SOME NOTES ON COST ALLOCATION IN MULTICASTING

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Abstract

We analyze the cost allocation strategies associated with the problem of broadcasting information from some source to a number of communication network users. A multicast routing chooses a minimum cost tree network that spans the source and all the receivers. The cost of such a network is distributed among its receivers who may be individuals or organizations with possibly conflicting interests. Providing network developers, users and owners with practical computable 'fair' cost allocation solution procedures is of great importance for network management. Consequently, this multidisciplinary problem was extensively studied by Operational Researchers, Economists, Mathematicians and Computer Scientists. The fairness of various proposed solutions was even argued in US courts. This presentation overviews some previously published, as well as some recent results, in the development of algorithmic mechanisms to efficiently compute 'attractive' cost allocation solutions for multicast networks. Specifically, we will analyze cooperative game theory based cost allocation models that avoid cross subsidies and/or are distance and population monotonic. We will also present some related open cost allocation problems and the potential contribution that such models might make to this problem in the future.

Keywords: *communication networks, cost allocation, cooperative combinatorial optimization games, multicast routing, Steiner trees.*

1 Introduction

In this work we are concerned with the cost allocation problem associated with the problem of broadcasting information from a source to specific network users. Each user is required to be connected, perhaps through other nodes, to a common source. With each link we associate the cost of using that link to send messages. The set of users should be linked to a common source at a minimum cost. The literature refers to this problem as to the minimum cost multicasting tree problem in networks. Sending message individually to each user might result in duplication and lack of efficiency. Specifically, to avoid sending several copies of the same message through the same link, the multicast routing is used. Multicast routing uses a tree connecting all the receivers to the source. Namely, whenever a message needs to be broadcast to a subset of receivers, multicast routing chooses a minimum cost tree that spans the source and all the receivers. Additional constraints might be added due to the specifics of network environment (for some relevant surveys see [23] and [3]).

There are various potential costs and gains associated with multicasting. It involves multiple network users who possibly have conflicting objectives. However, they might cooperate in order to decrease their joint cost. Clearly, these individuals or organizations will support a globally 'attractive' solution(s) only if their expectations for a 'fair share' of the cost are met. There is no general answer to the question of fairness. However, various cost allocation solution concepts might be proposed to encourage cooperation. For example, central to the game theory considerations is the core of the game which consists of stable cost allocation solutions that avoid cross subsidies. It is of great importance to identify a cost allocation scheme that would provide a stable solution if it exists. Further, given the dynamics of networks it is expected that the cost of certain network links might go down due to the development of technology. Hence, we would like to develop a so-called distance monotonic cost allocation scheme (see [14]). In such a scheme, the decrease of cost of any link would not increase the cost to any user. In addition, we would like that our cost allocation scheme support network growth. Namely, that it encourages new users to join the network, and does not give to any of already present users a reason to block others from joining. The cost allocation scheme that satisfies such property is said to be population monotonic or cross-monotonic (see [21], [32] and [34]).

The objective of this work is to overview some cost allocation issues associated with (potentially) practical algorithms for the multicasting problem. In Sections 2 we present some basic definitions. In Section 3 we discuss various approaches and results from the literature, including the basic idea for a primal-dual based cost allocation algorithm. In Section 4 we discuss our specific modification of the multicasting problem and discuss the application of the primal-dual algorithm to the associated multicasting cost allocation problem. In Section 5, we further modify this primal-dual cost allocation algorithm and discuss some cost allocation schemes that are distance and population monotonic. Finally, in Section 6, we discuss future research which should seek methods to combine the above algorithms in order to develop a cost allocation scheme that might simultaneously provide (almost) stable solutions and be distance and population monotonic.

2 Basic Definitions

Let P be the set of players and let for $S \subseteq P$, $c(S)$ be the characteristic function value, i.e. the cost for S. Then, the pair (P, c) is a game. We will consider the game with transferable utility in which the value $c(S)$ can be arbitrarily divided by the players in S. Let $x(S)$, for $S \subseteq P$, be the payment by S. If $x(P) = c(P)$, x is a cost allocation and the solution theory of cooperative games is concerned with the selection of a reasonable subset of cost allocation vectors.

Central to the solution theory of cooperative games is the concept of solution referred to as the core of a game. The *core*, $\mathcal{C}(P; c)$, of a game $(P; c)$ consists of all cost allocation vectors $x \in R^{|P|}$ $(x(P) = c(P))$, such that $x(S) \leq c(S)$ for all $S \subseteq P$. Observe that the core consists of all allocation vectors x which provide no incentive for any coalition to secede since there is no cross subsidization.

The characteristic function c is said to be non-decreasing if for $S, T \subseteq P$, $S \subseteq T \Rightarrow c(S) \leq c(T)$. It is submodular if $c(S \cap T) + c(S \cup T) \leq c(S) + c(T)$.

The vector $x = (x_{i,J})_{i \in J,\, J \subseteq P}$ is a *population monotonic* cost allocation scheme ([32]) of the game (P, c) if and only if it satisfies the following conditions:

$$\text{For each } J \subseteq P, \sum_{i \in J} x_{i,J} = c(J)$$
and
$$\text{For each } J, K \subseteq P \text{ and } i \in J, J \subseteq K \text{ implies } x_{i,J} \geq x_{i,K}.$$

In other words, the allocation scheme always allocates the full cost and for any player i in set J, i is allocated the same amount of cost or less when it is in a superset of J than it was allocated in J alone.

Distance monotonicity is another stability concept for network cost allocation schemes which we first introduced in [14]. A multicasting cost allocation scheme is said to be *distance monotonic* as long as no user's cost increases when the cost of any single link is decreased (and symmetrically no user's cost decreases when the cost of any single link is increased). This intuitively means that any revision of link costs would not cause unacceptable changes in allocations.

3 The Multicast Tree (*MT*) Game

Consider a connected undirected weighted network $G = (N \cup O,\, E,\, W(E))$ with a weight function $w\colon E \to R^+$. Let $D \subseteq N$ be the set of demand nodes (users) that are in need of service and let c be the cost (characteristic) function $c\colon 2^{|D|} \to R^+$, such that $c(\emptyset) = 0$. For $S \subseteq D$, let $c(S)$ be the cost of providing multicasting service to a set of users in D. The pair (D, c) is the multicast game.

There are various approaches and definitions to this problem that result in different computational complexity of optimization and cost allocation solutions. In the simplest approach to the above multicasting problem, we can assume the following calculation for the characteristic function (see [34]). The universal minimum cost spanning tree T containing all network nodes (source O and nodes in N) is constructed. Then we assume that the service to each coalition of users S, $S \subseteq D$ is provided via the cheapest subtree T_S of T which contains the source and all users in S. The characteristic function associated with this approach is non-decreasing and submodular.

The problem with this approach is that the cost of the minimum cost spanning tree T_S^* that spans $\{O\} \cup S$ might have arbitrarily smaller cost than T_S. Note that the characteristic function based on optimal tree T_S^* is not non-decreasing.

In general, this is further complicated if T_S^* could include the nodes out of S. This leads to the following interpretation. The minimum cost Multicasting Tree (MT) problem (or equivalently, Steiner Tree (*ST*) problem) in networks can be formulated as follows. Given a connected undirected weighted graph G and a set of nodes (receivers) D, $D \subseteq N$, find a tree MT in G, whose node set contains $D \cup \{O\}$ and whose total edge-weight is minimal. It is well known that this problem is *NP*-hard if $2 < |D| < |N|$ ([7]). Furthermore, the characteristic function associated with this approach is neither non-decreasing nor submodular.

The above minimum cost MT problem can also be formulated for directed graphs. The *minimum cost directed multicasting tree* problem is defined with respect to a complete directed weighted graph $G = (N \cup O,\, E,\, W(E))$. Namely, given a subset of nodes $D \subseteq N$; find a directed tree $MT = (N_{MT} \cup \{O\}, E_{MT})$ in G, rooted away from node O and whose node set contains D, such that the total edge-weight of MT is minimum. It is clear that any minimum cost

MT problem can be solved by considering an appropriate minimum cost directed *MT* problem, obtained by replacing each edge of the given network by two arcs of opposite directions.

It is well known that the directed multicasting tree (*MT*) can be formulated as an integer programming problem (see [25]). To describe the formulation, we need the following notation. Let $G=(N\cup O,\ E,\ W(E))$ be a complete weighted directed graph and $D,\ D\subseteq N$ the set of users. For a directed edge $l=(i,j)$ we refer to i as the tail and j as a head of l, and for a subset of vertices $S,\ S\subseteq N$, we denote by $\delta(S)$ the set of all directed edges having their heads, but not their tails, in S. A subset $S,\ S\subseteq N$, is said to be a *cut-set* of G, if $S\cap D\neq\emptyset$ and the subgraph $G(S)$ of G induced by S is connected. We denote by $\mathcal{S}_D$ the set of all cut-sets of G. The *minimum cost directed multicasting tree* problem can be formulated as the following integer programming problem:

$$IP(D){:}\ min\left\{wu:\ u(\delta(S))\geq 1,\ S\in\mathcal{S}_D,\ \ u\in\{0,1\}\right\},$$

where $u\equiv u(i,j)=1$ if $(i,j)\in E$ is used in the directed multicasting tree *MT*, and zero otherwise.

We now define the characteristic function $c_{MT}:\ 2^{|D|}\mapsto\mathbb{R}$ as follows: $c_{MT}(\emptyset)=0$, and for each $Q\subseteq D$, $c_{MT}(Q)$ is the value (the cost) of the optimal multicasting tree MT_Q spanning $Q\cup O$ in G. We will refer to (D,c_{MT}) as to the multicasting tree (*MT*) game.

Equivalently, the minimum cost of the directed multicasting tree *MT* is the best value of the objective function of the integer programming problem $IP(D)$. Denote by $IP(Q)$, for $Q\subseteq D$, the directed *MT* problem obtained from the original problem by simply replacing D by Q in network G. Then, the pair (D,c_{MT}), where $c_{MT}{:}\,2^{|D|}\rightarrow R^{+}$ is such that $c_{MT}(\emptyset)=0$ and for each $Q\subseteq D$ $c_{MT}(Q)$ is the minimum objective function value of $IP(Q)$, is a game to be referred to as the multicasting tree (*MT*) game.

The most used cooperative game theory solution concept in communication network cost allocation is the core of a cooperative game. Recall that the core consists of stable cost allocation solutions which provide no incentive for any coalition to secede and build their own subnetwork. The game theoretic solution concepts are generally hard to compute even for relatively small problems and there are no general practical algorithms for the computation of these solutions. Consequently network cost allocation studies are typically done in the context of a particular problem. Researchers have most extensively studied a special case of the multicasting game in which users (receivers) reside at all nodes. The associated cooperative games are referred to as the class of Minimum Cost Spanning Tree (*MCST*) games. The non-emptiness of the core of the *MCST* game was demonstrated and analyzed in the literature (see, for example, [1], [2], [4], [6], [8], [9], [10], [13] and[18]). Note, however, that up to date it is not possible to characterize the entire core.

It turns out that for the general case of the Multicasting game, i.e. when not all nodes are receivers, the stable cost allocations do not necessarily exist (see, for example, [18]). Some sufficient conditions for their existence, as well as a heuristic algorithm for finding core points of the Multicasting game were presented in [27]. Survey of results on some related combinatorial games were presented in [28] and [29]. For approximation results on the *STN* problem, and their application to associated Multicasting games, see [35] and [12], respectively. It seems that the combinatorial difficulty of the cost allocation problem has, at least temporarily, turned off the attention of the research community.

It is rare that theoretical terms appear in law statements. Therefore, it is particularly interesting that the certain Law Amendment of the US Federal Law (see [26]) discusses the compromise between what is desirable and what is practical to compute in multicasting cost allocation considerations. Namely, the Steiner tree approach was recognized as desirable, but due to its computational complexity not practical.

The main objective of the cost allocation analysis associated with multicasting problem in this paper is to overview some mathematical programming formulations of the associated cost allocation problem and a primal-dual based cost allocation scheme which have potential practical applications.

General idea of primal-dual cost allocation scheme is to formulate the multicasting problem as an integer programming problem (*IP*) and consider it's standard linear programming relaxation (LP). The cost shares correspond to dual variables. We initially set dual variables to zero and then in the main step of the scheme we keep increasing dual variables until dual constraint is tight. This determines an object in primal for which the cost can be “fairly” allocated. We proceed until primal and dual are close enough.

Consider now the linear programming relaxation $LP(D)$ of $IP(D)$ (the minimum cost directed Steiner tree problem as defined above),

$$LP(D)\text{: } min\left\{wu : u(\delta(S)) \geq 1,\, S \in \mathcal{S}_D,\; u \geq 0\right\}, \text{ and its dual } DP(D),$$

$$DP(D)\text{: } max\left\{\sum_{S \in \mathcal{S}_D} v_S : \sum_{S: e \in \delta(S)} v_S \leq w_e, \text{ for all } e \in E; S \in \mathcal{S}_D\,,\, v_S \geq 0 \text{ for all } S\right\}.$$

It turns out that the dual $DP(D)$ can be used to allocate a fraction of the total cost of the multicast tree MT, while satisfying the core constraints of the *MT* game. Namely, if $v_S, S \in \mathcal{S}_D$, is a feasible solution to $DP(D)$, and if for each $S \in \mathcal{S}_D$ the amount v_S is allocated arbitrarily to users in S, then the cost allocation vector x generated by this operation satisfies the core constraints. Indeed, for any subset of users $Q \subseteq D$, $c_{MT}(Q) = \sum_{e \in E_{MT_Q}} w(e)$, where E_{MT_Q} is the set of edges of the directed multicast tree $MT_Q = ((N_{MT_Q} \cup O, E_{MT_Q})$ that spans $Q \cup O$ in the complete network induced by nodes of the best known multicast tree MT. Then, by the construction of x and by dual feasibility:

$$x(Q) \leq \{\sum_{S \in \mathcal{S}_D} v_S : S \in \mathcal{S}_D \text{ and } S \cap Q \neq \emptyset\} \leq \sum_{e \in E_{MT_Q}} w(e) = c_{MT}(Q).$$

Clearly, the absence of a duality gap (i.e. if the objective value of $IP(D)$ and $LP(D)$ coincide) is a sufficient condition for the non-emptiness of the core and we could use dual to generate some core points.

Next we present an outline of the multicasting primal-dual cost allocation schema. This schema will initially set dual variables to zero and keep increasing them while maintaining feasibility. The assigned dual values will be allocated among users in particular cut-sets in such a way that the core constraints are not violated. The objective is to find a feasible dual which satisfies the complementary slackness conditions (hence is optimal) if possible (for more details and variations of this cost allocation schema see [13],[28],[30] and [31]).

The Outline of the Primal-Dual Cost Allocation Schema in Multicasting:

Input: A weighted complete network $G = (N \cup O, E)$. A set of users D, $D \subseteq N$. The best known minimum cost directed tree MT, $MT = (N_{MT} \cup \{O\}, E_{MT})$, spanning $D \cup O$.

Initialization: Set $x_p = 0$, for all $p \in D$, $w_{ij} = w_{ji} = W_{ij}$ for all node pairs (i, j), and $v_S = 0$ for all $S \in S_D$. (x_p, $p \in D$ is the cost allocated to a user p, and v_S, $S \in Q_D$ are the values of dual variables).

Main Algorithm

"Begin by finding a potential allocating set S".
Do for $k = 1, \ldots, |D|$,
Do for each node $p \in D$
Let $S = \{p\}$.
Do until $|S| = k$ or no additional nodes can be added to S.
If there exist $(i, j) \in \delta(S)$ such that $w(i, j) = 0$, $i \neq O$; let $S = S \cup \{i\}$.
EndDo
"Next we allocate as much as possible to users in $S \cap D$."
If for all $e \in \delta(S)$, $w(e) > 0$ and $|\delta(S) \cap E_{MT}| = 1$
Let $v_S = min\{w(e), e \in \delta(S)\}$.
Choose arbitrary a_p , $p \in S \cap D$ such that $\sum_{p \in S} a_p = v_S$.
Do for all $p \in S$
Let $x_p = x_p + a_p$
EndDo
Do for all $e \in \delta(S)$
Let $w(e) = w(e) - v_S$
EndDo
EndIf
EndDo
EndDo
End.

It was shown (see [5] and [13]) that for a special case of the multicasting game when $D = N$ (minimum spanning tree game) there is no integrality gap. Moreover, the above Primal-Dual schema finds optimal dual solution and consequently generates some core points of this game.

The work in [25] implies that the above integrality gap remains zero if underlying network G is series-parallel. Hence, the core of the multicast game is not empty if G is series-parallel.

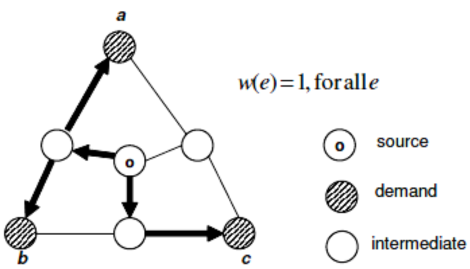

Figure 1 Empty core of the multicasting game

However, in general, when D is a proper subset of N, the above sufficient condition for the non-emptiness of the core is not always satisfied. It is known that the core of the multicasting game might be empty. Consider an example of the multicasting game associated with network informally presented in Figure 1 (this example was suggested in [33]). Clearly the entire cost of the multicasting network is $c(\{a,b,c\}) = 5$. On the other hand for any core allocation each two member coalition would not pay more than 3, $i.e.$ $x_a + x_b \leq 3$, $x_b + x_c \leq 3$, $x_a + x_c \leq 3$). The latter implies that the entire cost allocated could not exceed 4.5. Hence, the core is empty. Note, that there is an integrality gap in this example.

Observe, that the core might be non-empty even in the case of non-trivial duality gap. This can be demonstrated by the following example, originally presented in [27].

Indeed, consider the network $G = (N \cup \{O\}, E)$ in Fig. 2. Assume that all edge weights in G are 1 and let $D = \{5,6,7\}$ be the set of users. Let $(D; c)$ be the associated multicast game. An optimal solution to $IP(D)$ is indicated by bold arcs and has total weight $c(D) = 6$. It is easy to check that the vector $(x_5, x_6, x_7) = (2,2,2)$ is in the core $\mathcal{C}(D,c)$. On the other hand one can verify that $x^* \in R_+^E$ defined as follows: $x^*_{(0,1)} = 1$, $x^*_{(1,2)} = \frac{1}{2}$, $x^*_{(1,3)} = \frac{1}{2}$, $x^*_{(1,4)} = \frac{1}{2}$, $x^*_{(2,5)} = \frac{1}{2}$, $x^*_{(2,6)} = \frac{1}{2}$, $x^*_{(3,6)} = \frac{1}{2}$, $x^*_{(3,7)} = \frac{1}{2}$, $x^*_{(4,5)} = \frac{1}{2}$, $x^*_{(4,7)} = \frac{1}{2}$, and $x^*_{(i,j)} = 0$ otherwise, is feasible to $LP(D)$ associated with G and has the objective function value of 5.5.

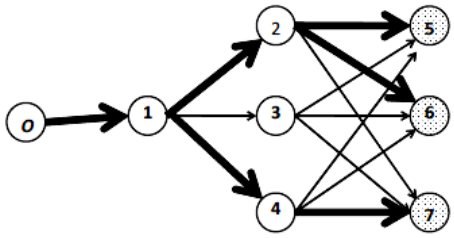

Figure 2 $G = (N \cup \{O\}, E)$

It was shown experimentally in [27] that in general case of multicasting game (i.e. when D is a proper subset N) the above Primal-Dual scheme becomes a good heuristic which quite often allocates most of the multicast cost in a subsidy free manner.

4 The Modified Multicast Tree Game and Cost Allocation Algorithm

In [31] we defined the *Modified Multicast Tree Game* as follows. The value of the characteristic function for each coalition is the cost of the optimal Multicast tree spanning all members of that coalition in the complete graph generated by all nodes (receivers and switching nodes) of MT. This definition is a modification of the characteristic function of the MT game. The profound difference between the two definitions is that, in the general MT game, each coalition is also allowed to use switching points which were not even used by the grand coalition.

In the *modified multicast game* each coalition restricts their considerations to the complete graph generated by nodes of the best-known solution of the grand coalition. Consequently, the value of the characteristic function for a particular coalition, as defined in the general *multicasting game*, might seem more rational, since it allows each coalition to consider the cheapest solution in the entire network. The modified multicasting game is more general and closer to reality than the widely used Minimum Cost Spanning Tree game. Since the best solution obtained by the grand coalition is "globally" the most attractive (cheapest known) solution, and in the absence of ability to compute the best solution for each possible coalition in the original network, it seems reasonable for users to agree to restrict their considerations to the complete network generated by nodes of such an optimal (or best known) network. Inspired by the studies from the management literature (see [15], [16] and [17]), it is argued in [31] that the users of multicasting network might agree to the above restriction in an attempt to balance exploitation and exploration in their strategic cost allocation decision-making. Namely, it appears that further exploitation of Min Cost Spanning Tree games for a class of multicasting tree cost allocation problems would move the cost allocation model too much away from reality, since it requires that all nodes of the network have users residing in them. On the other hand, the modified multicasting game explores more general solutions in which each coalition includes in their considerations all nodes (user and non-user nodes) of the best known multicasting tree of a grand coalition, as well as all possible links (which are not necessarily in the multicasting tree) between those nodes. In addition, it offers management cost allocation decisions which satisfy some critical performance targets (i.e. the existence and ease of finding a particular cost allocation solution). Moreover, further exploration of what might be achieved by the use of switching nodes outside the best Steiner tree of the grand coalition is known to be costly and time consuming, and it is known that it might lead to a potential failure in the existence of a "fair" solution we are seeking.

The most important practical problem with adopting the definition of the general multicasting game is that the core of that game may be empty. On the other hand, we demonstrated in [31] that the core of the modified multicasting game is not empty. Moreover, therein in the modified situation we were able to develop the tree based modification of the primal-dual cost allocation algorithm to efficiently compute some points in the core without explicitly specifying all core constraints. That algorithm uses the best known multicasting tree as input. It iterates from the leafs towards the source. The allocating sets are "grown" from single leaf users and in each iteration we try to allocate the cost of a single multicasting edge belonging to the cut-set under consideration (see [31] for details).

Consequently, the modified multicasting game model and the Primal-Dual Cost Allocation Schema from [31] which can be used to efficiently find some subsidy-free cost allocations have a good potential for practical application.

5 Distance and Population Monotonic Cost Allocation Schemes in Multicasting

First we demonstrate by example that a cost allocation scheme which produces core allocations is not necessarily distance or population monotonic. Consider the network in which all nodes are users $(N = D)$ and the scheme which allocates to each node the cost of the multicasting tree link connecting it to its predecessor. It can be shown that such a scheme produces core cost allocations (see [9]).

Consider now an example of an undirected network presented in Figure 3 with the source o and the initial node set $N = \{a, b, c, d\}$. Here we assume that the user set $D = N$ and that all links not indicated have infinite cost. (The dotted line indicates a link which might have been added later and/or its cost has been decreased. Note that the node e is added later.) The scheme would initially produce the multicast chain and the associated cost allocation $x = (x_a, x_b, x_c, x_d) = (4, 5, 4, 3)$. After the addition of another demand node e, the new multicast tree would be computed. It is easy to verify that the new cost allocation would become $x' = (x'_a, x'_b, x'_c, x'_d, x'_e) = (4, 4, 3, 4.5, 1)$. Since user d incurred increase in cost from 3 to 4.5, it would object the network expansion and the scheme is not population monotonic. Moreover, if we now decrease the cost of link (a, b) from 5 to 3.5, one can verify that the newly produced cost allocation becomes $x'' = (x''_a, x''_b, x''_c, x''_d, x''_e) = (4, 3.5, 4, 3, 1)$. Note that now the cost of c has been increased back from 3 to 4, in spite of the fact that the overall weight of the multicasting tree decreased. Consequently, this scheme is also not distance monotonic.

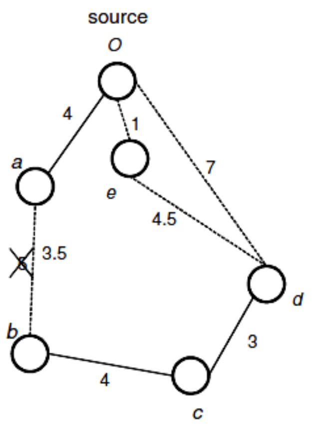

Figure 3 Network before and after adding nodes and/or decreasing the link cost

For submodular games population monotonic cost allocation schemes exist and population monotonicity is stronger than the core ([19]). But for most combinatorial optimization games the characterization functions are non-submodular and it seems to be very difficult to derive population monotonic schemes for those games (see [21] and [34]).

Note that the multicasting game is not submodular. Nevertheless, for the special case of multicasting game with $D = N$ (all nodes are users), some population monotonic schemes were presented in [2],[13] and [22]. Specifically, in [13] we constructed a primal-dual based cost allocation scheme which is in the core, and is distance and population monotonic. In [22] the authors considered a problem of finding minimum arborescence connecting every vertex via a directed path to the source and showed that the corresponding game does not have core population monotonic cost allocation scheme. In [20] they considered a simpler problem, namely finding a minimum arborescence in a directed acyclic graph, where every vertex is guaranteed to have a link to the source. Therein they developed a core-population monotonic cost allocation scheme for the corresponding cost allocation problem.

In case $D \neq N$ the optimization problem is NP-hard, the core might be empty, it might be hard to compute cost shares. Approximate approach, and the concept of α core was introduced in [12]. A cost allocation function x is in α-core if $\alpha c(S) \leq x(S) \leq c(S)$ for all $S \subseteq N$. They presented a 1/2-core population monotonic cost allocation scheme for the multicasting game. A general technique for turning a primal-dual algorithm into an α-core population monotonic cost allocation scheme was presented in [24].

We now modify the Primal-Dual algorithm from Section 3 and apply it to the modified Multicasting game from Section 4 as follows. Initially, in a preprocessing phase we treat all nodes as if they were receivers. This algorithm is not guided by the best known multicasting tree. Instead, it will search for the allocating sets of the smallest size. In each iteration, the algorithm will increase the value of the dual variable associated with the allocating set as much as possible and divide this part of the cost (the value of the dual variable) equally among nodes in the allocating cut set.

It was demonstrated in [13] that such an algorithm would identify the *MT* and divide the entire cost among its nodes. Since the real users are located only at nodes in D only, we need to further allocate the cost that was assigned to Steiner tree nodes in $N \backslash D$. We distribute this cost equally among receiver nodes in D. The rationale for distributing the cost of all switching nodes equally among users is the following. In practice, the broadcasting network is likely to dynamically change all the time. That might happen whenever cost of some link is changed and/or if a new node (user or non-user) joins the network and/or if any switching node becomes a user and/or if a receiver node also becomes a source node. Consequently, the individual position of any user node with respect to switching nodes is unpredictable and it seems somewhat reasonable for the users to agree up front to equally share the cost of switching nodes. We refer to the above algorithm as to the modified *MT* cost allocation scheme. It can be shown that this cost allocation scheme is distance and population monotonic. Clearly, the deficiency of such a scheme is that in a general case it does not produce core allocation and thus allows cross subsidies. Moreover, the negative results presented in [11] show that for many interesting combinatorial optimization games it is not even possible to achieve good core approximations combined with population monotonicity.

6 Conclusions

The general multicast cost allocation problem is a very challenging combinatorial problem. Finding the optimal multicasting tree is already $NP-$ hard problem and the core population and distance monotonic cost allocation scheme do not necessarily exist. Consequently, we modify the problem a little bit to make it technically tractable. The idea of the modification is to restrict our strategic considerations to only the nodes used in the best known multicast solution. Namely, when we analyze the best potential solution for a particular subset of users, we assume that they could use all nodes (receivers and switching points) of the best known multicast tree as their switching points. Consequently, each coalition would seek the cheapest multicast service in the complete network generated by nodes of the best known multicast tree. The value of the characteristic function for a particular coalition is defined as the cost of the best solution they could achieve under the above assumptions. It has been demonstrated that for modified multicast game stable cost allocation solution exits. We constructed a polynomial primal-dual based cost allocation scheme that finds some core points of the MT game.

Further analysis of the multicasting cost allocation problem using the modified multicasting game is needed. In particular, given the ever changing underlying communication networks it is of great interest to develop associated distance and population monotonic cost allocation schemes. It remains an open problem to characterize classes of optimization games for which core-distance-population monotonic cost allocation schemes exist. The objective of future work is to develop a distance and population monotonic cost allocation scheme(s) for the modified *MT* game which also minimizes cross-subsidies.

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