Kernel-Based Interior-Point Methods for Cartesian $P_*(\kappa)$-Linear Complementarity Problems over Symmetric Cones *

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* This article is dedicated to Professor Sanjo Zlobec on occasion of his 70th birthday

Abstract

We present an interior-point method (IPM) for Cartesian $P_*(\kappa)$- Linear Complementarity Problems over Symmetric Cones (SCLCPs). The Cartesian $P*(\kappa)$- SCLCPs have been recently introduced as the generalization of the more commonly known and more widely used monotone SCLCPs. The IPM is based on the barrier functions that are defined by a large class of univariate functions called eligible kernel functions which have recently been successfully used to design new IPMs for various optimization problems. Eligible barrier (kernel) functions are used in calculating the Nesterov-Todd search directions and the default step-size which leads to very good complexity results for the method. For some specific eligible kernel functions we match the best known iteration bound for the long-step methods while for the short-step methods the best iteration bound is matched for all cases.

Keywords: linear complementarity problem, Cartesian $P_*(\kappa)$ property, Euclidean Jordan algebras and symmetric cones, interior-point method, kernel functions, polynomial complexity.

1 Introduction

Let $(V, \circ)$ be an n-dimensional Euclidean Jordan Algebra (EJA) with rank r equipped with the standard inner product $\langle x, y\rangle = \mathrm{tr}(x \circ y)$. Let $\mathcal{K}$ be the corresponding symmetric cone. We consider the Linear Complementarity Problem over Symmetric Cone (SCLCP) in the *standard form*: Given an n-dimensional EJA $(V, \circ)$ and its associated symmetric cone of squares $\mathcal{K}$, find $(x, s) \in \mathcal{K} \times \mathcal{K}$ such that

$$s = Mx + q, \quad \langle x, y\rangle = 0, \tag{1}$$

where $M \in \mathbf{R}^{n\times n}$ and $q \in \mathbf{R}^n$. The second equation can be replaced by $x \circ y = 0$ due to the important fact that $\langle x, y\rangle = 0 \Leftrightarrow x \circ y = 0$. (Lemma 2.2 in [7]).

Although SCLCP is not an optimization problem, it is closely related to the optimization problems. One of the reasons is that the optimality conditions of several important optimization problems can be written in the form of SCLCP. For example, the solution of a Linear Optimization (LO) problem over symmetric cones (SCLO) can be formulated as SCLCP.

The algebraic structure of EJA has been known for quite same time. However, only in recent years it has been established [11] that symmetric cones, which are cones of squares of EJAs, serve as a unifying framework to which the important cases of cones used in optimization such as nonnegative orthant, second order cone, and positive semidefinite cone belong. The classical monograph of Faraut and Korany [6] provides a wealth of information on Jordan algebras, symmetric cones and related topics.

It is well known that SCLCP is an NP-hard problem for a general matrix M, even in the case when $\mathcal{K} = \mathbf{R}^n_+$, i.e. nonnegative orthant (see [3]). Thus, we need to restrict ourselves to classes of matrices for which polynomial IPMs exist. The class of monotone LCP where matrix M is positive semidefinite matrix appears most frequently in practice and therefore IPMs to solve this class were most frequently analyzed. Kojima et al. introduced in [12] a class of $P_*(\kappa)$-LCP (over nonnegative orthant) established the existence of the central path and designed IPM for solving this class of problems. The theoretical importance of this class lays in the fact that this is the largest class for which polynomial convergence of IPMs can be proved without additional conditions (such as boundedness of the level sets). They also discussed how is this class related to other classes of LCP. See also the monographs of Cottle et al. [4] and Fachinei and Pang [5].

While there is an extensive literature on IPMs for LCPs over the nonnegative orthant, there are fewer results on LCPs over the cone of semidefinite matrices, and even less so for LCP over general symmetric cones. Faybusovich was the first to analyze a short-step path-following IPM for SCLO and SCLCP [7, 8]. Rangarajan [20] proposed the first infeasible IPM for SCLCP. Yoshise [24, 25] was the first to analyze IPMs for Nonlinear Complementarity Problems (NCP) over symmetric cones. All these results are obtained for monotone complementarity problems. Luo and Xiu in [16] introduced Cartesian $P_*(\kappa)$-SCLCP as the generalization $P_*(\kappa)$-LCP over nonnegative orthant, proved the existence and uniqueness of the central path under the strict feasibility condition and analyzed short- and long- step IPMs obtaining favorable convergence results. Other classes of SCLCP were considered by Kong et al. [13] and Gowda et al. [9, 10].

All of the above mentioned results were obtained by using the classical logarithmic barrier function adapted for the framework of symmetric cones. The only result for IPMs with other types of barrier (kernel) functions, the eligible kernel functions, was obtained by Lesaja and Roos [15]. Vieira [23], considered IPMs based on the same class of kernel functions; however, only for SCLO. The problem under consideration in [15] was monotone SCLCP. In this paper we consider a generalization of kernel-based IPM discussed in [15] to Cartesian $P_*(\kappa)$-SCLCP. The obtained complexity results match the best known iteration bounds known for the small-update methods, $O((1+2\kappa)\sqrt{r}\log\frac{r}{\epsilon})$, and large-update methods, $O((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\epsilon})$.

The outline of the paper is as follows. In Section 2 we describe $P_*(\kappa)$-SCLCP. In Section 3 we give a Generic IPM to solve $P_*(\kappa)$-SCLCP that is based on the use of a general barrier function. In Section 4 the definition and few basic facts about the class of eligible kernel functions and corresponding barrier functions are mentioned. In Section 5 we outline the analysis and complexity of the Generic IPM stated in Section 3 when the barrier function is the eligible barrier (kernel) function. In Section 6 the iteration bounds are given for a number of specific eligible kernel functions. Concluding remarks are presented in Section 7.

2 Cartesian $P_*(\kappa)$-SCLCP

In this section we define Cartesian $P_*(\kappa)$-SCLCP. In what follows we assume that the reader is familiar with the basic concepts of EJA and symmetric cones. The detailed information can be found in the monograph of Faraut and Korany [6] and in [7, 21, 23, 15] (related to optimization).

Let V be a Cartesian product of a finite number of simple Euclidean Jordan algebras $(V_j, \circ)$ with dimension n_j and rank r_j for $j = 1, \ldots, m$, that is, $V = V_1 \times V_2 \times \cdots \times V_m$ with its cone of squares $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_m$, where $\mathcal{K}_j$ are corresponding cones of squares of V_j for for $j = 1, \ldots, m$. The dimension of V is $n = \sum_m^{j=1} n_j$ and the rank is $r = \sum_m^{j=1} r_j$. The bilinear map on V is defined as

$$x \circ s = \left(x^{(1)} \circ s^{(1)}, \ldots, x^{(m)} \circ s^{(m)}\right)^T, \tag{2}$$

where $x = \left(x^{(1)}, \ldots, x^{(m)}\right)^T$ and $s = \left(s^{(1)}, \ldots, s^{(m)}\right)^T$ in V with $x^{(j)}, s^{(j)} \in V_j$, $j = 1, \ldots, m$.
Let

$$x^{(j)} = \sum_{k=1}^{r_j} \lambda_k(x^{(j)}) c_k^{(j)}$$

be the spectral decomposition of $x^{(j)} \in V_j$ with the Jordan frame $c_1^{(j)}, \ldots, c_{r_j}^{(j)}$ and eigenvalues $\lambda_1(x^{(j)}), \ldots, \lambda_{r_j}(x^{(j)})$. Then

$$\det(x^{(j)}) = \prod_{k=1}^{r_j} \lambda_k(x^{(j)}), \quad \operatorname{tr}(x^{(j)}) = \sum_{k=1}^{r_j} \lambda_k(x^{(j)}), \quad \|x^{(j)}\|_F = \sqrt{\left\langle x^{(j)}, x^{(j)} \right\rangle} = \sqrt{\sum_{k=1}^{r_j} \lambda_k^2(x^{(j)})},$$

$$\lambda_{min}(x^{(j)}) = \min\{\lambda_k(x^{(j)}) : 1 \le k \le r_j\}, \quad \lambda_{max}(x^{(j)}) = \max\{\lambda_k(x^{(j)}) : 1 \le k \le r_j\}.$$

The spectral decomposition of $x = \left(x^{(1)}, \ldots, x^{(m)}\right)^T \in V$ is then defined accordingly. Furthermore, we have

$$\operatorname{tr}(x) = \sum_{j=1}^{m} \operatorname{tr}(x^{(j)}), \quad \det(x) = \prod_{j=1}^{m} \det(x^{(j)}), \quad \|x\|_F = \sqrt{\sum_{j=1}^{m} \|x^{(j)}\|_F^2},$$

$$\lambda_{min}(x) = \min\{\lambda_{min}(x^{(j)}) : 1 \le j \le m\}, \quad \lambda_{max}(x) = \max\{\lambda_{max}(x^{(j)}) : 1 \le j \le m\}.$$

The index F for the Frobenius norm $\| \quad \|_F$ will be omitted in the sequel whenever there is no danger of confusion. The Peirce decomposition of $x \in V$ can be defined in a similar manner by using Peirce decompositions of components $x^{(j)} \in V_j$.

The multiplication, $L(x)$, and quadratic representation, $P(x)$, operators are defined as follows:

$$L(x) = \operatorname{diag}\left(L(x^{(1)}), \ldots, L(x^{(m)})\right), \quad P(x) = \operatorname{diag}\left(P(x^{(1)}), \ldots, P(x^{(m)})\right).$$

The SLCP (1) is said to be Cartesian $P_*(\kappa)$-SCLCP if EJA V and the corresponding symmetric cone of squares $\mathcal{K}$ are Cartesian products of simple EJAs V_j and corresponding symmetric cones of squares $\mathcal{K}_j$, as described above. In addition, the following $P_*(\kappa)$ property is satisfied for each $x \in V$ if

$$(1+4\kappa) \sum_{j \in \mathcal{I}^+(x)} \left\langle x^{(j)}, (Mx)^{(j)} \right\rangle + \sum_{j \in \mathcal{I}^-(x)} \left\langle x^{(j)}, (Mx)^{(j)} \right\rangle \ge 0\,, \tag{3}$$

where κ is a nonnegative constant and

$$j \in \{1, \ldots, m\}, \quad \mathcal{I}^+(x) = \left\{j \;:\; \left\langle x^{(j)}, (Mx)^{(j)} \right\rangle > 0\right\} \quad \mathcal{I}^-(x) = \left\{i \;:\; \left\langle x^{(j)}, (Mx)^{(j)} \right\rangle < 0\right\}\,,$$

or equivalently

$$\langle x, Mx \rangle \ge -4\kappa \sum_{j \in \mathcal{I}^+(x)} \left\langle x^{(j)}, (Mx)^{(j)} \right\rangle\,.$$

Furthermore, we define

$$P_* := \bigcup_{\kappa \ge 0} P_*(\kappa)\,.$$

The SCLCP has monotone property if for each $x \in V$ the following holds: $\langle x, Mx \rangle \ge 0$. This is equivalent with the fact that matrix M is positive semidefinite with respect to the inner product in V. It is easy to see that Cartesian $P_*(0)$-SCLCP represents the monotone SCLCP. It is also straight-forward to see that in the case of nonnegative orthant, where $V = \mathbf{R}^n$ and $x \circ s = (x_1 s_1, \ldots, x_n s_n)$, the definition (3) reduces to the original definition of $P_*(\kappa)$ property given by Kojima et al. in [12] by taking $V_j = \mathbf{R}$ and $x^{(j)} \circ s^{(j)} = x_j s_j$ for $j = 1, \ldots, n$.

3 Interior - Point Method for Cartesian $P_*(\kappa)$-SCLCP

In this section we present a generic IPM for Cartesian $P_*(\kappa)$-SCLCP (1). The standard approach is to consider the perturbed system

$$\begin{aligned} s &= Mx + q, \\ x \circ s &= \mu e, \\ (x, s) &\in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K}. \end{aligned} \tag{4}$$

The main question that arises is whether this system has a solution. It has been shown that if (1) has a strictly feasible point, then the above system (4) has a unique solution, for each $\mu > 0$ (Lemma 3.2. in [16]). The strict feasibility condition, or as it is often called the interior-point condition (IPC), means that there exists a point $(x, s) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K}$ such that $s = Mx + q$. Thus, in the sequel we will assume that the Cartesian $P_*(\kappa)$-SCLCP (1) has IPC.

The unique solution of (4) for each μ is denoted as $(x(\mu)), s(\mu))$ and we call it the μ-center of Cartesian $P_*(\kappa)$-SCLCP. The set of μ-centers gives a path, which is called the *central path* of Cartesian $P_*(\kappa)$-SCLCP. Hence, if $\mu \to 0$, the limit of the central path exists and it is a solution of Cartesian $P_*(\kappa)$-SCLCP.

The limiting property of the central path, mentioned above, leads naturally to the main idea of the IPMs for solving Cartesian $P_*(\kappa)$-SCLCP: trace the central path while reducing μ at each iteration. However, tracing the central path exactly would be too costly and inefficient. It has been shown that it is sufficient to trace the central path approximately.

In the rest of the paper whenever possible we consider one component of the Cartesian structure of SCLCP and we omit the component index (for example, (11) and (15)). The complete structure can be built straightforwardly as described in Section 2.

Suppose that a strictly feasible point (x, s) that is 'close' to the μ-center $(x(\mu), s(\mu))$, for some positive μ, is known. The 'closeness' is measured using a barrier function, as discussed later in the paper. We then decrease μ to $\mu_+ := (1-\theta)\mu$, for some $\theta \in (0, 1)$ and, redefining $\mu := \mu_+$ calculate a search direction using Newton's method.

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ s \circ \Delta x + x \circ \Delta s &= -x \circ s + \mu e. \end{aligned} \tag{5}$$

As it was first discovered for Semidefinite Optimization (SDO) the above system unfortunately does not have a unique solution. The reason is the fact that x and s do not operator commute in general, that is, $L(x)L(s) \neq L(s)L(x)$. In order to fix this problem a scaling scheme had to be applied. Several different scaling schemes were proposed leading to different search directions.The generalization to the symmetric cones case was first proposed by Faybusovich [7, 8] and then further discussed by Alizadeh and Schmieta [21, 22].

The scaling scheme is based on the following fact (Lemma 28 in [22]): Let $u \in \operatorname{int}\mathcal{K}$. Then,

$$x \circ s = \mu e \quad \Leftrightarrow \quad P(u)x \circ P(u^{-1})s = \mu e.$$

Now replacing the second equation of (4) by $P(u)x \circ P(u^{-1})s = \mu e$ and applying Newton's method we obtain the system

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ P(u^{-1})s \circ P(u)\Delta x + P(u)x \circ P(u^{-1})\Delta s &= -P(u)x \circ P(u^{-1})s + \mu e. \end{aligned} \tag{6}$$

The appropriate choices of u that lead to obtaining unique search directions from system (6) can be generalized from the SDO case. We use the classical Nesterov-Todd scaling point to find unique Nesterov-Todd direction (NT-direction)

$$u = w^{-\frac{1}{2}}, \quad w = P(x)^{\frac{1}{2}}(P(x^{\frac{1}{2}})s)^{-\frac{1}{2}} \quad \left[= P(s^{-\frac{1}{2}})(P(s^{\frac{1}{2}})x)^{\frac{1}{2}} \right]. \tag{7}$$

This scaling point was first proposed by Nesterov and Todd for self-scaled cones [19] and then adapted by Faybusovich [8] for symmetric cones.

For the formulation and analysis of the generic IPM that we want to present, the introduction of the following *variance vector* is of critical importance

$$v := \frac{1}{\sqrt{\mu}} P(w)^{-\frac{1}{2}} x \quad \left[= \frac{1}{\sqrt{\mu}} P(w)^{\frac{1}{2}} s \right]. \tag{8}$$

Note that $x \circ s = \mu e \quad \Leftrightarrow v = e$ (Proposition 5.7.2 in [23]). Thus, μ-centers are obtained when $v = e$. Using the variance vector v the following scaled search directions are introduced

$$d_x = \frac{1}{\sqrt{\mu}} P(w)^{-\frac{1}{2}} \Delta x, \qquad d_s = \frac{1}{\sqrt{\mu}} P(w)^{\frac{1}{2}} \Delta s. \tag{9}$$

Using (9) the system (6) can be rewritten as

$$\begin{aligned} -\overline{M} d_x + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{10}$$

where $\overline{M} := P(w)^{\frac{1}{2}} M P(w)^{\frac{1}{2}}$. Given the fact that M has Cartesian $P_*(\kappa)$ property (3), it follows from

$$\langle u, \ \overline{M} u \rangle = \left\langle u, \ \left(P(w)^{\frac{1}{2}} M P(w)^{\frac{1}{2}} \right) u \right\rangle = \left\langle P(w)^{\frac{1}{2}} u, \ M \left(P(w)^{\frac{1}{2}} u \right) \right\rangle$$

that $\overline{M}$ also has Cartesian $P_*(\kappa)$ property.

Given the spectral decomposition of v with respect to the Jordan frame $(c_1, \cdots c_r)$, $v = \sum_{i=1}^{r} \lambda_i(v) c_i$, the *scaled logarithmic barrier function* is defined as

$$\Psi_c(v) := \sum_{i=1}^{r} \left(\frac{\lambda_i(v)^2 - 1}{2} - \log \lambda_i(v) \right). \tag{11}$$

A crucial observation is that the second equation in (10) can be written as

$$v^{-1} - v = -\nabla \Psi_c(v), \tag{12}$$

where $\nabla \Psi_c(v)$ denotes the gradient of $\Psi_c(v)$. This means that the logarithmic barrier function essentially determines the search direction. Another important observation is: since $\nabla \Psi_c(e) = 0$, it follows that $\Psi_c(v)$ attains its minimal value at $v = e$, with $\Psi_c(e) = 0$, that is, at μ-centers, $(x(\mu), s(\mu))$. In addition, $\Psi_c(v)$ is strictly convex function. The implication is that the function $\Psi_c(v)$ essentially serves as a *'proximity' measure* of closeness for (x, s) with respect to the μ-center.

Our generalization follows the argument that $\Psi_c(v)$ can be replaced by any function $\Psi(v)$, $v \in \mathrm{int}\mathcal{K}$, with the same properties as the scaled logarithmic barrier function (11), that is, $\Psi(v)$ is strictly convex and attains its minimum at $v = e$ with $\Psi(e) = 0$. The function $\Psi(v)$ is called a *scaled barrier function*. Since $\Psi(v) = 0 \Leftrightarrow \nabla \Psi(v) = 0 \Leftrightarrow v = e$, the function $\Psi(v)$ still serves as a proximity measure to the μ-center. Specifically, the inequality $\Psi(v) \leq \tau$ with $\tau > 0$ as a threshold value defines a τ-neighborhood of the μ-center. In addition, the following norm-based proximity measure $\delta(v) := \frac{1}{2} \|\nabla \Psi(v)\|$ will play an important role in the analysis of the algorithm:

The scaled centering equation in (10) becomes $d_x + d_s = -\nabla \Psi(v)$ and, hence, the search direction is calculated from the following system

$$\begin{aligned} -\overline{M} d_x + d_s &= 0, \\ d_x + d_s &= -\nabla \Psi(v). \end{aligned} \tag{13}$$

The above discussion can be summarized in the form of the Generic IPM presented in Figure 1.

Generic IPM for Cartesian $P_*(\kappa)$-SCLCP

Input:
A threshold parameter $\tau \geq 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter θ, $0 < \theta < 1$;
a starting point (x^0, s^0) with $\mu^0 = (x^0)^T s^0/n$ such that $\Psi(v^0) \leq \tau$
where v^0 is calculated using (7) and (8).

```
begin
  x := x^0; s := s^0; μ := μ^0;
  while nμ ≥ ε do
  begin
    μ := (1 − θ)μ;
    calculate v using (7) and (8);
    while Ψ(v) > τ do
    begin
      calculate search direction (Δx, Δs);
      determine the step size α;
      update (x, s) := (x, s) + α(Δx, Δs).
    end
  end
end
```

Figure 1: Generic IPM for Cartesian $P_*(\kappa)$-SCLCP

In what follows we will refer to the Generic IPM for Cartesian $P_*(\kappa)$-SCLCP simply as the *Algorithm*. The behavior of the Algorithm depends on the choice of the values for the input parameters as well as the choice of the barrier function and the step size. It is now standard to call IPM a *large-update method* if θ is a fixed constant, and $\theta \in (0,1)$ is independent of the dimension n of the problem, i.e., $\theta = O(1)$ and $\tau = O(n)$. If θ depends on the dimension of the problem with $\theta = O(\frac{1}{\sqrt{n}})$ and $\tau = O(1)$, IPM is called a *small-update method*.

4 Kernel-based barrier functions

As already observed in the previous section, the behavior of the Algorithm depends heavily on the choice of the barrier function. We can look at the log-barrier function defined in (11) from another angle by introducing the univariate function $\psi_c : (0, \infty) \to [0, \infty)$ defined as

$$\psi_c(t) = \frac{t^2 - 1}{2} - \log t. \tag{14}$$

Then $\Psi_c(v)$ in (11) can be written as $\Psi_c(v) = \sum_{i=1}^{r} \psi_c(\lambda_i(v))$, that is, as a separable function with identical univariate functions for each component. The univariate function ψ_c is called the *logarithmic kernel function*.

We consider barrier functions of the same type as the logarithmic barrier function, that is, separable functions

$$\Psi(v) := \sum_{i=1}^{r} \psi(\lambda_i(v)), \tag{15}$$

defined by the univariate function $\psi : (0,\infty) \to [0,\infty)$, called the *kernel function*. We require that ψ has the same properties as the logarithmic kernel function, that is,

$$\begin{aligned} \psi'(1) = \psi(1) &= 0, & \text{(16-a)} \\ \psi''(t) &> 0, & \text{(16-b)} \\ \lim_{t\downarrow 0}\psi(t) = \lim_{t\to\infty}\psi(t) &= \infty. & \text{(16-c)} \end{aligned}$$

In order to obtain favorable iterations bounds we introduce the following additional conditions on the kernel function.

$$\begin{aligned} t\psi''(t) + \psi'(t) &> 0, \quad t < 1, & \text{(17-a)} \\ \psi'''(t) &< 0, \quad t > 0, & \text{(17-b)} \\ 2\psi''(t)^2 - \psi'(t)\psi'''(t) &> 0, \quad t < 1, & \text{(17-c)} \\ \psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) &> 0, \quad t > 1,\ \beta > 1. & \text{(17-d)} \end{aligned}$$

The kernel functions that satisfy conditions (16-a)-(16-c) and (17-a)-(17-d) are called *eligible kernel functions*. They were first introduced for LO by Bai *et al.* in [1] and revisited for $P_*(\kappa)$-LCP over nonnegative orthant in [2, 14]. They were adapted for SCLO by Vieira [23] and Lesaja and Roos for monotone SCLCP [15].

Two inverse functions related to the kernel function have shown to be essential in the analysis of the Algorithm and are defined below.

Definition 4.1 *Given the kernel function ψ, we define*

(i) $\varrho : [0,\infty) \to [1,\infty)$ is the inverse function of $\psi(t)$ for $t \geq 1$;

(ii) $\rho : [0,\infty) \to (0,1]$ is the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \leq 1$.

In the first column of the Table 1 the ten frequently used eligible kernel functions are listed.

5 Analysis and complexity of the algorithm

The analysis of the Algorithm follows the same outline employed in papers on IPMs with eligible kernel functions for LO [1] and for $P_*(\kappa)$-LCP [14]) over nonnegative orthant. In addition, the analysis builds on the only work on IPMs with eligible kernel functions for SCLO [23] and monotone SCLCP [15]. The analysis consists of three major steps: finding the upper bound on the increase of the barrier function at the start of each outer iteration, determining the default step size, and finding the lower bound on the decrease of the barrier function during each inner iteration. The statements and proofs of the results can be adapted to the framework of Cartesian $P_*(\kappa)$-SCLCP without major difficulties from [14] and [15] once the following key lemma is proved.

Lemma 5.1 *The following inequalities hold:*

$$\|d_x\| \leq 2\delta\sqrt{1+2\kappa}, \qquad \|d_s\| \leq 2\delta\sqrt{1+2\kappa}.$$

Proof. Since M has a Cartesian $P_*(\kappa)$ property and, according to (5), $\Delta s = M\Delta x$, it follows that

$$\langle \Delta x, \Delta s\rangle = \langle \Delta x, M\Delta x\rangle \geq -4\kappa \sum_{j\in\mathcal{I}^+(\Delta x)} \left\langle \Delta x^{(j)}, (M\Delta x)^{(j)}\right\rangle = -4\kappa \sum_{j\in\mathcal{I}^+(\Delta x)} \left\langle \Delta x^{(j)}, \Delta s^{(j)}\right\rangle.$$

Because of (9) we have $\langle \Delta x, \Delta s\rangle = \mu\,\langle d_x, d_s\rangle$. Hence, it follows that

$$\langle d_x, d_s\rangle \geq -4\kappa \sum_{j\in\mathcal{I}^+(d_x)} \left\langle d_x^{(j)}, d_s^{(j)}\right\rangle \tag{18}$$

Using the arithmetic-geometric mean inequality $\langle a,b\rangle \leq \frac{1}{4}\langle a+b,a+b\rangle$ we obtain

$$\begin{aligned}\sum_{j\in\mathcal{I}^+(d_x)}\left\langle d_x^{(j)},d_s^{(j)}\right\rangle &\leq \tfrac{1}{4}\sum_{j\in\mathcal{I}^+(d_x)}\left\langle d_x^{(j)}+d_s^{(j)},d_x^{(j)}+d_s^{(j)}\right\rangle\\ &\leq \tfrac{1}{4}\sum_{j=1}^{m}\left\langle d_x^{(j)}+d_s^{(j)},d_x^{(j)}+d_s^{(j)}\right\rangle\\ &= \tfrac{1}{4}\|d_x+d_s\|^2=\tfrac{1}{4}\|-\nabla\Psi(v)\|^2=\delta^2,\end{aligned}$$

where we used (13) and the definition of δ. Substitution of this inequality into (18) yields

$$\langle d_x,d_s\rangle \geq -4\kappa\delta^2. \tag{19}$$

We may now write:

$$\begin{aligned}\|(d_x;d_s)\|^2 &= \sum_{j=1}^{m}(\|d_x^{(j)}\|^2+\|d_s^{(j)}\|^2)\\ &= \sum_{j=1}^{m}\left(\left\langle d_x^{(j)}+d_s^{(j)},d_x^{(j)}+d_s^{(j)}\right\rangle-2\left\langle d_x^{(j)},d_s^{(j)}\right\rangle\right)\\ &= \|d_x+d_s\|^2-2\langle d_x,d_s\rangle\\ &\leq 4\delta^2+8\kappa\delta^2=4(1+2\kappa)\delta^2.\end{aligned}$$

The inequality above is due to (13) and (19). The inequalities in the statement of the lemma immediately follow. □

Due to the page constraint the paper needs to satisfy, the rest of the analysis is omitted and reader is referred to [14, 15] for details. We only state the final complexity result.

Theorem 5.2 (Theorem 6.2 in [15]) *The total number of iterations N in the Algorithm is bounded above by*

$$N \leq \frac{\Psi_0^\gamma}{\theta\beta\gamma}\log\frac{r}{\varepsilon}. \tag{20}$$

The Ψ_0 is the value of the barrier function at the beginning of the outer iteration. This value is usually not known; however, the upper bound can be found. The constants β and γ are connected to the eligible kernel functions and they assume different values for different kernel functions.

6 Iteration bounds

The iteration bound for the specific eligible kernel function can be calculated using the above Theorem 5.2 and overall analysis. The process can be streamlined into the scheme similar to the one developed in [14, 15]. Sometime the skill and effort is needed in using the scheme for specific eligible kernel function. The obtained iteration bounds are listed in the Table 1.

For large-update methods, the resulting iteration bounds are summarized in the third column of the Table 1. For ψ_2 and ψ_7 the bound is minimal if we choose $q=\frac{1}{2}\log n$, and for ψ_{10} the bound is minimal if we choose $p=1$ and $q=\frac{1}{2}\log n$. This gives the best bound known so far for large-update IPMs:

$$O\left((1+2\kappa)\sqrt{r}\,(\log n)\log\frac{r}{\varepsilon}\right).$$

For the small-update methods the resulting iteration bounds all have the same order of magnitude. Thus, the best iteration bound is the same as the iteration bound for the logarithmic kernel function: namely

$$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$$

which is the best known iteration bound for these types of methods.

i	kernel functions ψ_i	small-update methods	large-update methods
1	$\frac{t^2-1}{2} - \log t$	$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)r\log\frac{r}{\varepsilon}\right)$
2	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1), \quad q>1$	$O\left((1+2\kappa)q\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)qr^{\frac{q+1}{2q}}\log\frac{r}{\varepsilon}\right)$
3	$\frac{t^2-1}{2} + \frac{(e-1)^2}{e}\frac{1}{e^t-1} - \frac{e-1}{e}$	$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)r^{\frac{3}{4}}\log\frac{r}{\varepsilon}\right)$
4	$\frac{1}{2}\left(t-\frac{1}{t}\right)^2$	$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)r^{\frac{2}{3}}\log\frac{r}{\varepsilon}\right)$
5	$\frac{t^2-1}{2} + e^{\frac{1}{t}-1} - 1$	$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)\sqrt{r}\log^2 r\log\frac{r}{\varepsilon}\right)$
6	$\frac{t^2-1}{2} - \int_1^t e^{\frac{1}{\xi}-1}d\xi$	$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)\sqrt{r}\log^2 r\log\frac{r}{\varepsilon}\right)$
7	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1}, \ q>1$	$O\left((1+2\kappa)q^2\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)qr^{\frac{q+1}{2q}}\log\frac{r}{\varepsilon}\right)$
8	$t-1+\frac{t^{1-q}-1}{q-1}, \quad q>1$	$O\left((1+2\kappa)q^2\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)qr\log\frac{r}{\varepsilon}\right)$
9	$\frac{t^{1+p}-1}{1+p} - \log t, \ p\in[0,1]$	$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)r\log\frac{r}{\varepsilon}\right)$
10	$\frac{t^{p+1}-1}{p+1} + \frac{t^{1-q}-1}{q-1}, \quad p\in[0,1], \quad q>1$	$O\left((1+2\kappa)q^2\sqrt{r}\log\frac{r}{\varepsilon}\right)$	$O\left((1+2\kappa)qr^{\frac{p+q}{q(1+p)}}\log\frac{r}{\varepsilon}\right)$

Table 1: Complexity results for ten eligible kernel functions

7 Concluding Remarks

In this paper the large- and small-update versions of the IPM for Cartesian $P_*(\kappa)$-SCLCP, described in Figure 1 are presented. The method is based on the large class of separable barrier functions defined by univariate eligible kernel functions. The class of eligible kernel functions is important because it is very general and includes the classical logarithmic kernel function, the self-regular kernel function, and a number of non-self-regular kernel functions as special cases.

The paper generalizes results obtained in two of our papers, [14] where we consider the same type of IPMs for $P_*(\kappa)$-LCPs over the nonnegative orthant, and [15] where we consider monotone SCLCP. It turns out that the iterations bounds are the same as for the nonnegative orthant except that n is replaced by r, the rank of the EJA. Although expected, this was not obvious and required the proof of the key Lemma 5.1 in the context of $P_*(\kappa)$-SCLCP . For the large-update methods the best iteration bound is $O\left(\sqrt{r}\,(\log r)\log\frac{r}{\varepsilon}\right)$ and for the small-update methods all iteration bounds have the same order of magnitude, namely, $O\left(\sqrt{r}\log\frac{r}{\varepsilon}\right)$. In both cases we were able to match the best known iteration bounds for these types of methods. Thus, the iteration bounds are as good as they can be in the current state-of-the-art.

The Cartesian $P_*(\kappa)$ property, as defined in (3), essentially depends on having more than one simple EJA V_j. Interesting topic for further research may be to look for the alternate definition of $P_*(\kappa)$ property that would be defined on one simple EJA. One possible direction in developing such property may be to consider eigenvalues of spectral decompositions of elements in EJA.

The other issue that deserves a comment is the assumption of strict feasibility of SCLCP. Strict feasibility, or as it is often called the Interior Point Condition (IPC), can be assumed without loss of generality for LCPs over nonnegative orthant (Kojima et al. [12]). For the LO case the same is true not only for the nonnegative orthant but also in general; Luo, Sturm and Zhang in [17] discussed self dual embedding for LO over symmetric cones. However, we are not aware of similar results for SCLCP. It certainly is a worthwhile research topic. Alternative direction for further research may be the development of infeasible kernel-based IPMs for SCLCP.

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