ON THE DERIVATIVE OF SMOOTH MEANINGFUL FUNCTIONS[1]

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EXTENDED ABSTRACT

The derivative of a function f in n variables at a point x^* is one of the most important tools in mathematical modelling. If this object exists, it is represented by the row n-tuple $\nabla f(x^*) = [\partial f/\partial x_i(x^*)]$ called the gradient of f at x^*, abbreviated: "the gradient". The evaluation of $\nabla f(x^*)$ is usually done in two stages, first by calculating the n partials and then their values at $x = x^*$. In this talk we give an alternative approach. We show that one can characterize the gradient without differentiation! The idea is to fix an arbitrary row n-tuple G and answer the following question: What is a necessary and sufficient condition such that G is the gradient of a given f at a given x^*? The answer is given after adjusting the quadratic envelope property introduced in [3].

We work with smooth, i.e., continuously differentiable, functions with a Lipschitz derivative on a compact convex set with a non-empty interior. Working with this class of functions is not a serious restriction. In fact, loosely speaking, "almost all" smooth meaningful functions used in modelling of real life situations are expected to have a bounded "acceleration" hence they belong to this class. In particular, the class contains all twice differentiable functions [1]. An important property of the functions from this class is that every f can be represented as the difference of some convex function and a convex quadratic function. This decomposition was used in [3] to characterize the zero derivative points. There we obtained reformulations and augmentations of some well known classic results on optimality such as Fermat′s extreme value theorem (known from high school) and the Lagrange multiplier theorem from calculus [2, 3]. In this talk we extend the results on zero derivative points to characterize the relation $G = \nabla f(x^*)$, where G is an arbitrary n-tuple. Some special cases: If G = O, we recover the results on zero derivative points. For functions of a single variable on I = [a, b], the choice $G = [f(b) - f(a)]/(b - a)$ yields characterizations of points c where the instantaneous and average rates of change coincide [4], etc. The celebrated mean value theorem [2] claims that at least one such point c exists but it does not characterize it. These ideas are illustrated by examples and a photograph of an overpass in Beijing. A successful implementation of the new approach requires familiarity with the basic theory of infinite sequences.

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[1] Full paper based on this talk, with the title: "Equivalent formulations of the gradient" is forthcoming in Journal of Global Optimization (published online January 25, 2011).