ANALYSIS OF THE CROATIAN PENSIONER'S DOUBT 2005th

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Abstract

This paper answers the question that was, at the end of year 2005, set in front of the majority of Croatian pensioners. The government put forward a plan of debt repayment to each pensioner, giving him/her an option of two models: A fast (accelerated) model and a regular model of debt repayment. The paper analyzes and compares these two models, depending on the pensioner's age, gender and individual rate of value of money i*. The paper presents mathematical expressions for calculating the present value of all payments for both models. The model with greater present value is considered to be better option for a person at age* x*. The goal is to find a critical (turning) rate* $\tilde{i}_x$ *at which the profitability of both models equalizes. Analysis of the conditions of existence and uniqueness of rate* $\tilde{i}_x$ *is done for any age* x*. Finally, for women, men and total population, turning point curve in the* x-i *plane is shown. Coordinates of the turning points* $(x, \tilde{i}_x)$*, are crucial values for which the present value of both models have the same amount, i.e. where the profitability goes from one to another model. In other words, the critical value of rate* $\tilde{i}_x$*, for each pensioner at age* x *with his/her own individual rate of value of money* i*, was found. If* $i > \tilde{i}_x$ *the better choice is the fast model, otherwise if* $i < \tilde{i}_x$ *the better choice is the regular model.*

Key words: *Actuarial model, retiree, pensioner's debt repayment, debt service models, Descartes' rule of signs, turning point curve, critical rate*

1. INTRODUCTION

At the end of the year 2005, the Croatian government put forward a plan to return its debt to each of pensioners. A notice was sent to the pensioners indicating a debt amount and requesting them to declare the model of debt repayment. They had two options: regular repayment model over the next eight years with two years delay, and accelerated (fast) repayment model in which case they would receive an amount of 50 percent of the total debt over next two years.

If they sign up for accelerated model, first two rates will be paid in 2006, and next two in 2007. If they decide for the regular model, the payments will begin at the end of 2007, and the full debt will be paid throughout next six years. Because this is not enough clear description of models of payments, we will refine the models.

The first model ('A model') is a model of accelerated payments of only half of the debt, during the next two years in 4 equal semi-annual payments, of which the first comes at the end of June 2006, and the others at the end of next 3 semi-annual periods. The second model ('B model') is a model of postponed payments of the entire amount of debt, that will be received through 6 equal annual payments, of which the first is coming at the end of year 2007, and the others at the end of following years.

For a use of actuarial calculation, hereafter, the models A and B are more precisely defined as follows:

- A model - a refund of half the debt in four equal payments, immediately, semi-annually, at the end of the payment period (accelerated model).
- B model - a refund of the entire debt in six equal payments, with a delay of two years, annually, at the beginning of the payment period (regular model).

So, the pensioner's doubt is how to answer a question, which model is more suitable for her/him.

The answer depends on the situation of each pensioner and the conditions in the market of goods and capitals. The pensioner's decision which option is better was, probably, partly influenced by a fear by writings in the media about the lack of assurance that their debt will be returned, especially for latter payments. In this work, this fear is ignored and we suppose that payments are guaranteed.

To compare these two models, the time interval in length of 7 years that begins at the end of 2005, and ends with the last payment by regular model, i.e. at the end of 2012 was observed.

2. THE CASE OF CERTAIN PAYMENTS

First, let's consider the situation when retired person is relatively young or in good health and believes that she/he will live long enough to receive all instalments according to both model (from the end of 2005 to the end of 2012). That is the case when the pensioner (retiree) in making of the decision ignores her/his own age, i.e. the likelihood of dying. In this case, age is not important, or more correctly the risk of dying is ignored, and all payments are consider to be received.

Also, in the situation when a care about the pensioner is taken by (young and healthy) a legal successor of the first order, who in the case of death of the pensioner will inherit unpaid debt, and the pensioner does not give importance if he/she does not receive all instalments because of his/her own dead, then the risk of dying can be ignored and all payments can be considered certain.

The question remains; which of the models, in this case, is more profitable?

Profitability depends on what the pensioner intends to do with the received amounts; what her/his value of money is during the observed interval of 7 years, i.e. what the rate is at which he/she can and wants to invest money, received in that interval.

For example, if received amounts, she/he held in 'socks' until 2012. (invested at the rate 0), means that for her/him, the time value of money[1] is independent of time, then, in the case of certain payments, it is obvious more profitable to wait for 2 years to receive the entire debt (the B model). Received amounts could be invested by saving in the bank at some interest rate, or invested by purchasing durable goods, value of which will grow at some annual rate as result of an inflation or due to increase of demand. The pensioner might spend received amounts to meet some urgent needs; this would mean that for her/him, money at the present time is worth much more than in the future. The value of money during the observed time interval is specific for each of retirees and reflects his/her needs for money and how he/she can invest it. The time value of money can be described by an expected annual rate of return from investments of the received payments, and it is specific for each case. Generally, this rate is a measure of pensioner's value of money (desperation for money) over a time[2]. Furthermore, we consider such rate (investment rate, interest rate, inflation rate, rate of growth …) and assume that it is a constant in our considered seven-year interval.

The question is which of two models is more profitable for some rate of the pensioner's time value of money? It is obvious that for a zero rate (time value of money is invariant over the time) the model B is more profitable. We compare the models by comparing the sum of present values of their payments. The present value is discounted value at the beginning of the seven year period, i.e. at the end of year 2005. Denote this rate by i. Hereafter, the end of 2005th is taken for a present time and denoted by $t=0$. Denote by $s.v.A_I$ and $s.v.B_I$ the sum of present values of payments for models A and B.

The model with greater present value is considered to be more profitable.

In general, the present value of C which is payable at time t is equal to $s.v. = v^t \cdot C$, where $v = v(i) = \frac{1}{(1+i)}$ is the discount factor, for $i \in (-1,+\infty)$, $v \in (0,+\infty)$ [3]. Then we have:

$$s.v.A_I = \frac{0.5 \cdot Dbt}{4} \cdot \left(v^{1/2} + v^1 + v^{3/2} + v^2\right) = \frac{Dbt}{8} \cdot v^{0.5} \cdot \frac{1-v^2}{1-v^{0.5}} , \tag{1}$$

$$s.v.B_I = \frac{Dbt}{6} \cdot \left(v^2 + v^3 + \cdots + v^7\right) = \frac{Dbt}{6} \cdot v^2 \cdot \frac{1-v^6}{1-v} . \tag{2}$$

Let us denote $S_I(v(i)) = s.v.B_I(v(i)) - s.v.A_I(v(i))$, then from (1) and (2) we have

$$S_I(v) = \frac{1}{24} Dbt \left[-3\,v^{1/2} - 3\,v^1 - 3\,v^{3/2} + 1\,v^2 + 4\,v^3 + 4\,v^4 + 4\,v^5 + 4\,v^6 + 4\,v^7\right] \tag{3}$$

[1] Money at the present time is worth more than the same amount in the future due to its earning capacity.
[2] For example if she/he prefers 1000 Euros today, rather than 1500 for one year, annual rate is over 50%.
[3] Note that as i grows from 0 to $+\infty$, v falls from 1 to 0.

Obviously, the model A is more profitable when $S_I < 0$, and the model B when $S_I > 0$. It is easy to see that for enough small rate $i \approx 0$ ($v \approx 1$) is $S_I > 0$, i.e. $s.v.A_I < s.v.B_I$.

A goal is to find a turning (critical) rate i_p at which both models are equally profitable, i.e. at which $S_I = 0$ ($s.v.A_I = s.v.B_I$).

Note that there is one-to-one correspondence between v and i. Thus, for the rate i_p there is a unique critical discounted factor v_p. Since the right side of (3) is a polynomial in a variable $\lambda = v^{1/2}$, according to the Descartes' Rule of Signs[4] (Kennedy, J., 2010), it has a unique positive zero $\lambda_p = v_p^{1/2}$, i.e. $S_I(v_p) = S_I(v(i_p)) = 0$, and $v_p^{1/2} > 0$.

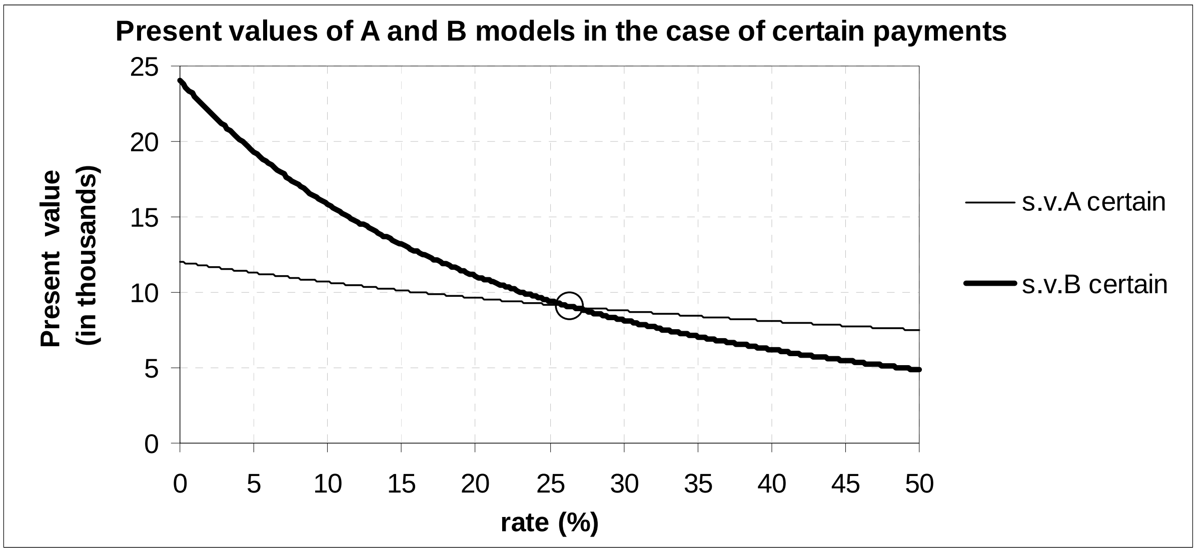

Figure 1:. Present value of both models, depending on the rate. Circle marks the point of intersection.

According to the Upper Bound Theorem[5] we have that $v_p^{1/2} < 1$. From the positivity of the leading coefficient of the polynomial and the fact that $f(x) = x^2$ is increasing, it follows that:

$S_I \le 0$ for $0 < v \le v_p < 1$, i.e. for $0 < i_p \le i < +\infty$, and

$S_I > 0$ for $v_p < v < +\infty$, i.e. for $-1 < i < i_p$.

Calculating[6] we get that $i_p \approx 0.2649049434$. So, in the case of certain payments, the B model will be

[4] Descartes' Rule of Sign: Let $P_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$ be a polynomial with real coefficients. The number of positives zeros (roots) of $P_n(x)$ is either equal to the number of variations in sign in $P_n(x)$ or less than this by an even number. Each zero counts as many times as it was multiples. (Note that when determining sign variations we can ignore terms with zero coefficients). Therefore we have: if the coefficients of the polynomial changes sign only once, the polynomial has only one strictly positive zero-point multiples of 1!

[5] Let $P_n(x)$ be any polynomial with a positive leading coefficient. If $a < 0$ and $P_n(a) < 0$ and if in applying synthetic substitution to compute $P_n(a)$ all numbers in the 3rd row are positive, then a is an upper bound for all the roots of $P_n(x) < 0$.

more profitable for $i < i_p$, but the A model for $i > i_p$, where $i_p \approx 26.49\%$ is the critical rate at which the profitability goes from one model to another. In examples presented *Dbt=24000.*

3. THE CASE WHEN PAYMENTS ARE NOT CERTAIN

In this case, the considerations are the same as in the previous case except that the payments were not certain and depend on the probability of survival of pensioner at the time of payments. Probability of survival for the overall, male and female populations were taken from "The mortality tables of Republic of Croatia, 2000-2002" (DZS, 2006).

Present value of amount C paid at the time t is equal to $s.v. = C \cdot v^t \cdot {}_t p_x$, where ${}_t p_x$ is the likelihood[7] of payment of the amount C at time $t > 0$, i.e. probability that the retiree, who was at the end of 2005 turned x years of life, survives to age $x+t$.

v^t is the discount factor for period length t. l_{x+t} is the number of persons who survive to age $x+t$. ${}_t q_x = 1 - {}_t p_x$ is the probability that the retiree dies before age $x+t$. For $t=1$ we write p_x instead of ${}_1 p_x$. The values of l_x and q_x, for integer values of $0 \le x \le 100$, are given in the table of mortality. (Francišković, D., 2008, DSZ RH, 2006) (for $x > 100$ $q_x = 1$). It is easy to see that

$$ {}_t p_x = \frac{l_{x+1}}{l_x} \cdot \frac{l_{x+2}}{l_{x+1}} \cdot \frac{l_{x+3}}{l_{x+2}} \cdot \cdots \cdot \frac{l_{x+t-1}}{l_{x+t-2}} \cdot \frac{l_{x+t}}{l_{x+t-1}} = p_x \, p_{x+1} \, p_{x+2} \cdots p_{x+t-1}. \tag{4} $$

The present value of all payments by A and B models are:

$$ s.v.A = s.v.A(v(i), x) = \frac{Dbt}{8}\left[{}_{1/2}p_x v^{1/2} + {}_1 p_x v^1 + {}_{3/2} p_x v^{3/2} + {}_2 p_x v^2\right], \tag{5} $$

$$ s.v.B = s.v.B(v(i), x) = \frac{Dbt}{6}\left[{}_2 p_x v^2 + {}_3 p_x v^3 + \cdots + {}_7 p_x v^7\right]. \tag{6} $$

As $v^t > {}_t p_x \cdot v^t$ for all x and $i > -1$, $s.v.A(v(i), x) < s.v.A_I(v(i))$ and $s.v.B(v(i), x) < s.v.B_I(v(i))$ (see Figure 2 [8]). Because function $f(t, v(i)) = v^t = (1+i)^{-t}$ is positive; monotonically decreasing for $i > -1$; and ${}_t p_x > 0$, follows that $s.v.A$ and $s.v.B$ are also positive; monotonically decreasing functions for $i > -1$, at every age x (see Figure 2).

[6] Calculations and graphs have been done using software MS Excel and Mathematica.

[7] $0 < {}_t p_x = \frac{l_{x+t}}{l_x} < 1$. Calculations assumes that all deaths come end of the year (Neill, A., 1989)..

[8] To calculate $s.v.A$ the approximation $s.v.\widetilde{A}$, define in (8), was used.

Figure 2 shows that the increase in age reduces the turning rate (the points of intersection move left) in which $s.v.A = s.v.B$. Note, that at the age of 90 there is no a positive turning rate.

In the mortality tables it can be found an age x_T, so that for every two ages a and b, where $x_T < a < b \leq 100$, is $q_a < q_b$, i.e. $p_a > p_b$. Now (4) gives ${}_t p_a > {}_t p_b$.

As a consequence we have that for each rate i, $s.v.A$ and $s.v.B$ are positive and monotonically declining in the variable x for $x \in (x_T, 100)$.

So, $s.v.A$ and $s.v.B$ will always be higher for younger retirees aged more than x_T (Figures 2 and 3).

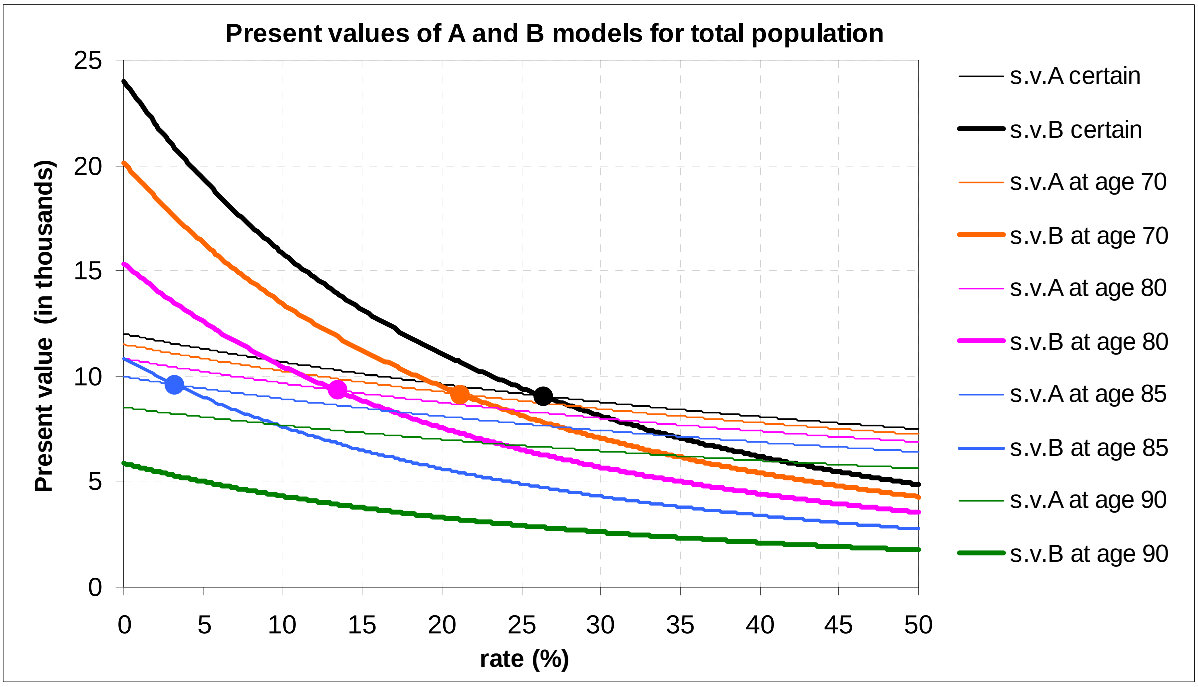

Figure 2:. Present value of both models, depending on the rate and pensioners age, regardless of their sex.

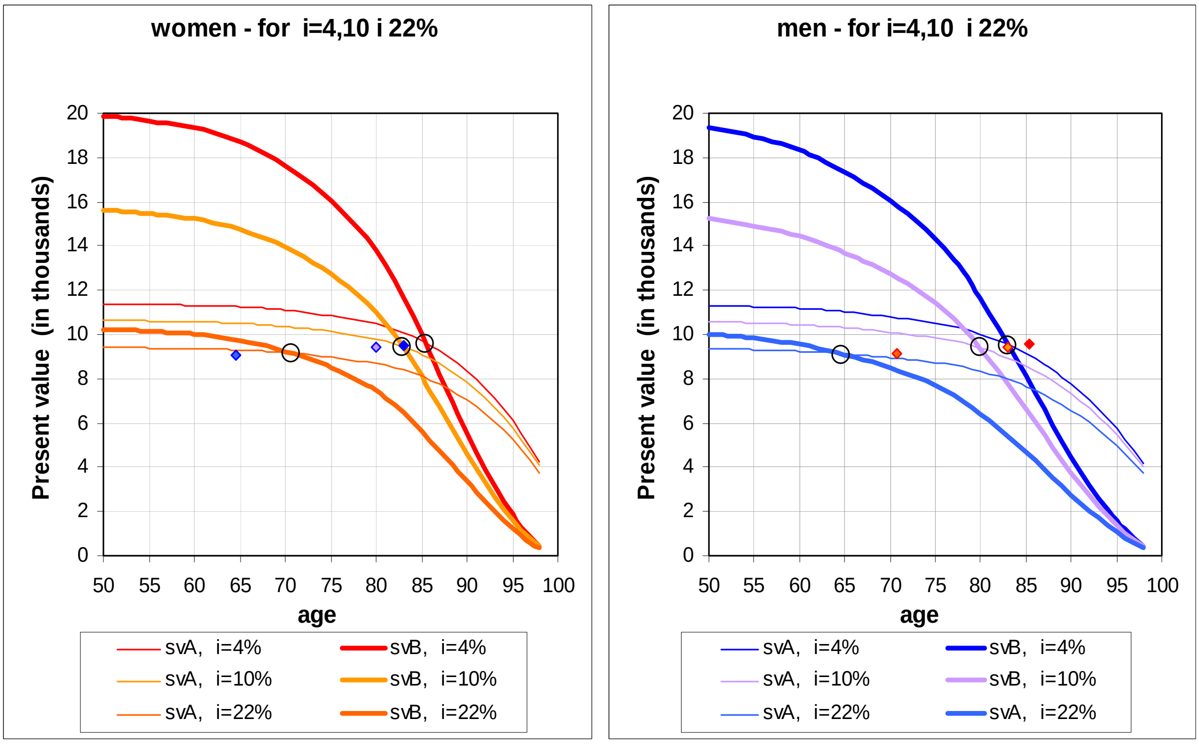

Figure 3:. Present value of both models, depending on gender and age of retirees, for rates 4, 10 and 22 %.

In the mortality tables we can find that for the entire populations $x_T = 33$; $x_T = 28$ for women and $x_T = 33$ for men.

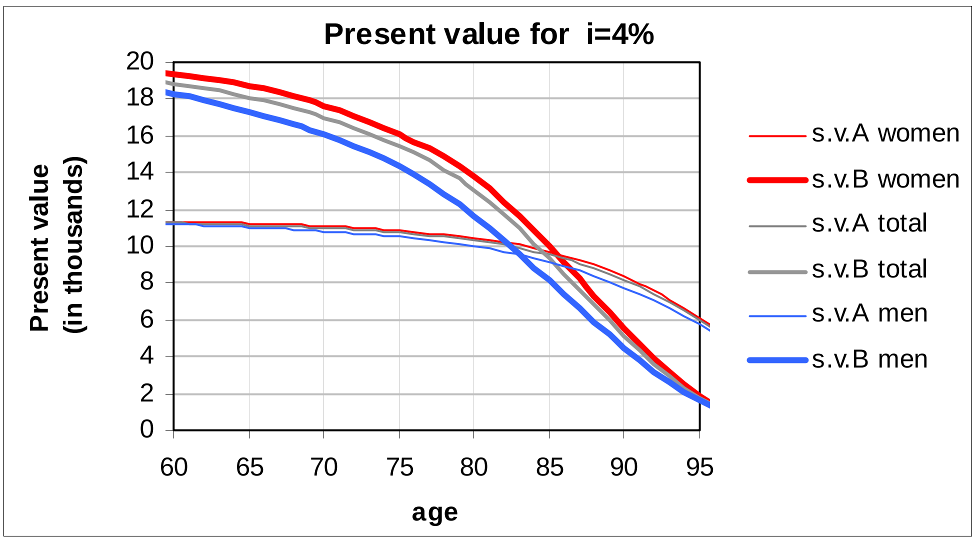

Figure 4:. Comparison of present values of both models for overall, male and female population based on age of pensioners at a rate of 4%.

Similarly, it can be shown that for $x \leq 98$, $s.v.A$ and $s.v.B$ have higher values for female than for male population (Figure 4), since the q_x in the male population is higher for $x \leq 98$.

By denoting $S = S(v(i),x) = s.v.B(v(i),x) - s.v.A(v(i),x)$, it is obvious that

$$S(v(i),x) = \frac{Dbt}{24}\left[-3\,{}_{1/2}p_x v^{1/2} - 3\,{}_1p_x v^1 - 3\,{}_{3/2}p_x v^{3/2} + 1\,{}_2p_x v^2 + 4\,{}_3p_x v^3 + \cdots + 4\,{}_7p_x v^7\right]. \quad (7)$$

As in the case of certain payments, we are interested in when the A model will be more profitable than the B model. The goal is, for each age x, to find a turning rate i_x, at which both models are equally profitable, i.e. when $S(v(i_x),x) = 0$. For $x \le x_M = 98$ ${}_2p_x \ne 0$, meaning that the coefficients of the polynomial change sign only once. Now, as in the case of certain payments according to the Descartes' Rule of Sign, for $x \le x_M$ there is unique turning rate $i_x \in (-1,+\infty)$.

Hence, for $-1 < i < i_x$, $S > 0$, i.e. the model B is more profitable, while for $i > i_x$, $S < 0$ and the model A is more profitable. For $x > x_M$ always is $S < 0$ and the model A is more profitable regardless of rate i.

The problem is that the values of ${}_{1/2}p_x$ and ${}_{3/2}p_x$ in (5) and (7) can not be accurately obtained from data in the tables of mortality, and i_x can not be found. Values of q_{x+t} are tabulated only if $x+t$ is a natural number, and from (4) and a fact that $p_x = 1 - q_x$, ${}_tp_x$ can be obtained only for natural number t. Because of this, ${}_{1/2}p_x v^{1/2}$ and ${}_{3/2}p_x v^{3/2}$ will be replaced with their linear approximations. Including ${}_{1/2}p_x v^{1/2} \approx \frac{1}{2}({}_0p_x v^0 + {}_1p_x v^1)$ and ${}_{3/2}p_x v^{3/2} \approx \frac{1}{2}({}_1p_x v^1 + {}_2p_x v^2)$ in (5) and (7) we get [9]

$$s.v.\widetilde{A} = s.v.\widetilde{A}(v(i),x) = \frac{Dbt}{8}\left[\frac{1}{2} + 2\,{}_1p_x v^1 + \frac{3}{2}\,{}_2p_x v^2\right], \quad (8)$$

$$\widetilde{S}(v(i),x) = \frac{1}{24}Dbt\left[-\frac{3}{2} - 6\,{}_1p_x v^1 - \frac{1}{2}\,{}_2p_x v^2 + 4\,{}_3p_x v^3 + \cdots + 4\,{}_7p_x v^7\right]. \quad (9)$$

$\widetilde{S} = s.v.B - s.v.\widetilde{A}$ is approximation of S and therefore the model B would be considered more profitable if $\widetilde{S} > 0$, and the model A if $\widetilde{S} < 0$.

With similar consideration as for S, for $x \le \widetilde{x}_M = 97$ there is unique turning rate $\widetilde{i}_x$. For $i \in (-1,\widetilde{i}_x)$ the model B is more profitable, while for $i \in (\widetilde{i}_x,+\infty)$ the model A is more profitable. For $x > \widetilde{x}_M$ the model A is always the best choice.

In the case ${}_tp_x = 1$ [10], for each t and x, (7) coincide with (3), or have the case of certain payments, i.e. $S = S_I$. In the case that in (9) ${}_tp_x = 1$ for every t and x, denote $\widetilde{S}$ by $\widetilde{S}_I$, so,

[9] The same result would come if we use formula (3.1.6), (2.2.3), (2.1.1) and (2.2.2) in the book "Life contingencies" (Neill, A., 1989).

[10] That would be a situation when we consider the person who will certainly live at least 107 years.

$$\widetilde{S}_I(v(i)) = \frac{1}{24} Dbt\left[-\tfrac{3}{2} - 6v^1 - \tfrac{1}{2}v^2 + 4v^3 + \cdots + 4v^7\right]. \tag{10}$$

Let us denote turning rate for $\widetilde{S}_I$ by $\widetilde{i}_p$. Similar as for i_p it is easy to show that $\widetilde{i}_p$ is unique.

For the turning rates i_p, $\widetilde{i}_p$, i_x and $\widetilde{i}_x$, can be shown that

$$i_p > \widetilde{i}_p > \widetilde{i}_{x1} > \widetilde{i}_{x2}, \qquad \text{for} \quad x_T < x1 < x2 \le \widetilde{x}_M, \tag{11}$$

$$i_p > i_{x1} > i_{x2}, \qquad \text{for} \quad x_T < x1 < x2 \le x_M. \tag{12}$$

The turning rates i_x and $\widetilde{i}_x$ are strictly descending in x, for $x > x_T$, and always lower than corresponding turning rates in the case of certain payments .

From (10) is obtained that $\widetilde{i}_p \approx 0.2631912437$, i.e. $\widetilde{i}_p \approx 26.32\%$. Earlier was shown that $i_p \approx 26.49\%$, and so relation $i_p > \widetilde{i}_p$ is proofed.

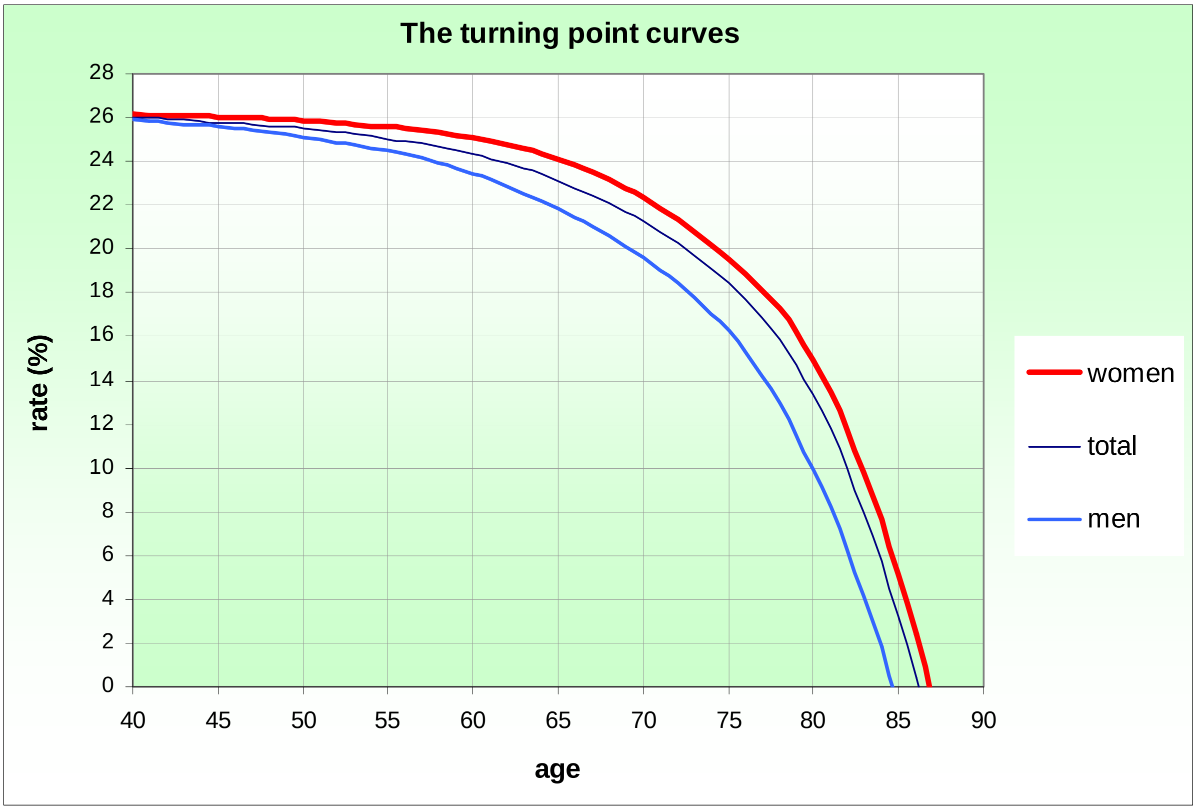

Figure 5:. The turning points curves for total, women and men population.

Other relations in (11) and (12) are consequence of the fact that p_x is strictly descending in x for $x > x_T$. Proof of that is not presented here because it is beyond the scope of this work.

In other words, as the age of a pensioner older than x_T is increasing, turning (critical) rate $\widetilde{i}_x$ is unique and strictly decreasing (see Figure 5).

Figure 5. shows a graph of function $\tilde{i}(x) = \tilde{i}_x$ defined implicitly with $\tilde{S}(v(\tilde{i}_x), x) = 0$ for $40 \le x \le \tilde{x}_M$, for total, women and men population. Each point on these graphs has coordinate $(x, \tilde{i}_x)$, and they are called turning points. The graphs are called turning points curves.

Table 1: The table shows numerical values of implicit defined function $\tilde{i}(x) = \tilde{i}_x$ for some ages (see Figure 5).

Marginal rate of profitability for model B ($\tilde{i}_x$ in %)								
age	**total**	**women**	**men**		**age**	**total**	**women**	**men**
28	26,22	26,27	26,17		76	17,70	18,86	15,30
40	26,01	26,14	25,88		77	16,86	18,11	14,20
50	25,49	25,86	25,11		78	15,90	17,26	12,93
56	24,93	25,48	24,32		79	14,74	16,23	11,50
60	24,33	25,08	23,46		80	13,35	14,94	9,93
64	23,40	24,35	22,21		81	11,76	13,44	8,19
66	22,79	23,82	21,44		82	9,98	11,74	6,27
68	22,07	23,15	20,57		83	7,98	9,81	4,14
70	21,25	22,34	19,59		84	5,73	7,63	1,79
72	20,28	21,36	18,45		85	3,21	5,17	-0,82
74	19,11	20,19	17,07		86	0,38	2,38	-3,70
75	18,44	19,55	16,25		87	-2,80	-0,78	-6,88

Source: The calculation of a (critical) turning rates, based on (9), using Matlab.

For all points (x, i) below the turning points curve the model B is more profitable and is better pensioner's choice. Otherwise, for all points (x, i) above the turning points curve the model A is more profitable. For example, if a rate is up to 10%, then a man no older then 79 and woman no older then 82 should choose regular payments of the entire amount with two years waiting period. Of course, if they are in good health as others in their gender group. At the rate up to 6%, these values are 82 for men and 84 for women.

Let us denote by $\tilde{x}_0$, an age for which the turning rate is a zero, i.e. for which $\tilde{S}(v(0), \tilde{x}_0) = 0$. For overall population $\tilde{x}_0 \approx 86{,}13$, for women $\tilde{x}_0 \approx 86{,}77$, and for men $\tilde{x}_0 \approx 84{,}70$.

4. CONCLUSION

Now, we answer the question, which model is better choice for a pensioner, depending on her/his rate $i \ge 0$. If this rate is greater than $i_p \approx 26{,}5\%$ the model A is the right choice regardless of her/his age. If she/he is at age $x < \tilde{x}_0$ and in good health as others in her/his gender group, first, we should find corresponding turning rate $\tilde{i}_x$ defined implicitly by $\tilde{S}(v(\tilde{i}_x), x) = 0$. Then, if $\tilde{i}_x > i$ the model B is more profitable, otherwise, if $\tilde{i}_x > i$ the model A is the better choice. If $x > \tilde{x}_0$ the model A is the better choice regardless of the positive rate i. Also, we found that the turning rate $\tilde{i}_x$ is unique and as the age of pensioner older than x_T is increasing, turning rate $\tilde{i}_x$ is strictly decreasing.

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