key obstacle to appreciating and understanding the relativistic geometry introduced by Einstein and Minkowski.

The spread $s(v, u)$ is unchanged if either v or u are multiplied by a non-zero number, and so we define the **spread** between any non-null lines l and m with direction vectors v and u to be $s(l, m) \equiv s(v, u)$. From the Parallel vectors theorem, the spread between parallel lines is 0. Two non-null lines l and m are perpendicular precisely when the spread between them is 1.

1.4 Triple spread formula

We now derive one of the basic formulas in the subject: the relation between the three spreads made by three (coplanar) vectors, and give a linear algebra proof, following the same lines as the papers [11] and [15].

Theorem 2 (Triple spread formula) *Suppose that v_1, v_2, v_3 are (planar) non-null vectors with respective spreads $s_1 \equiv s(v_2, v_3)$, $s_2 \equiv s(v_1, v_3)$ and $s_3 \equiv s(v_1, v_2)$. Then*

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4 s_1 s_2 s_3. \qquad (4)$$

Proof. We may that assume at least two of the vectors are linear independent, as otherwise all spreads are zero and the relation is trivial. So suppose that v_1 and v_2 linearly independent, and $v_3 = kv_1 + lv_2$. Suppose the bilinear form is given by the matrix

$$C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with respect to the ordered basis v_1, v_2. Then in this basis $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (k, l)$ and we may compute that

$$s_3 = \frac{ac - b^2}{ac} \qquad s_2 = \frac{l^2 (ac - b^2)}{a(ak^2 + 2bkl + cl^2)}$$

$$s_1 = \frac{k^2 (ac - b^2)}{b(ak^2 + 2bkl + cl^2)}.$$

Then (4) is an identity, satisfied for all a, b, c, k and l. □

We now mention three consequences of the Triple spread formula, taken from [10]. The *Equal spreads theorem* asserts that if $s_1 = s_2 = s$, then $s_3 = 0$ or $s_3 = 4s(1 - s)$. This follows from the identity $(s + s + s_3)^2 - 2\left(s^2 + s^2 + s_3^2\right) - 4s^2 s_3 = -s_3\left(s_3 - 4s + 4s^2\right)$. The *Complementary spreads theorem* asserts that if $s_3 = 1$ then $s_1 + s_2 = 1$. This follows by rewriting the Triple spread formula in the form $(s_3 - s_1 - s_2)^2 = 4 s_1 s_2 (1 - s_3)$.

And the *Perpendicular spreads theorem* asserts that if v and u are non-null planar vectors with perpendicular vectors $v^\perp$ and $u^\perp$, then $s(v, u) = s\left(v^\perp, u^\perp\right)$. This follows from the Complementary spreads theorem, since if $s\left(v, v^\perp\right) = s\left(u, u^\perp\right) = 1$, then $s\left(v^\perp, u^\perp\right) = 1 - s\left(v^\perp, u\right) = 1 - (1 - s(v, u)) = s(v, u)$.

1.5 Altitudes and orthocenters

Given a line l and a point P, there is a unique line n through P which is perpendicular to the line l; it is the line $n : P + \lambda w$, where w is a perpendicular vector to the direction vector v of l. We call n the **altitude to** l **through** P. Note that this holds true even if l is a null line; in this case a direction vector v of l is null, so the altitude to l through P agrees with the parallel to l through P.

We use the following conventions: a set $\{A, B\}$ of two distinct points is a **side** and is denoted $\overline{AB}$, and a set $\{l, m\}$ of two distinct lines is a **vertex** and is denoted $\overline{lm}$. A set $\{A_1, A_2, A_3\}$ of three distinct non-collinear points is a **triangle** and is denoted $\overline{A_1 A_2 A_3}$. The triangle $\overline{A_1 A_2 A_3}$ has lines $l_3 \equiv A_1 A_2$, $l_2 \equiv A_1 A_3$ and $l_1 \equiv A_2 A_3$ (by assumption no two of these are parallel), sides $\overline{A_1 A_2}, \overline{A_1 A_3}$ and $\overline{A_2 A_3}$, and vertices $\overline{l_1 l_2}, \overline{l_1 l_3}$ and $\overline{l_2 l_3}$.

The triangle $\overline{A_1 A_2 A_3}$ also has three **altitudes** n_1, n_2, n_3 passing through A_1, A_2, A_3 and perpendicular to the opposite lines $A_2 A_3$, $A_1 A_3$, $A_1 A_2$ respectively. The following holds both for affine and projective geometries: we give a short and novel proof here for the general affine case.

Theorem 3 (Orthocenter) *For any triangle $\overline{A_1A_2A_3}$ the three altitudes n_1, n_2, n_3 are concurrent at a point H.*

Proof. Suppose that a_1, a_2, a_3 are the associated position vectors to A_1, A_2, A_3 respectively. Since no two of the lines of the triangle $\overline{A_1A_2A_3}$ are parallel, the Perpendicular spreads corollary implies that no two of the three altitude lines are parallel. Define H to be the meet of n_1 and n_2, with h the associated position vector. In the identity

$$(h-a_1)(a_3-a_2)+(h-a_2)(a_1-a_3)=(h-a_3)(a_1-a_2)$$

the left hand side equals 0 by assumption, so the right hand is also equal to 0, implying that $h-a_3$ is perpendicular to the line a_1a_2. Therefore, the three altitude lines n_1, n_2, n_3 are concurrent at the point H. □

We call H the **orthocenter** of the triangle $\overline{A_1A_2A_3}$.

1.6 Change of coordinates and an explicit example

If we change coordinates via either an affine transformation in the original affine space $\mathbb{A}^2$, or equivalently a linear transformation in the associated vector space $\mathbb{V}^2$, then the matrix for the form changes in the familiar fashion. Suppose $\phi : V \to V$ is a linear transformation given by an invertible 2×2 matrix M, so that $\phi(v) = vM = w$, with inverse matrix N, so that $wN = v$. Define a new bilinear form $\circ$ by

$$\begin{aligned} w_1 \circ w_2 &\equiv (w_1N)\cdot(w_2N) = (w_1N)\,C\,(w_2N)^T \\ &= w_1(NCN^T)w_2^T. \end{aligned} \qquad (5)$$

So the matrix C for the original bilinear form $\cdot$ becomes the matrix $D \equiv NCN^T$ for the new bilinear form $\circ$.

Example 1 *We illustrate these abstractions in a concrete example that will be used throughout in our diagrams. Our basic Triangle shown in Figure 2 has points $A_1 \equiv [3,1]$, $A_2 \equiv [4,4]$ and $A_3 \equiv [47/5, 29/5]$, and lines $A_1A_2 = \langle -3:1:8\rangle$, $A_1A_3 = \langle -3:4:5\rangle$ and $A_2A_3 = \langle 1:-3:8\rangle$. The bilinear form we will consider is that of green geometry in the language of chromogeometry ([12], [13]), determined by the symmetric matrix $C_g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and corresponding quadrance $Q_{(x,y)} = 2xy$. After translation by $(-3,-1)$ we obtain $\widetilde{A}_1 = [0,0]$, $\widetilde{A}_2 = [1,3]$, $\widetilde{A}_3 = [32/5, 24/5]$. The matrix N and its inverse M*

$$N = \begin{pmatrix} 1 & 3 \\ \frac{32}{5} & \frac{24}{5} \end{pmatrix} \qquad M = N^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{5}{24} \\ \frac{4}{9} & -\frac{5}{72} \end{pmatrix}$$

send $[1,0]$ and $[0,1]$ to $\widetilde{A}_2$ and $\widetilde{A}_3$, and $\widetilde{A}_2$ and $\widetilde{A}_3$ to $[1,0]$ and $[0,1]$ respectively. So the effect of translation followed by multiplication by M is to send the original triangle to the ***standard triangle*** *with points $[0,0]$, $[1,0]$ and $[0,1]$. The bilinear form in these new standard coordinates is given by the matrix $N\mathbf{C}_gN^T$ which is, up to a multiple,*

$$C = \begin{pmatrix} \frac{1}{4} & 1 \\ 1 & \frac{64}{25} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

We will shortly see that the Orthocenter in standard coordinates is $(ac-b^2)^{-1}[b(c-b), b(a-b)]$. In our example this would be the point $\left[-\frac{13}{3}, \frac{25}{12}\right]$, and to convert that back into the original coordinates, we would multiply by N to get

$$\begin{bmatrix} -\frac{13}{3} & \frac{25}{12} \end{bmatrix} N = \begin{bmatrix} 9 & -3 \end{bmatrix}$$

and translate by $(3,1)$ to get the original orthocenter $H = [12,-2]$. This is shown in Figure 2, along with the Centroid $G = [82/15, 18/5]$ and the Circumcenter $C = [11/5, 32/5]$—we will meet these points shortly.

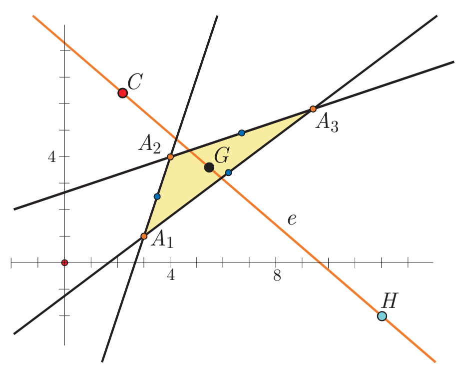

Figure 2: *Euler line in green geometry*

1.7 Bilines

A **biline** of the non-null vertex $\overline{l_1l_2}$ is a line b which passes through l_1l_2 and satisfies $s(l_1,b) = s(l_2,b)$. The existence of bilines depends on number-theoretical considerations of a particularly simple kind.

Theorem 4 (Vertex bilines) *If v and u are linearly independent non-null vectors, then there is a non-zero vector w with $s(v,w) = s(u,w)$ precisely when $1-s(v,u)$ is a square. In this case we may renormalize v and u so that $Q_v = Q_u$, and then there are exactly two possibilities for w up to a multiple, namely $v+u$ and $v-u$, and these are perpendicular.*

Proof. Since v and u are linearly independent, any vector can be written uniquely as $w = kv + lu$ for some numbers

k and l. The condition $s(v,w) = s(u,w)$ amounts to

$$\frac{(vw)^2}{Q_v Q_w} = \frac{(uw)^2}{Q_u Q_w} \iff u^2\left(kv^2 + lvu\right)^2 = v^2\left(kvu + lu^2\right)^2$$
$$\iff u^2\left(k^2\left(v^2\right)^2 + 2lkv^2(vu) + l^2(vu)^2\right) =$$
$$= v^2\left(k^2(vu)^2 + 2lku^2(vu) + l^2\left(u^2\right)^2\right)$$
$$\iff k^2u^2\left(v^2\right)^2 + l^2u^2(vu)^2 = k^2v^2(vu)^2 + l^2v^2\left(u^2\right)^2$$
$$\iff \left(v^2u^2 - (vu)^2\right)\left(k^2v^2 - l^2u^2\right) = 0.$$

Since v and u are by assumption not parallel, the first term is non-zero by the Parallel vectors theorem, and so the condition $s(v,w) = s(u,w)$ is equivalent to $k^2v^2 = l^2u^2$. Since v,u are non-null, v^2 and u^2 are non-zero, so k and l are also, since by assumption $w = kv + lu$ is non-zero.
So if $s(v,w) = s(u,w)$ then we may renormalize v and u so that $v^2 = u^2$ (by for example setting $\widetilde{v} = kv$ and $\widetilde{u} = lu$, and then replacing $\widetilde{v},\widetilde{u}$ by v,u again), and then $1 - s(v,u) = (vu)^2 / \left(v^2\right)^2$ is a square. There are then two solutions: $w = v + u$ and $w = v - u$, corresponding to $l = \pm k$. Since $(v+u)(v-u) = v^2 - u^2 = 0$, these vectors are perpendicular. The converse is straightforward along the same lines. □

Example 2 *In our example triangle of Figure 2,* $v_1 = \overrightarrow{A_2A_3} = (27/5, 9/5)$, $v_2 = \overrightarrow{A_1A_3} = (32/5, 24/5)$ *and* $v_3 = \overrightarrow{A_1A_2} = (1,3)$, *so*

$$s(v_2,v_3) = 1 - \frac{\left(v_2C_gv_3^T\right)^2}{\left(v_2C_gv_2^T\right)\left(v_3C_gv_3^T\right)} = \frac{25}{16}$$

is a square, so the vertex at A_1 *has bilines. Since* $Q_{v_2} = v_2C_gv_2^T = 1536/25$ *and* $Q_{v_3} = v_3C_gv_3^T = 6$, *we can renormalize* v_2 *by scaling it by* $5/16$ *to get* $u_2 = \overrightarrow{A_1B} = (2, 3/2)$ *so that now* $Q_{u_2} = Q_{v_3}$. *This means that* $u_2 + v_3 = \overrightarrow{A_1C_1}$ *and* $u_2 - v_3 = \overrightarrow{A_1C_2}$ *are the direction vectors for the bilines of the vertex at* A_1.

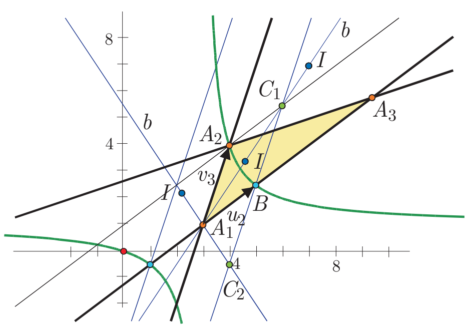

Figure 3 : *Green bilines b at* A_1

These are shown in Figure 3, along with three of the four Incenters I (the other two vertices also have bilines, and they are mutually concurrent). Naturally this triangle has been chosen carefully to ensure that Incenters do exist. In green geometry, a vertex formed from a light-like line and a time-like line will not have bislines, not even approximately over the rational numbers.

2 Standard coordinates and triangle geometry

Our principle strategy to study triangle geometry is to apply an affine transformation to move a general triangle to *standard position*:

$$A_1 = [0,0] \qquad A_2 = [1,0] \qquad \text{and} \qquad A_3 = [0,1]. \tag{6}$$

With this convention, $\overline{A_1A_2A_3}$ will be called the (standard) **Triangle**, with **Points** A_1, A_2, A_3. The **Lines** of the Triangle are then

$$l_1 \equiv A_2A_3 = \langle 1:1:-1\rangle \qquad l_2 \equiv A_1A_3 = \langle 1:0:0\rangle$$

$$l_3 \equiv A_2A_1 = \langle 0:1:0\rangle.$$

All further objects that we define with capital letters refer to this standard Triangle, and coordinates in this framework are called *standard coordinates*. In general the standard coordinates of points and lines in the plane of the original triangle depend on the choice of affine transformation—we are in principle free to permute the vertices—but triangle centers and central lines will have well-defined standard coordinates independent of such permutations.
Since we have performed an affine transformation, whatever metrical structure we started with has changed as in (5). So we will assume that the new metrical structure, in standard coordinates, is determined by a bilinear form with generic symmetric matrix

$$C \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{7}$$

We assume that the form is non-degenerate, so that the **determinant**

$$\Delta \equiv \det C = ac - b^2$$

is non-zero. Another important number is the **mixed trace**

$$d \equiv a + c - 2b.$$

It will also be useful to introduce the closely related secondary quantities

$$\overline{a} \equiv c - b \qquad \overline{b} \equiv a - c \qquad \overline{c} \equiv a - b$$

to simplify formulas. For example $d = \overline{a} + \overline{c}$.

Theorem 5 (Standard triangle quadrances and spreads) *The quadrances and spreads of $\overline{A_1A_2A_3}$ are*

$$Q_1 \equiv Q(A_2,A_3) = d \qquad Q_2 \equiv Q(A_1,A_3) = c$$
$$Q_3 \equiv Q(A_1,A_2) = a$$

and

$$s_1 \equiv s(A_1A_2,A_1A_3) = \frac{\Delta}{ac} \qquad s_2 \equiv s(A_2A_3,A_2A_1) = \frac{\Delta}{ad}$$
$$s_3 \equiv s(A_3A_1,A_3A_2) = \frac{\Delta}{cd}.$$

Furthermore

$$1-s_1 = \frac{b^2}{ac} \qquad 1-s_2 = \frac{(\overline{c})^2}{ad} \qquad 1-s_3 = \frac{(\overline{a})^2}{cd}.$$

Proof. Using the definition of quadrance,

$$Q_1 \equiv Q(A_2,A_3) = Q_{\overrightarrow{A_2A_3}} = (-1,1)C(-1,1)^T$$
$$= a+c-2b = d$$

and similarly for Q_2 and Q_3. Using the definition of spread,

$$s_1 \equiv s(A_1A_2,A_1A_3) = s((1,0),(0,1))$$
$$= 1 - \frac{\left((1,0)C(0,1)^T\right)^2}{\left((1,0)C(1,0)^T\right)\left((0,1)C(0,1)^T\right)}$$
$$= 1 - \frac{1}{ac}b^2 = \frac{\Delta}{ac}$$

and similarly for s_2 and s_3. □

2.1 Basic affine objects in triangle geometry

We now write down some basic central objects which figure prominently in triangle geometry, all with reference to the standard triangle $\overline{A_1A_2A_3}$ in the form (6). The derivations of these formulas are mostly immediate using the two basic operations of joins (2) and meets (3). We begin with some purely affine notions, independent of the bilinear form.

The **Midpoints** of the Triangle are

$$M_1 = \left[\frac{1}{2},\frac{1}{2}\right] \qquad M_2 = \left[0,\frac{1}{2}\right] \qquad M_3 = \left[\frac{1}{2},0\right].$$

The **Medians** are

$$d_1 \equiv A_1M_1 = \langle 1:-1:0\rangle \qquad d_2 \equiv A_2M_2 = \langle 1:2:-1\rangle$$
$$d_3 \equiv A_3M_3 = \langle 2:1:-1\rangle.$$

The **Centroid** is the common meet of the Medians

$$G = \left[\frac{1}{3},\frac{1}{3}\right].$$

The **Circumlines** are the lines of the **Medial triangle** $\overline{M_1M_2M_3}$, these are

$$b_1 \equiv M_2M_3 = \langle 2:2:-1\rangle \qquad b_2 \equiv M_3M_1 = \langle 2:0:-1\rangle$$
$$b_3 \equiv M_1M_2 = \langle 0:2:-1\rangle.$$

The **Double triangle** of $\overline{A_1A_2A_3}$ (usually called the **anti-medial triangle**) is formed from lines through the Points parallel to the opposite Lines. This is $\overline{D_1D_2D_3}$ where

$$D_1 = [1,1] \qquad D_2 = [-1,1] \qquad D_3 = [1,-1].$$

The lines of $\overline{D_1D_2D_3}$ are

$$D_2D_3 = \langle 1:1:0\rangle \qquad D_1D_3 = \langle 1:0:-1\rangle$$
$$D_1D_2 = \langle 0:1:-1\rangle.$$

Figure 4 shows these objects for our example Triangle.

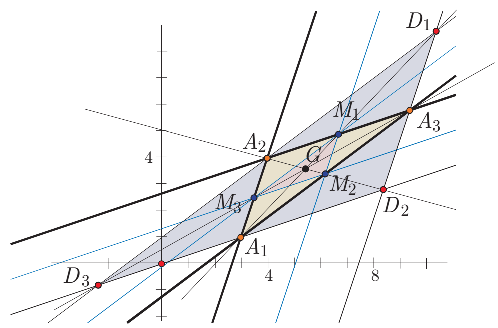

Figure 4: *The Medial triangle $\overline{M_1M_2M_3}$ and Double triangle $\overline{D_1D_2D_3}$*

2.2 The Orthocenter hierarchy

We now introduce some objects involving the metrical structure, and so the entries a,b,c of C from (7). Recall that $\overline{a} \equiv c-b$ and $\overline{c} = a-b$.

The **Altitudes** of $\overline{A_1A_2A_3}$ are the lines

$$n_1 = \langle \overline{c} : -\overline{a} : 0\rangle \qquad n_2 = \langle b:c:-b\rangle \qquad n_3 = \langle a:b:-b\rangle.$$

Theorem 6 (Orthocenter formula) *The three Altitudes meet at the* ***Orthocenter***

$$H = \frac{b}{\Delta}[\overline{a},\overline{c}].$$

Proof. We know that the altitudes meet from the Orthocenter theorem. We check that n_1 passes through H by computing $b\Delta^{-1}(\overline{a}\overline{c} - \overline{a}\overline{c}) = 0$.

Also n_2 passes through H since

$$\frac{b}{\Delta}(b\overline{a}+c\overline{c}) - b = \frac{b}{\Delta}\left(b(c-b)+c(a-b)-ac+b^2\right) = 0$$

and similarly for n_3. □

The **Midlines** m_1, m_2 and m_3 are the lines through the midpoints M_1, M_2 and M_3 perpendicular to the respective sides— these are usually called **perpendicular bisectors**. They are also the altitudes of $\overline{M_1M_2M_3}$:

$$m_1 = \left\langle -2\overline{c} : 2\overline{a} : \overline{b} \right\rangle \qquad m_2 = \langle 2b : 2c : -c \rangle$$
$$m_3 = \langle 2a : 2b : -a \rangle .$$

Theorem 7 (Circumcenter) *The Midlines m_1, m_2, m_3 meet at the **Circumcenter***

$$C = \frac{1}{2\Delta}\left[c\overline{c}, a\overline{a}\right].$$

Proof. We check that m_1 passes through C by computing

$$\frac{1}{2\Delta}\left(-2\overline{c}^2 c + 2\overline{a}^2 a\right) + \overline{b}$$
$$= \frac{1}{2\left(ac - b^2\right)}\left(-2\left(a-b\right)^2 c + 2\left(c-b\right)^2 a\right) + \left(a-c\right) = 0$$

and similarly for m_2 and m_3. □

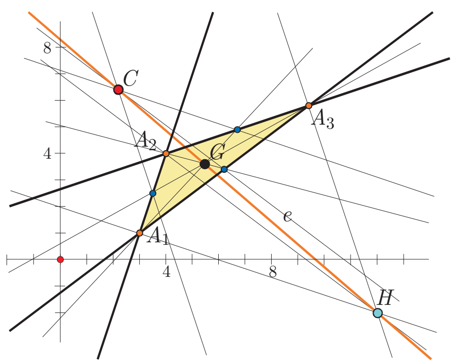

Figure 5: *The Euler line of a triangle*

As Gauss realized, this is also a consequence of the Orthocenter theorem applied to the Medial triangle $\overline{M_1M_2M_3}$, since the altitudes of the Medial triangle are the Midlines of the original Triangle.
The three altitudes of the Double triangle $\overline{D_1D_2D_3}$ are

$$t_1 = \left\langle \overline{c} : -\overline{a} : -\overline{b} \right\rangle \quad t_2 = \left\langle b : c : -\overline{a} \right\rangle \quad t_3 = \left\langle a : b : -\overline{c} \right\rangle .$$

Theorem 8 (Double orthocenter formula) *The three altitudes of the Double triangle meet in the De Longchamps point*

$$X_{20} \equiv \frac{1}{\Delta}\left[b^2 - 2bc + ac, b^2 - 2ab + ac\right].$$

Proof. We check that t_1 passes through X_{20} by computing

$$\frac{1}{\Delta}\left(\overline{c}\left(b^2 - 2bc + ac\right) - \overline{a}\left(b^2 - 2ab + ac\right)\right) - \overline{b}$$
$$= \frac{1}{\Delta}\left(\left(a-b\right)\left(b^2 - 2bc + ac\right) - \left(c-b\right)\left(b^2 - 2ab + ac\right)\right.$$
$$\left. - \left(a-c\right)\left(ac - b^2\right)\right) = 0$$

and similarly for t_2 and t_3. □

The existence of an Euler line in relativistic geometries was established in [13], here we extend this to the general case.

Theorem 9 (Euler line) *The points H, C and G are concurrent, and satisfy $G = \frac{1}{3}H + \frac{2}{3}C$. The Euler line $e \equiv CH$ is*

$$e = \left\langle \Delta - 3b\overline{c} : -\Delta + 3b\overline{a} : b\overline{b} \right\rangle .$$

Proof. Using the formulas above for H and C, we see that

$$\frac{1}{3}H + \frac{2}{3}C = \left(\frac{1}{3}\right)\frac{1}{\Delta}\left[b\overline{a}, b\overline{c}\right] + \left(\frac{2}{3}\right)\frac{1}{2\Delta}\left[c\overline{c}, a\overline{a}\right]$$
$$= \frac{1}{3\Delta}\left[ac - b^2, ac - b^2\right] = \frac{1}{3}\left[1,1\right] = G.$$

Computing the equation for the Euler line CH is straightforward. □

In Figure 5 we illustrate the situation with our basic example triangle with the Altitudes, Medians and Midlines meeting to form the Orthocenter H, Centroid G and Circumcenter C respectively on the Euler line e.
The **bases of altitudes** of $\overline{M_1M_2M_3}$ are:

$$E_1 = \frac{1}{2d}\left[\overline{c}, \overline{a}\right] \qquad E_2 = \frac{1}{2c}\left[c, \overline{a}\right] \qquad E_3 = \frac{1}{2a}\left[\overline{c}, a\right].$$

The joins of Points and corresponding bases of altitudes of $\overline{M_1M_2M_3}$ are

$$A_1E_1 = \left\langle \overline{a} : -\overline{c} : 0 \right\rangle \qquad A_2E_2 = \left\langle \overline{a} : c : -\overline{a} \right\rangle$$
$$A_3E_3 = \left\langle a : \overline{c} : -\overline{c} \right\rangle .$$

Theorem 10 (Medial base perspectivity) *The three lines A_1E_1, A_2E_2, A_3E_3 meet at the point*

$$X_{69} = \frac{1}{a + c - b}\left[\overline{c}, \overline{a}\right].$$

Proof. Straightforward. □

2.3 Bilines and Incenters

We now introduce the *Incenter hierarchy*. Unlike the Orthocenter hierarchy, this depends on number-theoretical conditions. Recall that $d \equiv a + c - 2b$.

Theorem 11 (Existence of Triangle bilines) *The Triangle $\overline{A_1A_2A_3}$ has Bilines at each vertex precisely when we can find numbers u, v, w in the field satisfying*

$$ac = u^2 \qquad ad = v^2 \qquad cd = w^2. \tag{8}$$

Proof. From the Vertex bilines theorem, bilines exist precisely when the spreads s_1, s_2, s_3 of the Triangle have the property that $1-s_1, 1-s_2, 1-s_3$ are all squares. From the Standard triangle quadrances and spreads theorem, this occurs for our standard triangle $\overline{A_1A_2A_3}$ precisely when we can find u, v, w satisfying (8). □

There is an important flexibility here: the three **Incenter constants** u,v,w are only determined up to a sign. The relations imply that

$$d^2u^2 = v^2w^2 \qquad c^2v^2 = u^2w^2 \qquad a^2w^2 = u^2v^2.$$

So we may choose the sign of u so that $du = vw$, and multiplying by u we get

$$acd = uvw.$$

From this we deduce that

$$du = vw \qquad cv = uw \qquad \text{and} \qquad aw = uv. \tag{9}$$

The **quadratic relations** (8) and (9) will be very important for us, for they reveal that the existence of the Incenter hierarchy is a number-theoretical issue which depends not only on the given triangle and the bilinear form, but also on the nature of the field over which we work, and they allow us to simplify many formulas involving u,v and w. Because only quadratic conditions are involved, we may always extend our field by adjoining (algebraic!) square roots to ensure that a given triangle has bilines.

The quadratic relations carry an important symmetry: we may replace any two of u,v and w with their negatives, and the relations remain unchanged. So if we have a formula F_0 involving u,v,w, then we may obtain related formulas F_1,F_2,F_3 by replacing v,w with their negatives, u,w with their negatives, and u,v with their negatives respectively. Adopting this convention allows us to exhibit the single formula F_0, since then F_1,F_2,F_3 are determined—we refer to this as **quadratic symmetry**, and will make frequent use of it in the rest of this paper.

From now on our working assumption is that: *the standard triangle $\overline{A_1A_2A_3}$ has bilines at each vertex, implying that we have Incenter constants u,v and w satisfying* (8) *and* (9). So u,v and w now become ingredients in our formulas for various objects in the Incenter hierarchy, along with the numbers a,b and c (and d) from the bilinear form $C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Theorem 12 (Bilines) *The Bilines of the Triangle are $b_{1+} \equiv \langle v:w:0\rangle$ and $b_{1-} \equiv \langle v:-w:0\rangle$ through A_1, $b_{2+} \equiv \langle u:u+w:-u\rangle$ and $b_{2-} \equiv \langle u:u-w:-u\rangle$ through A_2, and $b_{3+} \equiv \langle u-v:u:-u\rangle$ and $b_{3-} \equiv \langle u+v:u:-u\rangle$ through A_3.*

Proof. We use the Bilines theorem to find bilines through $A_1 = [0,0]$. The lines meeting at A_1 have direction vectors $v_1 = (0,1)$ and $v_2 = (1,0)$, with $Q_{v_1} = (0,1)C(0,1)^T = c$ and $Q_{v_2} = (1,0)C(1,0)^T = a$. Now we renormalize and set $u_1 = \frac{v}{w}v_1$ to get $Q_{u_1} = \frac{v^2}{w^2}c = a = Q_{v_2}$. So the bilines at A_1 have direction vectors

$$u_1+v_2 = \frac{v}{w}(0,1)+(1,0) = \left(1,\frac{v}{w}\right) \qquad \text{and}$$
$$u_1-v_2 = \frac{v}{w}(0,1)-(1,0) = \left(-1,\frac{v}{w}\right)$$

and the bilines are $b_{1+} \equiv \langle v:w:0\rangle$ and $b_{1-} \equiv \langle v:-w:0\rangle$. Similarly you may check the other bilines through A_2 and A_3. □

Theorem 13 (Incenters) *The triples $\{b_{1+},b_{2+},b_{3+}\}$, $\{b_{1+},b_{2-},b_{3-}\}$, $\{b_{1-},b_{2+},b_{3-}\}$ and $\{b_{1-},b_{2-},b_{3+}\}$ of Bilines are concurrent, meeting respectively at the four **Incenters***

$$I_0 = \left[\frac{-uw}{uv-uw+vw}, \frac{uv}{uv-uw+vw}\right] = \frac{1}{(d+v-w)}[-w,v]$$
$$I_1 = \left[\frac{uw}{-uv+uw+vw}, \frac{-uv}{-uv+uw+vw}\right] = \frac{1}{(d-v+w)}[w,-v]$$
$$I_2 = \left[\frac{uw}{uv+uw+vw}, \frac{uv}{uv+uw+vw}\right] = \frac{1}{(d+v+w)}[w,v]$$
$$I_3 = \left[\frac{uw}{uv+uw-vw}, \frac{uv}{uv+uw-vw}\right] = \frac{1}{(d-v-w)}[-w,-v].$$

Proof. We may check concurrency of the various triples by computing

$$\det\begin{bmatrix} v & w & 0 \\ u & u+w & -u \\ u-v & u & -u \end{bmatrix} = \det\begin{bmatrix} v & w & 0 \\ u & u-w & -u \\ u+v & u & -u \end{bmatrix}$$
$$= \det\begin{bmatrix} v & -w & 0 \\ u & u-w & -u \\ u-v & u & -u \end{bmatrix} = \det\begin{bmatrix} v & -w & 0 \\ u & u+w & -u \\ u+v & u & -u \end{bmatrix} = 0.$$

The corresponding meet of $\langle v:w:0\rangle$, $\langle u:u+w:-u\rangle$ and $\langle u-v:u:-u\rangle$ is

$$\begin{aligned} b_{1+}b_{2+}b_{3+} &= \left[\frac{-uw}{uv-uw+vw}, \frac{uv}{uv-uw+vw}\right] \\ &= \frac{u}{aw-cv+du}[-w,v] \\ &= \frac{1}{(d+v-w)}[-w,v] \equiv I_0. \end{aligned}$$

We have used the quadratic relations, and the last equality is valid since

$$u(d+v-w)-(aw-cv+du) = cv-aw+uv-uw = 0.$$

The computations are similar for the other Incenters. □

The reader should check that the formulas for I_1, I_2, I_3 may also be obtained from I_0 by the quadratic symmetry rule described above. From now on in such a situation we will only write down the formula corresponding to I_0, and we will also often omit algebraic manipulations involving the quadratic relations.

The **Incenter altitude** t_{ij} is the line through the Incenter I_j and perpendicular to the Line l_i of our Triangle. There are twelve Incenter altitudes; three associated to each Incenter. The Incenter altitudes associated to I_0 are

$$t_{10} = \langle \overline{c}\,(d+v-w) : -\overline{a}\,(d+v-w) : \overline{a}v+\overline{c}w \rangle$$
$$t_{20} = \langle b\,(d+v-w) : c\,(d+v-w) : bw-cv \rangle$$
$$t_{30} = \langle a\,(d+v-w) : b\,(d+v-w) : aw-bv \rangle .$$

The **Contact points** C_{ij} are the meets of corresponding Incenter altitudes t_{ij} and Lines l_i. There are twelve Contact points; three associated to each Incenter. The Contact points associated to the Incenter I_0 are

$$C_{10} = \frac{1}{(d+v-w)}\left[\overline{a}-w, \overline{c}+v\right]$$
$$C_{20} = \frac{1}{c\,(d+v-w)}\left[0, cv-bw\right]$$
$$C_{30} = \frac{1}{a\,(d+v-w)}\left[bv-aw, 0\right].$$

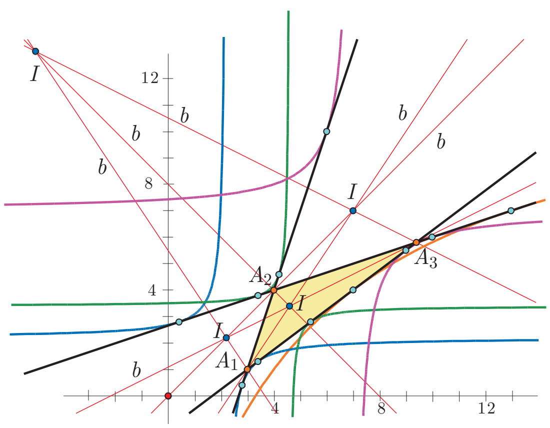

Figure 6: *Green bilines b, Incenters I, Contact points and Incircles*

In Figure 6 we see our standard example Triangle in the green geometry with Bilines b at each vertex, meeting in threes at the Incenters I. The Contact points are also shown, as are the Incircles, which are the circles with respect to the metrical structure centered at the Incenters and passing through the Contact points: they have equations in the variable point X of the form $Q(X,I) = Q(C,I)$ where I is an incenter and C is one of its associated Contact points. In this green geometry such circles appear as rectangular hyperbolas, with axes parallel to the coordinate axes.

2.4 New points

One of the main novelties of this paper is the introduction of the four *New points* L_j associated to each Incenter I_j. It is surprising that these points have seemingly slipped through the radar: they deserve to be among the top twenty in Kimberling's list, in our opinion.

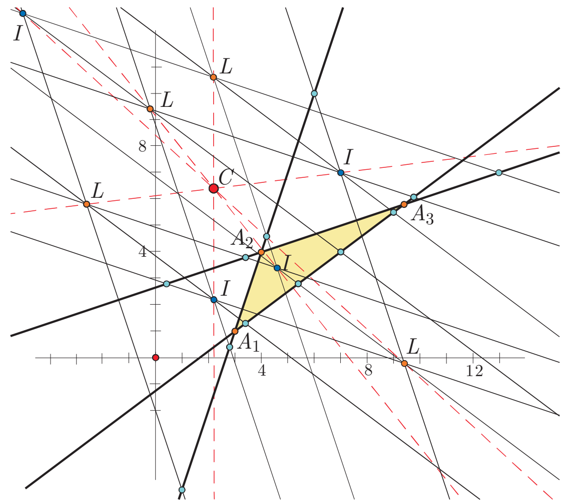

Figure 7: *Green Incenter altitudes, New points L and In-New center C*

Theorem 14 (New points) *The triples* $\{t_{11}, t_{22}, t_{33}\}$, $\{t_{10}, t_{23}, t_{32}\}$, $\{t_{20}, t_{13}, t_{31}\}$ *and* $\{t_{30}, t_{12}, t_{21}\}$ *of Incenter altitudes are concurrent. Each triple is associated to the Incenter which does not lie on any of the lines in that triple. The points where these triples meet are the* ***New points*** L_i*; for example* $\{t_{11}, t_{22}, t_{33}\}$ *meet at*

$$L_0 = \frac{1}{2\Delta}\left[\overline{a}u+cv+bw+c\overline{c}, \overline{c}u-bv-aw+a\overline{a}\right].$$

Proof. We check that L_0 as defined is incident with $t_{11} = \langle \overline{c}\,(d-v+w) : -\overline{a}\,(d-v+w) : -\overline{a}v-\overline{c}w \rangle$ by computing

$$((c-b)u+cv+bw+c\,(a-b))\,(a-b)\,(a+c-2b-v+w)$$
$$+((a-b)u-bv-aw+a\,(c-b))\,(-(c-b)\,(a+c-2b-v+w))$$
$$-2\left(ac-b^2\right)((c-b)v+(a-b)w)$$
$$= a^3c+2ab^3-2a^2cb-a^2b^2-ac^3+2ac^2b+c^2b^2$$
$$-2cb^3+b^2v^2-b^2w^2-acv^2+acw^2$$
$$= \left(ac-b^2\right)\left(a^2-c^2-2ab+2cb-v^2+w^2\right) = 0$$

using the quadratic relations (8). The computations for the other Incenter altitudes and L_1, L_2, L_3 are similar. □

The **In-New lines** are the joins of corresponding Incenter points and New points. The In-New line associated to I_0 is

$$I_0L_0 = \langle -a\overline{a}d+\left(ac+ab-2b^2\right)v+a\overline{a}w :$$
$$c\overline{c}d+c\overline{c}v+\left(ac+cb-2b^2\right)w : -a\overline{a}w-c\overline{c}v \rangle.$$

Theorem 15 (In-New center) *The four In-New lines I_jL_j are concurrent and meet at the circumcenter*

$$C = \frac{1}{2\Delta}\left[c\overline{c}, a\overline{a}\right],$$

and in fact C is the midpoint of $\overline{I_jL_j}$.

Proof. We check that C is the midpoint of $\overline{I_jL_j}$ by computing

$$\frac{1}{2}I_j+\frac{1}{2}L_j = \left(\frac{1}{2}\right)\frac{1}{(d+v-w)}\left[-w,v\right]$$
$$+\left(\frac{1}{2}\right)\frac{1}{2\Delta}\left[(c-b)u+cv+bw+c(a-b),(a-b)u-bv-aw+a(c-b)\right]$$
$$=\frac{1}{4\Delta(d+v-w)}\left[2c(a-b)(d+v-w),2a(c-b)(d+v-w)\right]$$
$$=\frac{1}{2\Delta}\left[c\overline{c}, a\overline{a}\right]=C.$$

□

The In-New center theorem shows that what we are calling the In-New lines are also the In-Circumcenter lines, the standard one which is labelled $L_{1,3}$ in [6]. The Incenter altitudes, New points and In-New lines are shown in Figure 7.

The proofs in these two theorems are typical of the ones which appear in the rest of the paper. Algebraic manipulations are combined with the quadratic relations to simplify expressions. Although sometimes long and involved, the verifications are in principle straightforward, and so *from now on we omit the details for results such as these.*

3 Transformations

Important classical transformations of points associated to a triangle include dilations in the centroid, and the isogonal and isotomic conjugates. It is useful to have general formulae for these in our standard coordinates.

3.1 Dilations about the Centroid

The dilation δ of factor λ centered at the origin takes $[x,y]$ to $\lambda[x,y]$. This also acts on vectors by scalar multiplying, and in particular it leaves spreads unchanged and multiplies any quadrance by a factor of λ^2. Similarly the dilation centered at a point A takes a point B to $A+\lambda\overrightarrow{AB}$. Any dilation preserves directions of lines, so preserves spreads, and changes quadrances between points proportionally.

Given our Triangle $\overline{A_1A_2A_3}$ with centroid G, define the **central dilation** $\delta_{-1/2}$ to be the dilation by the factor $-1/2$ centered at G. It takes the three Points of the Triangle to the midpoints M_1, M_2, M_3 of the opposite sides. This medial triangle $\overline{M_1M_2M_3}$ then clearly has lines which are parallel to the original triangle.

Since the central dilation preserves spread, the three altitudes of $\overline{A_1A_2A_3}$ are sent by $\delta_{-1/2}$ to the three altitudes of the medial triangle, which are the midlines/perpendicular bisectors of the original Triangle, showing again that $\delta_{-1/2}$ sends the orthocenter H to the circumcenter C, and as in the Euler line theorem it follows that G lies on $e=HC$, dividing $\overline{HC}$ in the affine ratio $2:1$.

We will see later that the central dilation also explains aspects of the various Nagel lines (there are four), since $\delta_{-1/2}$ takes any Incenter I_i to an incenter of the Medial triangle, called a **Spieker point** S_i. It follows that the four joins of Incenters and corresponding Spieker points all pass through G, and G divides each side $\overline{I_iS_i}$ in the affine ratio $2:1$.

The inverse of the central dilation $\delta_{-1/2}$ is δ_{-2}, which takes the Points of $\overline{A_1A_2A_3}$ to the points of the Double triangle $\overline{D_1D_2D_3}$, which has $\overline{A_1A_2A_3}$ as its medial triangle.

Theorem 16 (Central dilation formula) *The central dilation takes $X=[x,y]$ to*

$$\delta_{-1/2}(X)=\frac{1}{2}\left[1-x,1-y\right]$$

while the inverse central dilation δ_{-2} takes X to $\delta_{-2}(X)=[1-2x,1-2y]$.

Proof. If $Y=\delta_{-1/2}(X)$ then affinely $\frac{1}{3}X+\frac{2}{3}Y=G$ so that

$$Y=\frac{3}{2}G-\frac{1}{2}X=\frac{1}{2}\left[1-x,1-y\right].$$

Inverting, we get the formula for $\delta_{-2}(X)$. □

Example 3 *The central dilation of the Orthocenter is*

$$\delta_{-1/2}(H)=\frac{1}{2}\left[1-\frac{b(c-b)}{\Delta},1-\frac{b(a-b)}{\Delta}\right]$$
$$=\frac{1}{2\Delta}\left[c(a-b),a(c-b)\right]=\frac{1}{2\Delta}\left[c\overline{c},a\overline{a}\right]=C$$

which is the Circumcenter.

Example 4 *The inverse central dilation of the Orthocenter is the* ***De Longchamps point*** *X_{20}—the orthocenter of the Double triangle $\overline{D_1D_2D_3}$*

$$\delta_{-2}(H)=X_{20}\equiv\left[1-\frac{2b(c-b)}{\Delta},1-\frac{2b(a-b)}{\Delta}\right]$$
$$=\frac{1}{\Delta}\left[b^2-2cb+ac,b^2-2ab+ac\right].$$

3.2 Reflections and Isogonal conjugates

Suppose that v is a non-null vector, so that v is not perpendicular to itself. It means that we can find a perpendicular vector w so that v and w are linearly independent. Now if u is an arbitrary vector, write $u=rv+sw$ for some unique numbers r and s, and define the **reflection of** u **in** v to be

$$r_v(u)\equiv rv-sw.$$

If we replace v with a multiple, the reflection is unchanged. Now suppose that l and m are lines which meet at a point A, with respective direction vectors v and u. Then the **reflection of** m **in** l is the line through A with direction vector $r_v(u)$. It is important to note that if n is the perpendicular to l through A, then

$$r_l(m) = r_n(m).$$

Our standard triangle $\overline{A_1A_2A_3}$ determines an important transformation of points.

Theorem 17 (Isogonal conjugate) *If X is a point distinct from A_1,A_2,A_3, then the reflections of the lines A_1X, A_2X, A_3X in the bilines at A_1,A_2,A_3 respectively meet in a point $i(X)$, called the* ***isogonal conjugate*** *of X. If $X=[x,y]$ then*

$$i(X) = \frac{x+y-1}{ax^2+2bxy+cy^2-ax-cy}[cy,ax].$$

Proof. First we reflect the vector $a=(x,y)$ in the bilines $\langle v:w:0\rangle$ and $\langle v:-w:0\rangle$ through A_1. We do this by writing $(x,y)=r(w,v)+s(w,-v)=(rw+sw,rv-sv)$ and solving to get $r=(2vw)^{-1}(vx+wy)$ and $s=(2vw)^{-1}(vx-wy)$. The reflection is then

$$r(w,v)-s(w,-v) = \frac{1}{2vw}(vx+wy)(w,v) - \frac{1}{2vw}(vx-wy)(w,-v) = \left(\frac{wy}{v},\frac{vx}{w}\right)$$

which is, up to a multiple and using the quadratic relations,

$$\left(w^2y, v^2x\right) = (cdy, adx) = d(cy,ax).$$

So reflection in the biline at A_1 takes the line A_1X to the line $A_1+\lambda_1(cy,ax)$. Similarly, by computing the reflections of $(x-1,y)$ and $(x,y-1)$ in the bilines at A_2 and A_3, we find that the lines A_2X and A_3X get sent to the lines $A_2+\lambda_2(ax+(a-d)y-a,-ax-ay+a)$ and $A_3+\lambda_3(-cx-cy+c,x(c-d)+cy-c)$ respectively. It is now a computation that these three reflected lines meet at the point $i(X)$ as defined above. □

Example 5 *The isogonal conjugate of the centroid G is the* ***symmedian point***

$$K \equiv i\left(\left[\frac{1}{3},\frac{1}{3}\right]\right) = \frac{1}{2(a+c-b)}[c,a] = X_6.$$

Example 6 *The isogonal conjugate of the Orthocenter H is the Circumcenter:*

$$i\left(\left[\frac{b(c-b)}{\Delta},\frac{b(a-b)}{\Delta}\right]\right) = \frac{1}{2\Delta}[c(a-b),a(c-b)] = C = X_3.$$

3.3 Isotomic conjugates

Theorem 18 (Isotomic conjugates) *If X is a point distinct from A_1,A_2,A_3, then the lines joining the points A_1,A_2,A_3 to the reflections in the midpoints M_1,M_2,M_3 of the meets of A_1X,A_2X,A_3X with the lines of the Triangle are themselves concurrent, meeting in the* ***isotomic conjugate*** *of X. If $X=[x,y]$ then*

$$t(X) = \left[\frac{y(x+y-1)}{x^2+xy+y^2-x-y},\frac{x(x+y-1)}{x^2+xy+y^2-x-y}\right].$$

Proof. The point $X\equiv[x,y]$ has Cevian lines which meet the lines A_2A_3, A_1A_3, A_1A_2 respectively in the points

$$\left[\frac{x}{x+y},\frac{y}{x+y}\right] \qquad \left[0,\frac{y}{1-x}\right] \qquad \left[\frac{x}{1-y},0\right].$$

These three points may be reflected respectively in the midpoints $[1/2,1/2], [0,1/2], [1/2,0]$ to get the points

$$\left[\frac{y}{x+y},\frac{x}{x+y}\right] \qquad \left[0,\frac{1-x-y}{1-x}\right] \qquad \left[\frac{1-x-y}{1-y},0\right].$$

The lines $\langle x:-y:0\rangle$, $\langle 1-x-y:1-x:-1+x+y\rangle$ and $\langle 1-y:1-x-y:-1+x+y\rangle$ joining these points to the original vertices meet at $t(X)$ as defined above. □

Example 7 *The isotomic conjugate of the Orthocenter H is*

$$t\left(\left[\frac{(c-b)b}{ac-b^2},\frac{(a-b)b}{ac-b^2}\right]\right) = \left[\frac{a-b}{a+c-b},\frac{c-b}{a+c-b}\right] \equiv X_{69}.$$

4 Strong concurrences

4.1 Sight Lines, Gergonne and Nagel points

We now adopt the principle that algebraic verifications of incidence, using the quadratic relations, will be omitted.

A **Sight line** s_{ij} is the join of a Contact point C_{ij} with the Point A_i opposite to the Line that it lies on, and is naturally associated with the Incenter I_j. There are twelve Sight lines; three associated to each Incenter:

$$s_{10} = \langle \overline{c}+v : -\overline{a}+w : 0\rangle$$
$$s_{20} = \langle cv-bw : c(d+v-w) : -cv+bw\rangle$$
$$s_{30} = \langle a(d+v-w) : -aw+bv : aw-bv\rangle$$
$$s_{11} = \langle \overline{c}-v : -\overline{a}-w : 0\rangle$$
$$s_{21} = \langle -cv+bw : c(d-v+w) : cv-bw\rangle$$
$$s_{31} = \langle a(d-v+w) : aw-bv : -aw+bv\rangle$$
$$s_{12} = \langle \overline{c}+v : -\overline{a}-w : 0\rangle$$
$$s_{22} = \langle cv+bw : c(d+v+w) : -cv-bw\rangle$$
$$s_{32} = \langle a(d+v+w) : aw+bv : -aw-bv\rangle$$
$$s_{13} = \langle \overline{c}-v : -\overline{a}+w : 0\rangle$$
$$s_{23} = \langle -cv-bw : c(d-v-w) : cv+bw\rangle$$
$$s_{33} = \langle a(d-v-w) : -aw-bv : aw+bv\rangle.$$

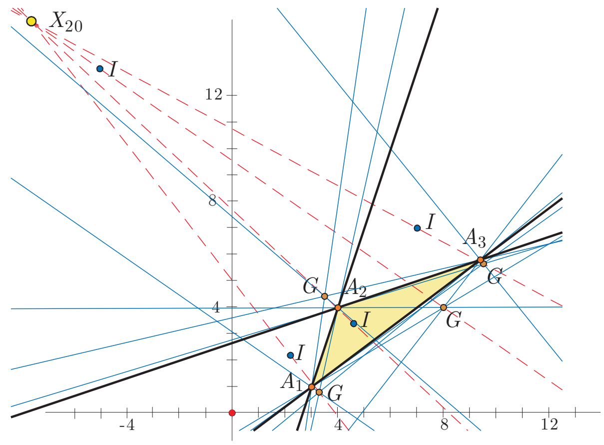

Figure 8: *Green Sight lines, Gergonne points G, In-Gergonne lines and In-Gergonne center X_{20}*

Here we introduce a well-known center of the triangle, the Gergonne point (see for example [2], [9]).

Theorem 19 (Gergonne points) *The triples $\{s_{10}, s_{20}, s_{30}\}$, $\{s_{11}, s_{21}, s_{31}\}$, $\{s_{12}, s_{22}, s_{32}\}$ and $\{s_{13}, s_{23}, s_{33}\}$ of Sight lines are concurrent. Each triple is associated to an Incenter, and the meets of these triples are the* ***Gergonne points*** *G_j. The Gergonne point associated to I_0 is*

$$G_0 = \frac{b-u}{2(du-cv+aw)-\Delta}\left[w-\overline{a}, -v-\overline{c}\right].$$

The join of a corresponding Incenter I_j and Gergonne point G_j is an **In-Gergonne line** or **Soddy line**. There are four Soddy lines, and

$$\begin{aligned} I_0G_0 = &\langle 2b\overline{c}v + (\Delta - 2b\overline{c})\, w - (\Delta - 2b\overline{c})\, d : \\ &2b\overline{a}w + (\Delta - 2b\overline{a})\, v + (\Delta - 2b\overline{a})\, d : \\ &-(\Delta - 2b\overline{a})\, v - (\Delta - 2b\overline{c})\, w\rangle. \end{aligned}$$

Theorem 20 (In-Gergonne center) *The four In-Gergonne/Soddy lines I_jG_j are concurrent, and meet at the De Longchamps point*

$$X_{20} = \frac{1}{\Delta}\left[b^2 - 2cb + ac, b^2 - 2ab + ac\right]$$

which is the orthocenter of the Double triangle. Furthermore the midpoint of $\overline{HX_{20}}$ is the Circumcenter C, so that X_{20} lies on the Euler line.

Proof. The concurrency of the In-Gergonne/Soddy lines I_jG_j is as usual. The equation

$$\begin{aligned} &\frac{1}{2\Delta}\left[b(c-b), b(a-b)\right] + \frac{1}{2\Delta}\left[b^2 - 2cb + ac, b^2 - 2ab + ac\right] \\ &= \frac{1}{2\Delta}\left[c(a-b), a(c-b)\right] = C \end{aligned}$$

shows that $C = \frac{1}{2}H + \frac{1}{2}X_{20}$. Since the Euler line is $e = CH$, X_{20} lies on e. □

Figure 8 shows the Gergonne points G and the In-Gergonne lines meeting at X_{20}.

Theorem 21 (Nagel points) *The triples $\{s_{11}, s_{22}, s_{33}\}$, $\{s_{10}, s_{32}, s_{23}\}$, $\{s_{20}, s_{31}, s_{13}\}$ and $\{s_{30}, s_{21}, s_{12}\}$ of Sight lines are concurrent. Each triple involves one Sight line associated to each of the Incenters, and so is associated to the Incenter with which it does not share a Sight line. The points where these triples meet are the* ***Nagel points*** *N_j. For example, $\{s_{11}, s_{22}, s_{33}\}$ meet at*

$$N_0 = \frac{1}{\Delta}\left[(b+u)\overline{a} + cv + bw, (b+u)\overline{c} - bv - aw\right].$$

Proof. We check that N_0 as defined is incident with $\langle \overline{c} - v : -\overline{a} - w : 0\rangle$ by computing

$$\begin{aligned} &\frac{(b+u)\overline{a} + cv + bw}{\Delta}(\overline{c} - v) + \frac{(b+u)\overline{c} - bv - aw}{\Delta}(-\overline{a} - w) \\ &= \frac{-cduv^2 + aduw^2 + c\overline{c}v^2w + a\overline{a}vw^2 - ac\overline{a}dv - ac\overline{c}dw}{\Delta} = 0 \end{aligned}$$

using the quadratic relations, (8) and (9).

The computations for the other Sight lines and N_1, N_2, N_3 are similar. □

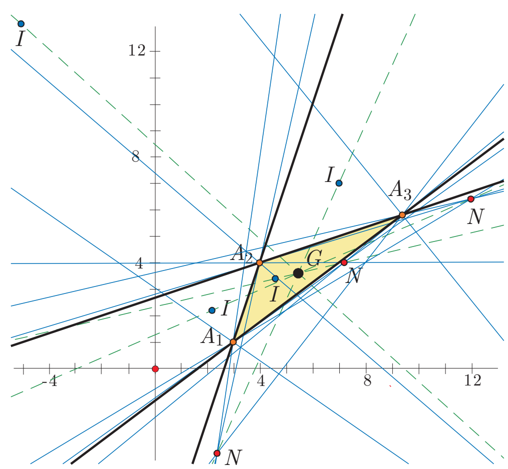

Figure 9: *Green Sight lines, Nagel points N, In-Nagel lines and In-Nagel center $G = X_2$*

The join of a corresponding Incenter and Nagel point is an **In-Nagel line.** There are four In-Nagel lines, and

$$I_0N_0 = \langle 2v + w - d : v + 2w + d : -v - w\rangle.$$

In classical triangle geometry, the line I_0N_0 is called simply the *Nagel line*.

Theorem 22 (In-Nagel center) *The four In-Nagel lines I_jN_j are concurrent, and meet at the Centroid $G = X_2$, and in fact $G = \frac{2}{3}I_j + \frac{1}{3}N_j$.*

Proof. Using the formulas above for I_0 and N_0, we see that

$$\frac{2}{3}I_0+\frac{1}{3}N_0=\left(\frac{2}{3}\right)\frac{1}{(d+v-w)}[-w,v]$$
$$+\left(\frac{1}{3}\right)\frac{1}{\Delta}[(b+u)\overline{a}+cv+bw,(b+u)\overline{c}-bv-aw]$$
$$=\frac{1}{3\Delta(d+v-w)}[\Delta(d+v-w),\Delta(d+v-w)]$$
$$=\frac{1}{3}[1,1]=G.$$

□

The join of a corresponding Gergonne point G_j and Nagel point N_j is a **Gergonne-Nagel line**. There are four Gergonne-Nagel lines, and

$$G_0N_0=\langle-\overline{a}u+\overline{a}v+aw:\overline{c}u+cv+\overline{c}w:-\overline{c}w-\overline{a}v\rangle.$$

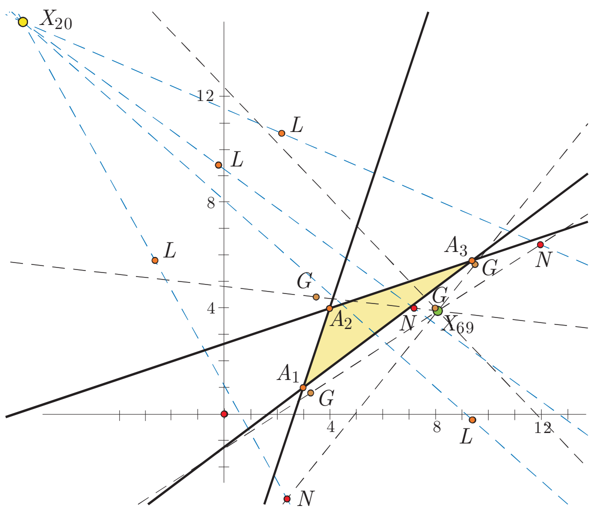

Figure 10: *Green Gergonne-Nagel center X_{69} and Nagel-New center X_{20}*

Theorem 23 (Gergonne-Nagel center) *The four Gergonne-Nagel G_jN_j lines are concurrent, and meet at the isotomic conjugate of the Orthocenter,*

$$X_{69}=\frac{1}{a+c-b}[\overline{c},\overline{a}].$$

The join of a corresponding New point L_j and Nagel point N_j is a **Nagel-New line**. There are four Nagel-New lines, and the one associated to I_0 is

$$L_0N_0=\langle ac-3ab+2b^2-\overline{c}u+bv+aw:$$
$$3cb-ac-2b^2+\overline{a}u+cv+bw:$$
$$(a-c)b+(a-c)u-\overline{a}v+\overline{c}w\rangle.$$

Theorem 24 (Nagel-New center) *The four Nagel-New lines N_jL_j meet in the De Longchamps point X_{20}, and in fact $L_j=\frac{1}{2}N_0+\frac{1}{2}X_{20}$.*

Proof. We check that

$$\frac{1}{2}X_{20}+\frac{1}{2}N_0=\left(\frac{1}{2}\right)\frac{1}{\Delta}[b^2-2cb+ac,b^2-2ab+ac]$$
$$+\left(\frac{1}{2}\right)\frac{1}{\Delta}[(b+u)\overline{a}+cv+bw,(b+u)\overline{c}-bv-aw]$$
$$=\frac{1}{2\Delta}[\overline{a}u+cv+bw+c\overline{c},\overline{c}u-bv-aw+a\overline{a}]=L_0.$$

□

4.2 InMid lines and Mittenpunkts

The join of an Incenter I_j with a Midpoint M_i is an **InMid line**. There are twelve InMid lines:

$$I_0M_1=\langle v+w-d:v+w+d:-v-w\rangle$$
$$I_0M_2=\langle v+w-d:2w:-w\rangle$$
$$I_0M_3=\langle 2v:v+w+d:-v\rangle$$
$$I_1M_1=\langle v+w+d:v+w-d:-v-w\rangle$$
$$I_1M_2=\langle v+w+d:2w:-w\rangle$$
$$I_1M_3=\langle 2v:v+w-d:-v\rangle$$
$$I_2M_1=\langle v-w-d:v-w+d:-v+w\rangle$$
$$I_2M_2=\langle v-w-d:-2w:w\rangle$$
$$I_2M_3=\langle 2v:v-w+d:-v\rangle$$
$$I_3M_1=\langle -v+w-d:-v+w+d:v-w\rangle$$
$$I_3M_2=\langle -v+w-d:2w:-w\rangle$$
$$I_3M_3=\langle -2v:-v+w+d:v\rangle.$$

Theorem 25 (InMid lines) *The triples of InMid lines $\{I_1M_1,I_2M_2,I_3M_3\}$, $\{I_0M_1,I_2M_3,I_3M_2\}$, $\{I_0M_2,I_1M_3,I_3M_1\}$ and $\{I_0M_3,I_1M_2,I_2M_1\}$ are concurrent. Each triple involves one InMid line associated to each of three Incenters, and so is associated to the Incenter which does not appear. The points where these triples meet are the* ***Mittenpunkts*** *D_j. For example, $\{I_1M_1,I_2M_2,I_3M_3\}$ meet at*

$$D_0=\frac{1}{2(a+c-b+u-v+w)}[c+u+w,a+u-v].$$

The join of a corresponding Incenter I_j and Mittenpunkt D_j is an **In-Mitten line**. There are four In-Mitten lines, and

$$I_0D_0=\langle(c+d)v+aw-ad:cv+(a+d)w+cd:-aw-cv\rangle.$$

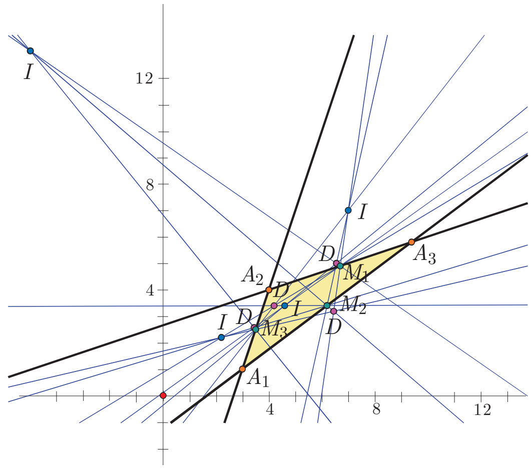

Figure 11: *Green InMid lines and Mittenpunkts D*

Theorem 26 (In-Mitten center) *The four In-Mitten lines are concurrent and meet at the* ***symmedian point*** *(see Example 3)*

$$K = X_6 = \frac{1}{2(a+c-b)}[c,a].$$

The join of a corresponding Gergonne point G_j and Mittenpunkt D_j is a **Gergonne-Mitten** line. There are four Gergonne-Mitten lines and

$$D_0G_0 = \left\langle \begin{array}{c} (\Delta-4bd)\,u+(4c\overline{c}-\Delta)\,v+2\,(4a\overline{a}-\Delta)\,w+(a-2\overline{a})\,\Delta: \\ -(\Delta-4bd)\,u+2\,(4c\overline{c}-\Delta)\,v+(4a\overline{a}-\Delta)\,w-(c-2\overline{c})\,\Delta: \\ (\Delta-4c\overline{c})\,v+(\Delta-4a\overline{a})\,w-\overline{b}\Delta \end{array} \right\rangle.$$

Theorem 27 (Gergonne-Mitten center) *The four Gergonne-Mitten lines G_jD_j meet in the Centroid $G = X_2$, and in fact $G = \frac{2}{3}D_j + \frac{1}{3}G_j$.*

Proof. We use the formulas above for D_0 and G_0 to compute

$$\begin{aligned} &\frac{2}{3}D_0 + \frac{1}{3}G_0 \\ &= \left(\frac{2}{3}\right)\frac{1}{2(a+c-b+u-v+w)}[c+u+w, a+u-v] \\ &+ \left(\frac{1}{3}\right)\frac{b-u}{2(du-cv+aw)-\Delta}[w-(c-b), -v-(a-b)] \\ &= \frac{1}{3}[1,1] = G. \end{aligned}$$

□

The join of a corresponding Mittenpunkt D_j and New point L_j is a **Mitten-New** line. There are four Mitten-New lines and

$$D_0L_0 = \langle av+bw-\overline{c}d : bv+cw+\overline{a}d : -b(v+w)\rangle.$$

Theorem 28 (Mitten-New center) *The four Mitten-New lines D_jL_j are concurrent, and meet at the Orthocenter*

$$H = \frac{1}{\Delta}[b\overline{a}, b\overline{c}].$$

Figure 12 shows the four In-Mitten lines meeting at $K = X_6$, the four Gergonne-Mitten lines meeting at $G = X_2$ and the four Mitten-New lines meeting at $H = X_4$.

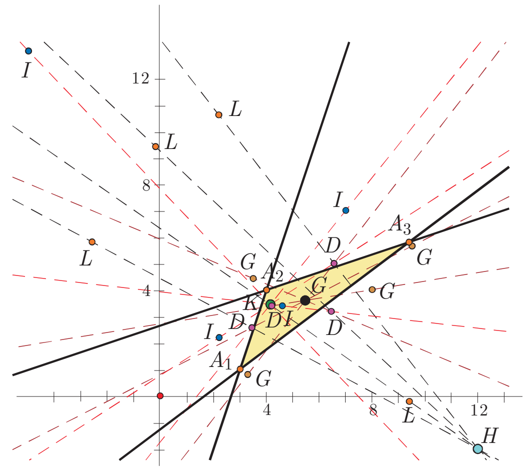

Figure 12: *Green Mitten-New center H, Gergonne-Mitten center G and In-Mitten center K*

4.3 Spieker points

The central dilation of an Incenter is a **Spieker point**. There are four Spieker points S_0, S_1, S_2, S_3 which are central dilations of I_0, I_1, I_2, I_3 respectively.

Theorem 29 (Spieker points) *The four Spieker points are*

$$\begin{aligned} S_0 &= \frac{1}{2}\frac{1}{(d+v-w)}[v+d, -w+d] \\ S_1 &= \frac{1}{2}\frac{1}{(d-v+w)}[-v+d, w+d] \\ S_2 &= \frac{1}{2}\frac{1}{(d+v+w)}[v+d, w+d] \\ S_3 &= \frac{1}{2}\frac{1}{(d-v-w)}[-v+d, -w+d]. \end{aligned}$$

Proof. We use the central dilation formula which takes $I_0 = (d+v-w)^{-1}[-w,v]$ to the point

$$\begin{aligned} S_0 \equiv \delta_{-1/2}(I_0) &= \frac{1}{2}\left[1-\frac{-w}{d+v-w}, 1-\frac{v}{d+v-w}\right] \\ &= \frac{1}{2(d+v-w)}[v+d, -w+d] \end{aligned}$$

and similarly for the other Spieker points. □

Theorem 30 (Spieker-Nagel lines) *The Spieker points lie on the corresponding In-Nagel lines, and in particular S_0, S_1, S_2, S_3 are the midpoints of the sides $\overline{I_0N_0}$, $\overline{I_1N_1}$, $\overline{I_2N_2}$, $\overline{I_3N_3}$ respectively.*

Proof. We check that in fact S_0 is the midpoint of $\overline{I_0N_0}$ by computing

$$\frac{1}{2}I_0+\frac{1}{2}N_0=\frac{1}{2}\frac{1}{(d+v-w)}[-w,v]$$
$$+\frac{1}{2}\frac{1}{\Delta}[(b+u)(c-b)+cv+bw,(b+u)(a-b)-bv-aw]$$
$$=S_0.$$

The computations for the other In-Nagel lines and S_1, S_2, S_3 are similar. □

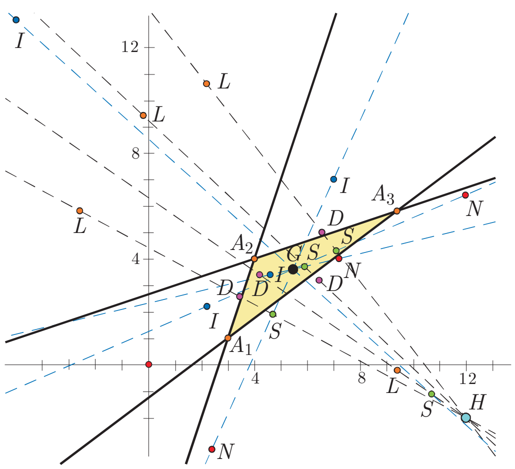

Figure 13: *Green Spieker points S and Mitten-Spieker center H*

The joins of corresponding Mittenpunkts D_j and Spieker points S_j are the **Mitten-Spieker lines**. There are four Mitten-Spieker lines, and

$$D_0S_0=\langle av+bw-\overline{c}d:bv+cw+\overline{a}d:-b(v+w)\rangle.$$

Theorem 31 (Mitten-Spieker center) *The four Mitten-Spieker lines D_jS_j are concurrent and meet at the Orthocenter $H=X_4$.*

Theorem 32 (New Mitten-Spieker) *The Spieker point S_j is the midpoint of $\overline{HL_j}$, so that the corresponding New point L_j also lies on the corresponding Mitten-Spieker line.*

Proof. The midpoint of $\overline{HL_0}$ is

$$\begin{aligned}\frac{1}{2}H+\frac{1}{2}L_0&=\frac{1}{2\Delta}[b(c-b),b(a-b)]\\&\quad+\frac{1}{4\Delta}[(c-b)u+cv+bw+c(a-b),\\&\qquad(a-b)u-bv-aw+a(c-b)]\\&=\frac{1}{4(ac-b^2)}[ac-b^2+(c-b)(u+b)+cv+bw,\\&\qquad ac-b^2+(a-b)(u+b)-aw-bv]\\&=\frac{1}{4(ac-b^2)}[ac-b^2+(c-b+w)(u+b),\\&\qquad ac-b^2+(a-b-v)(u+b)]\\&=\frac{1}{4}\left[1+\frac{(c-b+w)}{u-b},1+\frac{(a-b-v)}{u-b}\right]\\&=\frac{1}{4(u-b)}[c-2b+u+w,a-2b+u-v].\end{aligned}$$

Now a judicious use of the quadratic relations, which we leave to the reader, shows that this is S_0. The computations for the other Spieker points are similar. □

The proof shows in fact that there is quite some variety possible in the formulas for the various points and lines in this paper.

5 Future Directions

This paper might easily be the starting point for many more investigations, as there are lots of additional points in the Incenter hierarchy that might lead to similar phenomenon. In a related but slightly different direction, the basic idea of *Chromogeometry* ([12], [13]) is that we can expect wonderful relations between the corresponding geometrical facts in the *blue* (Euclidean bilinear form $x_1x_2+y_1y_2$), *red* (bilinear form $x_1x_2-y_1y_2$) and *green* (bilinear form $x_1y_2+y_1x_2$) geometries.

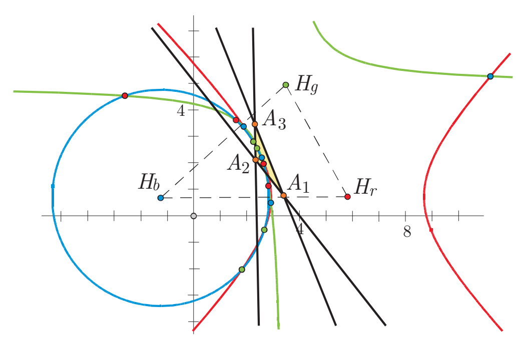

Figure 14: *Blue, red and green Incenter circles*

A spectacular illustration of this is the following, which we will describe in detail in a future work: if we have a triangle

$\overline{A_1A_2A_3}$ that has both blue, red and green Incenters (a rather delicate issue, as it turns out), then remarkably the four red Incenters and four green Incenters lie on a conic, in fact a *blue circle*, as in Figure 14. Similarly, the four red Incenters and four blue Incenters lie on a green circle, and the four green Incenters and four blue Incenters lie on a red circle. The centers of these three coloured Incenter circles are exactly the respective orthocenters H_b, H_r, H_g which form the *Omega triangle* of the given triangle $\overline{A_1A_2A_3}$, introduced in [12].

In particular the four green Incenters I that have appeared in our diagrams are in fact *concyclic in a Euclidean sense, as well as in a red geometry sense*. By applying central dilations, we may conclude similar facts about circles passing through Nagel points and Spieker points. Many more interesting facts wait to be discovered.

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Nguyen Le
e-mail: n.h.le@unsw.edu.au

Norman John Wildberger
e-mail: n.wildberger@unsw.edu.au

School of Mathematics and Statistics UNSW
Sydney 2052 Australia