MAGIC MOORE-PENROSE INVERSES AND PHILATELIC MAGIC SQUARES WITH SPECIAL EMPHASIS ON THE DANIELS–ZLOBEC MAGIC SQUARE

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Abstract
We study singular magic matrices in which the numbers in the rows and columns and in the two main diagonals all add up to the same sum. Our interest focuses on such magic matrices for which the Moore–Penrose inverse is also magic. Special attention is given to the “Daniels–Zlobec magic square’’ introduced by the British magician and television performer Paul Daniels (b. 1938) and considered by Zlobec (2001); see also Murray (1989, pp. 30–32). We introduce the concept of a “philatelic magic square” as a square arrangement of images of postage stamps so that the associated nominal values form a magic square. Three philatelic magic squares with stamps especially chosen for Sanjo Zlobec are presented in celebration of his 70th birthday; most helpful in identifying these stamps was an Excel checklist by Männikkö (2009).

Key words: *magic matrix, Moore–Penrose inverse, Paul Daniels, philatelic magic square.*

This paper is dedicated to Sanjo Zlobec in celebration of his 70th birthday.

Invited talk to be presented in the Session devoted to the 70th birthday of Professor Sanjo Zlobec at the 13th International Conference on Operational Research (KOI-2010), Split, Croatia, 30 September 2010. Based, in part, on Chu *et al.* (2010a) and research supported, in part, by the Natural Sciences and Engineering Research Council of Canada. This Word-file edited on 19 September 2010.

1. TWO DANIELS–ZLOBEC MAGIC MATRICES

Zlobec (2001) commented that:

> "In my undergraduate lectures on magic squares I sometimes mention Paul Daniels's magic square: Paul Daniels is a well-known British magician and TV personality who I often saw on Canadian TV in the 1970s. I copied the matrix for his magic square once from TV." It is

$$Z = \begin{pmatrix} 24 & 11 & 22 & 17 \\ 21 & 18 & 23 & 12 \\ 15 & 20 & 13 & 26 \\ 14 & 25 & 16 & 19 \end{pmatrix}.$$

We will call Z the *Daniels–Zlobec magic matrix*.

In a Festschrift for Martin Gardner (1914–2010) we find this "warning" from Persi Diaconis: "Martin Gardner has turned hundreds of mathematicians into magicians and hundreds of magicians into mathematicians!" (Morgan 2008, p. 3).

The rows, columns and two main diagonals of the Daniels–Zlobec magic matrix Z all add up to the *magic sum* 74. If, however, we subtract 1 from each element of Z we obtain

$$Z_1 = \begin{pmatrix} 23 & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix}$$

with magic sum 70. We will call Z_1 the *adjusted* Daniels–Zlobec magic matrix.

In an $n \times n$ magic matrix the numbers in the rows and columns and in the two main diagonals all add up to the same number, the *magic sum*. The $n \times n$ array defined by a magic matrix is called a *magic square*. When the entries are consecutive positive integers, then it is said to be *classic* (or *natural* or *normal*). Magic squares are over 1000 years old but the term "magic matrix" seems to have originated just over 50 years ago with the 1956 paper by Charles Fox (1897–1977), Professor of Mathematics at McGill University from 1949–1967. George Styan joined McGill in 1969 and Sanjo Zlobec came a year later in 1970, Sam Drury joined McGill in 1972. And Ka Lok Chu received his Ph.D. degree from McGill in 2004.

Murray (1989, pp. 30) gives the magic matrix M_d with parameter d defined by

$$M_d = \begin{pmatrix} d & 1 & 12 & 7 \\ 11 & 8 & d-1 & 2 \\ 5 & 10 & 3 & d+2 \\ 4 & d+1 & 6 & 9 \end{pmatrix},$$

which we call the *Daniels–Murray magic matrix.* Let E denote the 4×4 matrix with every element equal to 1. Then the Daniels–Zlobec magic matrices

$$Z = \begin{pmatrix} 24 & 11 & 22 & 17 \\ 21 & 18 & 23 & 12 \\ 15 & 20 & 13 & 26 \\ 14 & 25 & 16 & 19 \end{pmatrix} = M_{14} + 10E, \qquad Z_1 = \begin{pmatrix} 23 & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix} = M_{14} + 9E = Z - E.$$

We find that the determinant of the Daniels–Murray magic matrix

$$\det M_d = (d - 14)(d - 6)(d + 2)(d + 20)$$

and so M_d is singular if and only if d = 14, 6, –2 or –20. While M_d is magic for all d, it is classic magic only when d = 14 confirming that the Daniels–Zlobec magic matrices Z = M_{14} + 10E and Z_1 = M_{14} + 9E are both classic. Moreover, the Moore–Penrose inverse M_d^+ is magic only when d = 14 and so only when it is classic magic, and hence the Moore–Penrose inverse

$$Z^+ = \frac{1}{38480}\begin{pmatrix} 1277 & 167 & -55 & -869 \\ -1979 & 1055 & -647 & 2091 \\ 315 & 1129 & -1017 & 93 \\ 907 & -1831 & 2239 & -795 \end{pmatrix} = (M_{14} + 10E)^+$$

of the Daniels–Zlobec matrix Z is also magic, as observed by Zlobec (2001). The adjusted Daniels–Zlobec magic matrix Z_1 is also classic magic and its Moore–Penrose inverse is also magic.

A main goal in our research is to identify magic matrices, such as Z and Z_1, that have magic Moore–Penrose inverses. A key result is (Chu *et al.,* 2010a, 2010b):

The Moore–Penrose inverse of a V–associated magic matrix is also V–associated, and when the involutory matrix V is centrosymmetric, then the Moore–Penrose inverse is magic.

We have not found a magic matrix with a magic Moore–Penrose inverse that is not V-associated.

The symmetric *involutory* matrix V satisfies $V^2 = I$, the identity matrix, and the row totals of V are all equal to 1. The $n \times n$ magic matrix A with magic sum m is V-associated whenever

$$AV + VA = 2mE^* \quad \text{or equivalently} \quad A + VAV = 2mE^*,$$

where $E^* = (1/n)E$ has every element equal to $1/n$.

The involutory matrix V is *centrosymmetric* whenever it commutes with the flip matrix, i.e.,

$$FV = VF \quad \text{or equivalently} \quad FVF = V.$$

The symmetric *flip matrix F* is the identity matrix with its columns reversed and is involutory with row totals all equal to 1.

Magic matrices that are F-associated have been widely studied and are called (just) *associated* (or *associative, regular* or *symmetrical*) [Heinz and Hendricks (2000, pp. 8, 136)].

We define the special $2h \times 2h$ involutory matrix:

$$H = \begin{pmatrix} 0 & I_h \\ I_h & 0 \end{pmatrix},$$

which has all row totals equal to 1. It is easy to show that the 4x4 magic matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with magic sum m is H-associated whenever

$$M_{11} + M_{22} = M_{12} + M_{21} = (1/2)mE,$$

where $M_{11}, M_{12}, M_{21}, M_{22}$ are all 2x2, and E is the 2x2 matrix with every element equal to 1.

The adjusted Daniels–Zlobec matrix

$$Z_1 = \begin{pmatrix} \underline{23} & 10 & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & \underline{12} & 25 \\ 13 & 24 & 15 & 18 \end{pmatrix}$$

is H-associated and so, in particular, the underlined entries add to $m/2 = 35$.

We have shown that all V-associated magic matrices of even order are singular. It follows at once that the Daniels–Murray magic matrix cannot be H-associated if it is nonsingular.

In their wonderful book *Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration*, Ollerenshaw and Brée (1998, p. 20) define the $n \times n$ magic matrix $P = \{p_{i,j}\}$ with $n = 4h$ and magic sum m to be *most-perfect* whenever P is H-associated and the "blocks of 4" property holds:

$$p_{i,j} + p_{i,j+1} + p_{i+1,j} + p_{i+1,j+1} = m \text{ for all } i, j = 1, ..., n = 4h$$

with the subscripts taken modulo n (McClintock 1897).

A magic matrix is said to be *pandiagonal* whenever the numbers in the broken diagonals all add up to the magic sum. Pandiagonal magic squares are sometimes called *diabolic, diabolical, Nasik,* or *panmagic*. When $n = 2h$, then H-associated magic matrices are *pandiagonal*. Moreover, as Ollerenshaw and Brée (1998, pp. 21–22) show, with $n = 4h$ all most-perfect magic matrices are pandiagonal. We have recently shown the stronger result that all H-associated magic matrices with $n = 2h$ are pandiagonal (Chu et al, 2010a, 2010b).

All 4 x 4 pandiagonal matrices are most-perfect (Heinz and Hendricks, p. 97) and hence H-associated. When the order $n = 4h$ with $h \geq 2$, however, a pandiagonal matrix need not be H-associated and hence not most-perfect. For example, the 8 x 8 classic magic matrix

$$\begin{pmatrix} 29 & 54 & 42 & 35 & 6 & 22 & 46 & 26 \\ 31 & 7 & 64 & 10 & 20 & 52 & 62 & 14 \\ 17 & 50 & 47 & 2 & 28 & 44 & 60 & 12 \\ 1 & 32 & 38 & 61 & 24 & 40 & 8 & 56 \\ 59 & 58 & 4 & 39 & 36 & 11 & 23 & 30 \\ 45 & 13 & 3 & 51 & 34 & 43 & 16 & 55 \\ 37 & 21 & 5 & 53 & 63 & 15 & 18 & 48 \\ 41 & 25 & 57 & 9 & 49 & 33 & 27 & 19 \end{pmatrix}$$

is pandiagonal but not H-associated, and so not most-perfect.

The adjusted Daniels–Zlobec magic matrix and its Moore–Penrose inverse are, respectively,

$$Z_1 = \begin{pmatrix} 23 & \underline{10} & 21 & 16 \\ 20 & 17 & 22 & 11 \\ 14 & 19 & 12 & \underline{25} \\ 13 & 24 & 15 & 18 \end{pmatrix}, \quad Z_1^+ = \frac{1}{7280}\begin{pmatrix} 243 & \underline{33} & -9 & -163 \\ -373 & 201 & -121 & 397 \\ 61 & 215 & -191 & \underline{19} \\ 173 & -345 & 425 & -149 \end{pmatrix},$$

and both Z_1 are Z_1^+ are pandiagonal, most-perfect and H-associated.

Let M denote a 4×4 magic matrix with magic sum m. When its characteristic polynomial

$$\det(\lambda I - M) = \lambda(\lambda - m)(\lambda^2 - \kappa); \qquad \kappa \neq 0,$$

then we say that M is *keyed* with *magic key* $\kappa = (\mathrm{tr}M^2 - m^2)/2 \neq 0$, and then for $p = 0, 1, \ldots$

$$M^{2p+1} = \kappa^p M + m(m^{2p} - \kappa^p)E^*$$

and so the odd powers M^{2p+1} are all magic and inherit all the patterns present in the parent matrix M.

We find that Z_1 is keyed with magic key $\kappa = -48$ and for $p = 0, 1, \ldots$ we have

$$Z_1^{2p+1} = (-48)^p Z_1 + 70(70^{2p} - (-48)^p)E^*.$$

The $n \times n$ matrix A has *index* equal to 1 whenever rank(A^2) = rank(A). The *group inverse* $A^\#$ of a matrix A with index 1 is the unique matrix $A^\#$ which satisfies

$$AA^\# A = A, \qquad A^\# AA^\# = A^\#, \qquad AA^\# = A^\# A.$$

We find that Z_1 has index 1 with group inverse

$$Z_1^\# = \frac{1}{\kappa} Z_1 + \left(\frac{1}{m} - \frac{m}{k}\right)E^* = \frac{1}{3360}\begin{pmatrix} -373 & 537 & -233 & 117 \\ -163 & 47 & -303 & 467 \\ 257 & -93 & 397 & -513 \\ 327 & -443 & 187 & -23 \end{pmatrix},$$

which is both pandiagonal and H-associated. The group inverse $Z_1^\#$ does not, however, coincide with the Moore–Penrose inverse Z_1^+.

2. THREE PHILATELIC MAGIC SQUARES

Following a suggestion by Loly (2008), we define a *philatelic magic square* (PMS) as a square arrangement of images of postage stamps so that the associated nominal values form a magic square.

Figure 1: *Special 4x4 philatelic magic square for the adjusted Daniels–Zlobec magic matrix Z_1.*

Figure 2: *Special 3x3 Croatian-diagonal philatelic magic square for Sanjo Zlobec's 70^{th} birthday.*

Our first PMS (Figure 1) is based on the adjusted Daniels–Zlobec matrix Z_1 with magic sum 70. Featured are (row 1) Halley's comet, al-Biruni, again Halley's comet (with telescopes from Newton, Cassegrain, Ritchey and Galileo), Pythagoras; (row 2) Nunes, Kepler, Babbage, Franklin; (row 3) Karamata, Napoleon Bonaparte, Terradas, Chagall's "Magician of Paris"; and (bottom row) Dürer, Tesla, Banach, and Poincaré.

Our next two PMS (Figures 2 and 3) have been specially created for Sanjo Zlobec in celebration of his 70^{th} birthday. The Croatian-diagonal PMS shown in Figure 2 features on the diagonal in the (1,1) position: *Arithmetika Horvatszka,* the first arithmetic book in Croatian, published in 1758; stamp issued by Croatia 2008 to celebrate the 250th publication anniversary; (2, 2) Rudjer Joseph Boscovich (1711–1787), issued by Croatia 1943; (3,3) Nikola Tesla (1856–1943), stamp issued by Croatia 2006 based on the 1901 photograph

"Nikola Tesla, with Rudjer Boscovich's book *Theoria Philosophiae Naturalis,* in front of the spiral coil of his high-frequency transformer at East Houston Street, New York" by George Grantham Bain (1865–1944). Tesla was an important contributor to the birth of commercial electricity, and is best known for his many revolutionary developments in the field of electromagnetism. [Many thanks to Sanjo for getting this stamp for us.]

Featured on the off-diagonal are in the (1,2) position Halley's comet; (1,3) Silk-thread writings by the Muslim polymath Al-Biruni (973–1048); (2,1) Emanuel Lasker (1868–1941), German mathematician who was World Chess Champion for 27 years (1894–1921); (2,3) Niels Henrik Abel (1802–1829), Norwegian mathematician after whom the Abel Prize is named; (3,1) Francisco Ruiz Lozano (1607–1677), Peruvian astronomer and mathematician; (3, 2) Ernest Rutherford, 1st Baron Rutherford of Nelson (1871–1937), who held the chair of physics at McGill University from 1898–1907. His work at McGill gained him the Nobel Prize in 1908 in *chemistry*, which is ironic since he had said that "All science is either physics or stamp collecting" (Blackett 1962, p. 108).

Figure 3: Special 4×4 Farkas–Tesla–Napoleon philatelic magic square for Sanjo Zlobec's 70th birthday.

The Farkas–Tesla–Napoleon PMS (Figure 3) has 70 in the top-left corner and 40 in the top-right. These two stamps in the top row were issued for *Farkas* Bolyai (1775–1856), while the stamp in position (2,3) depicts his son János Bolyai (1802–1860). Both Bolyai *père et fils* were Hungarian mathematicians, as was

Julius *Farkas* (1847–1930) for whom we have not found a stamp. The Farkas lemma [Zlobec (2001, Cor. 3.8, p. 32)] is:

Let A be an m×n matrix and b an m x 1 *vector. Then there is an n* x 1 *vector* $x \geq 0$ *such that* $Ax = b$ *if and only if* $A'y \geq 0$ *implies* $b'y \geq 0$.

The stamps in positions (2,4) and (3,1) depict Nikola Tesla (1856–1943). The stamp in position (2,2) depicts Longwood House, the residence of Napoleon Bonaparte (1769–1821) during his exile on the island of Saint Helena, 1815–1821. The stamp in position (3,2) is based on the painting "Napoléon sur son lit de mort d'après Rouget"; Inti Zlobec *et al.* (Lugli *et al.* 2007) recently found that the cause of Napoleon's death "was strongly suggestive of stomach cancer".

Napoleon is best known as a military genius and Emperor of France but it seems he was also an outstanding mathematics student. His favourite topic apparently was geometry. From Coxeter and Greitzer (1967, Th. 3.36, p. 63) we find Napoleon's Theorem:

If equilateral triangles are erected externally on the sides of any triangle, their centres form an equilateral triangle.

The stamp in position (1,2) depicts Benjamin Franklin (1706–1790), one of the Founding Fathers of the United States. Also printer, postmaster and scientist, Franklin "worked" on magic squares:

"I confess that in my younger days, having once some leisure time which I still think I might have employed more usefully, I amused myself in making a kind of magic squares ..."

In Position (1,3) is a new local mail stamp from Sakhalin (Russia) for Rudjer Joseph Boscovich (1711–1787), mathematician, physicist and astronomer born in Ragusa (now Dubrovnik).

The Farkas–Tesla–Napoleon magic matrix Z_2 and its Moore–Penrose inverse $Z_2^{\#}$,

$$Z_2 = \begin{pmatrix} 70 & 65 & 15 & 40 \\ 45 & 10 & 60 & 75 \\ 20 & 35 & 85 & 50 \\ 55 & 80 & 30 & 25 \end{pmatrix}, \qquad Z_2^{+} = \frac{1}{25840} \begin{pmatrix} 205 & 167 & -213 & -23 \\ 15 & -251 & 129 & 243 \\ -175 & -61 & 319 & 53 \\ 91 & 281 & -99 & -137 \end{pmatrix}$$

are both F-associated. Moreover, Z_2 has index 1 and hence it has a group inverse $Z_2^{\#}$, which we find coincides with the Moore–Penrose inverse.

Another goal in our research is to identify magic matrices with a magic Moore–Penrose inverse that are also EP. The square matrix A with index 1 is said to be EP whenever $\operatorname{rank}(A : A') = \operatorname{rank}(A)$, i.e., whenever the column spaces of A and its transpose A' coincide.

The group inverse $A^{\#}$ of the index 1 matrix A equals the Moore–Penrose inverse A^{+} if and only if A is EP [Ben-Israel and Greville (2003, p. 157)]. The Farkas–Tesla–Napoleon magic matrix Z_2 is, therefore, EP, but the adjusted Daniels–Zlobec magic matrix Z_1 is not.

We believe that the term EP was introduced in 1950 in the First Edition of Schwerdtfeger's *Introduction to Linear Algebra and the Theory of Matrices* [2nd edition (1961, pp. 130–131)]. Hans Schwerdtfeger (1902–1990) was Professor of Mathematics at McGill University from 1960–1983.

The stamps in positions (2,1) and (4,3) are part of a set issued by several countries in the Middle East to celebrate Arabic achievements in various fields including specifically "algebra". The stamp in position (3,3) depicts the 39th Mersenne prime number $2^{13466917} - 1$ discovered in 2001 (stamp issued by Liechtenstein 2004). The mathematician and physicist Baron Jurij Bartolomej Vega (1754–1802), born in Zagorica (now in Slovenia), is depicted in position (3,4), while a slide rule is shown in the stamp in position (4,1) issued by Romania for the 2nd Congress of the Society of Engineers and Technicians, held in Bucharest, 29–31 May 1957. The stamp in Position (4,3) depicts Jacob Bernoulli (1654–1705), who was born in Basel, Switzerland, and who first proved the law of large numbers (illustrated on the stamp), while the last stamp shows the Polish astronomer and geodesist Tadeusz Banachiewicz (1882–1954), who invented "Cracovians"—a special kind of matrix algebra. Banachiewicz is credited with the following formula of
the inverse of a partitioned matrix

$$Q = \begin{pmatrix} R & S \\ T & U \end{pmatrix}^{-1} = \begin{pmatrix} R^{-1} + R^{-1}S(Q/R)^{-1}TR^{-1} & -R^{-1}S(Q/R)^{-1} \\ -(Q/R)^{-1}TR^{-1} & (Q/R)^{-1} \end{pmatrix}$$

where the Schur complement $(Q/R) = U - TR^{-1}S$ [Grala, Markiewicz and Styan (2000)].

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