INFEASIBLE FULL NEWTON-STEP INTERIOR-POINT METHOD FOR LINEAR COMPLEMENTARITY PROBLEMS

Goran Lešaja
Department of Mathematical Sciences, Georgia Southern University
Statesboro, Georgia, USA
E-mail: goran@georgiasuthern.edu

A. Drummer
Department of Mathematics, East Georgia College
Swainsboro, Georgia, USA
E-mail: amdrummer@ega.edu

Ljiljana Miletić
Department of Mathematics and Computer Science
Požega High School, Croatia
E-mail: Ljiljana.Miletic@public.carnet.hr

Abstract

In this paper we consider an Infeasible Full Newton-step Interior-Point Method (IFNS-IPM) for monotone Linear Complementarity Problems (LCP). The method does not require a strictly feasible starting point. In addition, the method avoids calculation of the step size and instead takes full Newton-steps at each iteration. Iterates are kept close to the central path by suitable choice of parameters. The algorithm is globally convergent and the iteration bound matches the best known iteration bound for these types of methods.

1. INTRODUCTION

We consider a class of Linear Complementarity Problems (LCP) formulated in the standard form:

Given a matrix $M \in R^{nxn}$ and a vector $q \in R^n$, find a pair of vectors $(x, s) \in R^{2n}$ such that

$$(P) \qquad s = Mx + q,\ x^T s = 0,\ (x, s) \geq 0. \tag{1.1}$$

Note that $x^T s = 0 \Leftrightarrow xs = 0$ where xs denotes the component-wise (Hadamard) product of the vectors x and s.

Although LCP is feasibility and not an optimization problem, it is closely related to the optimization problems. It is well known that the Karush-Kuhn-Tucker (KKT) optimality conditions for Linear Optimization (LO) and Convex Quadratic Optimization (CQO) can be written as LCP. Moreover, LCP also has a close connection to variational inequalities: some classes of variational inequalities can be formulated as an LCP and vice versa. In addition, many important practical problems in economics theory (equilibrium problems), game theory, transportation planning (assignment problems), optimal control, engineering, etc. can be directly formulated as LCP. For a comprehensive treatment of LCP theory and practice we refer the reader to the monographs of Cottle et al. [4], Fachinei and Pang [6] and Kojima et al. [7].

It is well known that for general matrices M the problem is NP-complete [3]. Therefore, it is natural to look for classes of matrices M for which the corresponding LCP can be solved in polynomial time. Most common and most studied is the class of monotone-LCPs, where matrix M is a positive-semidefinite matrix. This is largely due to the fact that the Karush-Kuhn-Tucker conditions (KKT) of the Quadratic Optimization problem with the quadratic objective function defined by the positive-semidefinite matrix can be formulated as monotone-LCP. In addition, most practical problems that can be directly formulated as LCP are usually monotone-LCP. We also recall that in the special case of LO problem, matrix M becomes a skew-symmetric matrix. For these reasons, in this paper we consider a monotone LCP.

Due to the theoretical and practical importance of LCP, efficient methods for solving LCPs are of a significant interest. The existing tradition of generalizing results for LO to LCP dates back to the early days of the development of simplex-type algorithms (pivoting algorithms) and it continues to this day. The Interior-Point Methods (IPMs) that have been a great success for LO are no exception. Various IPMs for LO have been successfully generalized to LCP. Besides the aforementioned monographs, and without any attempt to be complete, we mention few other relevant references [1, 2, 8, 10].

IPMs can be grouped in two groups, feasible algorithms which require the knowledge of a strictly feasible starting point and infeasible algorithms that do not. The algorithm presented in this paper falls into the second category. Furthermore, most IPMs control the closeness to the central path, which then reflects on the iteration bound, by appropriate choice of the step size along the search directions. A full Newton-step, where step size is always one, would be preferable; however, without other safeguards this may cause even infeasibility and divergence of the algorithm. By appropriate choice of parameters we manage to design an IPM that uses full Newton-steps, while still preserving global convergence and matching the best known complexity for these types of IPMs, namely, $O(n \log \frac{n}{\varepsilon})$. The method is a generalization of the infeasible full Newton-step IPM for LO discussed in [9].

The paper is organized as follows. The outline of the method is given in Section 2. The analysis and complexity of the algorithm are presented in key Section 3. Section 4 contains brief concluding remarks.

2. INFEASIBLE FULL NEWTON-STEP IPM

In this section we present an outline of the Infeasible Full Newton-step IPM (IFNS-IPM) for monotone LCP. We assume that the LCP (1.1) has a solution $\left(x^*, s^*\right)$ such that

$$\left\|x^*\right\|_\infty \le \rho_p, \qquad \max\left\{\left\|s^*\right\|_\infty, \rho_p\left\|Me\right\|_\infty, \left\|q\right\|_\infty\right\} \le \rho_d, \tag{2.1}$$

which basically means that the solution is inside a certain cube. The additional conditions are not restrictive and are added the ease of the analysis. Furthermore, we define

$$x^0 = \rho_p e,\ s^0 = \rho_d e,\ \mu_0 = \rho_p \rho_d, \tag{2.2}$$

as the initial starting point. Then, the initial residual is given by $r^0 = s^0 - Mx^0 - q$.

We consider a family of perturbed LCPs (P_υ) for $\upsilon \in (0,1]$

$$(P_\upsilon) \qquad s - Mx - q = \upsilon r^0,\ xs = 0,\ (x,s) \ge 0. \tag{2.3}$$

Note that for $\upsilon = 1$ the (x^0, s^0) is a strictly feasible solution of (P_υ). Moreover, we have the following lemma.

Lemma 2.1. If the original LCP, (P), is feasible, than (P_υ) is strictly feasible for $\upsilon \in (0,1]$.

As it is customary in the theory of IPMs, we perturb the complementarity equation in (P_υ) (2.3) and consider the perturbed problem.

$$s - Mx - q = \upsilon r^0,\ xs = \mu e,\ (x,s) > 0. \tag{2.4}$$

Given the fact that (P_υ) is strictly feasible, the above system has a unique solution, for each $\mu > 0$. This solution is denoted as $(x(\upsilon,\mu), s(\upsilon,\mu))$ and we call it the $\mu-$*center* of LCP (P_υ). The set of $\mu-$centers forms a trajectory (homotopy path), which is called a *central path* of the LCP. Moreover, if $\mu \to 0$, the limit of the central path exists and it is a solution of the LCP (P_υ). Thus, we have a family of central paths for $\upsilon \in (0,1]$.

However, we are not interested in the solution of (P_υ), we are interested in the solution of (P). Thus, we approximately trace a family of central paths while simultaneously reducing μ and υ at each iteration. The 'engine' to accomplish this task is a Newton Method (NM).

The μ and υ are connected in the following way: $\upsilon = \frac{\mu}{\mu_0}$. The proximity measure of (x,s) to the μ - center is measured by norm-based measure

$$\delta(x,s,\mu) := \delta(v) := \frac{1}{2}\left\|v - v^{-1}\right\|, \text{ where } v := \sqrt{\frac{xs}{\mu}} \tag{2.5}$$

is called a *variance* vector. It is easy to see $v = e \Leftrightarrow xs = \mu e$, i.e., $v = e$ iff (x,s) is a μ - center.

The method can be started because (x^0, s^0) is a μ_0-center of the (P_υ) for $\upsilon = 1$ and $\delta(x^0, s^0, \mu_0) = 0$. We assume that the iterate, (x,s) is given in the 'neighborhood' of the μ - center $(x(\upsilon,\mu), s(\upsilon,\mu))$, where the *neighborhood* is determined by $\delta(x,s,\mu) \le \tau$ for some (small) *threshold* parameter $\tau > 0$. Next, μ is decreased $\mu_+ = (1-\theta)\mu$ for some *barrier* parameter $\theta \in (0,1)$. Thus, υ is decreased for the same amount, $\upsilon_+ = (1-\theta)\upsilon$. A new iterate (x^+, s^+) is calculated so that it satisfies the same inequalities as above with x, s, μ, υ replaced by $x^+, s^+, \mu_+, \upsilon_+$. This is accomplished by performing a *feasibility* step and several *centering* steps at each iteration. First, a feasibility step is performed that moves from the old iterate (x,s) in (P_υ) to a point (x^f, s^f) in (P_{υ^+}) targeting new μ_+ - center. However, this point, although feasible, may not be close enough to the μ_+ -center; thus, several centering steps are performed to obtain a new iterate that is in the τ -neighborhood $\delta(x^+, s^+, \mu_+) \le \tau$ of the μ_+ -center.

In what follows we describe a feasibility part and centering part of the iteration in more details.

Feasibility step

Given $\mu_+ = (1-\theta)\mu$, a direct application of the Newton's method to the system (2.4) leads to the following Newton system for the search direction $(\Delta^f x, \Delta^f s)$:

$$\begin{aligned} M\Delta^f x - \Delta^f s &= \theta\upsilon r^0 \\ s\Delta^f x + x\Delta^f s &= (1-\theta)\mu e - xs. \end{aligned} \tag{2.6}$$

Given the assumptions this system has a unique solution for any $(x,s) > 0$ because the matrix of the system is non-singular if M is positive-semidefinite matrix (and even for more general matrices). By taking a full Newton-step along the search direction $(\Delta^f x, \Delta^f s)$, one constructs a new point (x^f, s^f) with

$$x^f = x + \Delta^f x, \;\; s^f = s + \Delta^f s$$

In the next section we will show that with the appropriate choice of values (small enough) for barrier parameter θ and threshold parameter τ we can guarantee that a point (x^f, s^f) is strictly feasible in (P_{υ^+}) and moreover $\delta(x^f, s^f, \mu_+) \leq \frac{1}{\sqrt{2}}$.

Centering steps

While performing centering steps we stay in (P_{υ^+}), that is, we do not change μ_+ and υ_+; however, we rename them to new μ and υ, i.e. $\mu := \mu_+$ and $\upsilon := \upsilon_+$. We start from the strictly feasible point $(x^f, s^f) := (x, s)$ in (P_υ). The goal is to come closer to the μ-center in (P_υ). A direct application of Newton's method to the system (2.4) leads to the following Newton system for the centering search direction $(\Delta^c x, \Delta^c s)$

$$\begin{aligned} M\Delta^c x - \Delta^f s &= 0 \\ s\Delta^c x + x\Delta^c s &= \mu e - xs. \end{aligned} \tag{2.7}$$

The first equation has zero on the right hand side because we search for a new strictly feasible point in the same LCP (P_υ) and the term $1-\theta$ is not present in the second equation because we are targeting the same μ-center. Again, for the same reasons as in feasibility step, this system has a unique solution for M positive semidefinite matrix. By taking a full Newton-step along the search direction $(\Delta^c x, \Delta^c s)$, one constructs a new pair (x^c, s^c) with

$$x^c = x + \Delta^c x, \quad s^c = s + \Delta^c s$$

Moreover, we will show in the next section that $\delta(x, s, \mu) \leq \frac{1}{\sqrt{2}}$ is a neighborhood where centering steps achieve quadratic convergence. Hence, the number of centering steps required to get to a $\delta(x, s, \mu) \leq \tau$ neighborhood of the μ-center is very small.

The algorithm is stopped when we obtain an iterate (x, s) for which the norm of the residual $r := \upsilon r^0$ and $x^T s$ are very small, i.e., $n\mu \leq \varepsilon$ and $\|r\| \leq \varepsilon$ for a small accuracy parameter $\varepsilon > 0$. This will certainly be achieved in a finite (even polynomial) number of steps because μ and υ are reduced at each iteration for a factor $1-\theta$.

The outline of the IFNS-IPM for LCP (1.1) is given in the Figure 1 below.

Algorithm 2.1 IFNS-IPM for LCP

Input:

accuracy parameter $\varepsilon > 0$;
barrier update parameter $0 < \theta < 1$;
a threshold parameter $\tau > 0$;
starting point (x^0, s^0) with $\mu_0 = (x^0)^T (s^0) / n$;

begin

$x := x^0, s := s^0, \mu := \mu_0, \upsilon := \upsilon_0 = 1$;
while $\max\{n\mu, \upsilon\|r^0\|\} > \varepsilon$
$\mu := (1-\theta)\mu,\ \upsilon := (1-\theta)\upsilon$;
calculate feasibility direction $(\Delta^f x, \Delta^f s)$ from (2.6);
update $x^f = x + \Delta^f x,\ \ s^f = s + \Delta^f s$;
calculate $v := v^f$ from (2.5);
while $\delta(v) > \tau$
calculate centering direction $(\Delta^c x, \Delta^c s)$ from (2.7);
update $x^c = x + \Delta^c x, \quad s^c = s + \Delta^c s$;
endwhile
endwhile
end

Figure 1: IFNS-IPM for LCP

3. ANALYSIS AND COMPLEXITY OF THE ALGORITHM

In this section we determine the appropriate values of the barrier parameter θ and threshold parameter τ for which the IFNS-IPM in Figure 1, in the sequel called just *Algorithm*, is globally convergent. We also derive the iteration bound for the Algorithm that matches the best known iteration bound for these types of methods.

Feasibility step

The major part of the analysis is devoted to finding the values of θ and τ that will guarantee a strict feasibility of (x^f, s^f). In order to facilitate the analysis we introduce the scaled search directions

$$d_x^f = \frac{v\Delta^f x}{x}, \quad d_s^f = \frac{v\Delta^f s}{s}, \tag{3.1}$$

where v is defined in (2.5). Using the above scaled search directions the system (2.6) can be transformed to the system

$$\begin{aligned} \tilde{M}\, d_x^f - d_s^f &= \theta \upsilon \frac{1}{\sqrt{\mu}} D r^0 \\ d_x^f + d_s^f &= (1-\theta) v^{-1} - v, \end{aligned} \tag{3.2}$$

where $\tilde{M} = DMD$ and $D = X^{1/2} S^{1/2}$. The matrices X and S represent diagonal matrices of vectors x and s, i.e. $X = diag(x)$ and $S = diag(s)$.

The following key lemma gives sufficient and necessary conditions for strict feasibility of (x^f, s^f).

Lemma 3.1 The point (x^f, s^f) is strictly feasible iff $(1-\theta)e + d_x^f d_s^f > 0$.

The more practical, however, just sufficient condition for strict feasibility is given in the following corollary.

Corollary 3.2 The point (x^f, s^f) is strictly feasible if $\left\| d_x^f d_s^f \right\|_\infty < 1-\theta$.

Next, we give an upper bound on the proximity measure $\delta(v^f)$ after the feasibility step.

Lemma 3.3 If $\left\| d_x^f d_s^f \right\|_\infty < 1-\theta$, then

$$4(\delta(v^f))^2 \le \frac{\left\| \frac{d_x^f d_s^f}{1-\theta} \right\|^2}{1 - \left\| \frac{d_x^f d_s^f}{1-\theta} \right\|_\infty}.$$

Recall that we want to obtain $\delta(x^f, s^f, \mu_+) = \delta(v^f) \le \frac{1}{\sqrt{2}}$. It immediately follows that this will be satisfied if the right hand side of the above inequality is less than or equal 2. Given the fact that $\left\| d_x^f d_s^f \right\|_\infty \le \frac{1}{2}\left(\left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2 \right)$, we have

$$4(\delta(v^f))^2 \le \frac{\left\| \frac{d_x^f d_s^f}{1-\theta} \right\|^2}{1 - \left\| \frac{d_x^f d_s^f}{1-\theta} \right\|_\infty} \le \frac{\frac{1}{4}\left(\frac{\left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2}{1-\theta} \right)^2}{1 - \frac{1}{2}\frac{\left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2}{1-\theta}} \le 2.$$

If we denote $u := \frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta}$ the above inequality becomes

$$\frac{1}{4}u^2 \le 2(1-\frac{1}{2}u) \Leftrightarrow u^2 + 4u - 8 \le 0$$

with solution

$$0 \le u \le -2 + \sqrt{12} \approx 1.46 \Leftrightarrow \|d_x^f\|^2 + \|d_s^f\|^2 \le 1.46(1-\theta). \tag{3.3}$$

Note that this is a stronger result than the condition on u that follows from Corollary 3.2

$$\frac{\|d_x^f d_s^f\|_\infty}{1-\theta} \le \frac{1}{2}\frac{\|d_x^f\|^2 + \|d_s^f\|^2}{1-\theta} \le 1 \Leftrightarrow u \le 2 \Leftrightarrow \|d_x^f\|^2 + \|d_s^f\|^2 \le 2(1-\theta).$$

Thus, the problem of finding the conditions for $\delta(v^f) \le \frac{1}{\sqrt{2}}$ to hold reduces to finding the upper bound on $\|d_x^f\|^2 + \|d_s^f\|^2$. In order to find this upper bound we have to examine the system (3.2) using the following lemma.

Lemma 3.4 Given a system $\tilde{M}u - z = \tilde{a}, \quad u + z = \tilde{b}$, the following hold

(1) $Du = (I + \tilde{M})^{-1}(a+b), \qquad Dz = b - Du,$

(2) $\|Du\| \le \|a+b\|$

(3) $\|Du\|^2 + \|Dz\|^2 \le \|b\|^2 + 2\|a+b\|\|a\|,$

where $\tilde{M} = DMD$, $D = X^{1/2}S^{1/2}$ and $a = D\tilde{a}$, $b = D\tilde{b}$.

Applying the above lemma to the system (3.2) we obtain the following bound

$$\|d_x^f\|^2 + \|d_s^f\|^2 \le \|(1-\theta)v^{-1} - v\|^2 + 2\left(\left\|\theta\upsilon\frac{1}{\sqrt{\mu}}Dr^0\right\| + \|(1-\theta)v^{-1} - v\|\right)\left\|\theta\upsilon\frac{1}{\sqrt{\mu}}Dr^0\right\| \tag{3.4}$$

Now, we need to find bounds for $\|(1-\theta)v^{-1} - v\|$ and $\left\|\theta\upsilon\frac{1}{\sqrt{\mu}}Dr^0\right\|$. Using the definitions of D, v, and relationship between norms, we obtain

$$\left\|\theta\upsilon\frac{1}{\sqrt{\mu}}Dr^0\right\| \le \frac{\theta}{\mu_0}\frac{1}{v_{\min}}\|r^0\|_\infty\|x\|_1,$$

where $v_{\min}$ is the minimal component of the vector v. Since we assumed (2.2), we have $r^0 = s^0 - Mx^0 - q = \rho_d e, -\rho_p Me - q$ implying

$$\|r^0\|_\infty = \rho_d \left\| e, -\frac{\rho_p}{\rho_d} Me - \frac{1}{\rho_d} q \right\|_\infty \le \rho_d \left(1 + \frac{1}{\rho_d} \rho_p \|Me\|_\infty + \frac{1}{\rho_d} \|q\|_\infty \right).$$

Using the assumption (2.1) we obtain

$$\left\| \theta \upsilon \frac{1}{\sqrt{\mu}} D r^0 \right\| \le \frac{3\theta}{\rho_p} \frac{\|x\|_1}{v_{\min}}, \tag{3.5}$$

which explains the reason for the second part of the assumption (2.1). The upper bound for the second term is obtained as follows.

$$\begin{aligned} \left\|(1-\theta)v^{-1} - v\right\|^2 &= (1-\theta)^2 \left\|v^{-1}\right\|^2 - 2(1-\theta)(v^{-1})^T v + \|v\|^2 = (1-\theta)^2 \left\|v^{-1}\right\|^2 - 2(1-\theta)n + \|v\|^2 \\ &\le \left\|v^{-1}\right\|^2 - 2n + \|v\|^2 + 2\theta n = \left\|v^{-1} - v\right\|^2 + 2\theta n = 4\delta^2(v) + 2\theta n. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into the (3.4) we obtain

$$\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2 \le 4\delta^2(v) + 2\theta n + 2\left(\frac{3\theta}{\rho_p} \frac{\|x\|_1}{v_{\min}} + \sqrt{4\delta^2(v) + 2\theta n} \right) \frac{3\theta}{\rho_p} \frac{\|x\|_1}{v_{\min}}. \tag{3.7}$$

Now we need an upper bound on $\|x\|_1$ and lower bound on $v_{\min}$. This is given in the lemma below.

Lemma 3.5 The following inequalities hold

(1) $\sigma^{-1}(\delta) \le v_i \le \sigma(\delta)$

(2) $\|x\|_x \le (2 + \sigma(\delta)) n \rho_p$, $\|s\|_x \le (2 + \sigma(\delta)) n \rho_d$,

where $\sigma(\delta) = \delta + \sqrt{\delta^2 + 1}$ and $\delta = \delta(v)$.

Applying the lemma to (3.7) we obtain

$$\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2 \le 4\delta^2(v) + 2\theta n + 18\big(\theta n (2 + \sigma(\delta))\sigma(\delta)\big)^2 + 6\theta n \sqrt{4\delta^2(v) + 2\theta n}\,(2 + \sigma(\delta))\sigma(\delta) \tag{3.8}$$

Combining (3.3) and (3.8) we have

$$4\delta^2(v) + 2\theta n + 18\big(\theta n(2+\sigma(\delta))\sigma(\delta)\big)^2 + 6\theta n\sqrt{4\delta^2(v) + 2\theta n}(2+\sigma(\delta))\sigma(\delta) \le 1.46(1-\theta) \tag{3.9}$$

Finally, we have to determine the values of τ (since $\delta \le \tau$) and θ such that (3.9) is satisfied. Numerical examination shows that the values that make the left hand side very close to the right hand side are

$$\tau = \frac{1}{4}, \qquad \theta = \frac{1}{12n}. \tag{3.10}$$

The above discussion can be summarized in the following theorem:

Theorem 3.6 Let $\tau = \frac{1}{4}$, $\theta = \frac{1}{12n}$ and (x, s) be the starting iteration near μ-center with $\delta(x, s, \mu) \le \tau = \frac{1}{4}$. Then, after the feasibility step, we obtain a point (x^f, s^f) that is strictly feasible in (P_{υ^+}) with $\delta(x^f, s^f, \mu_+) \le \frac{1}{\sqrt{2}}$.

Centering step

Our next goal is to perform several centering steps to get sufficiently close to the μ_+-center of the (P_{υ^+}). Since (x^f, s^f) is the starting point we denote it as (x, s), and μ_+ as μ and we denote $\delta(x^f, s^f, \mu_+)$ as δ. Similarly, as in feasibility step, we use scaled centering directions

$$d_x^c = \frac{v\Delta^c x}{x}, \qquad d_s^c = \frac{v\Delta^c s}{s}, \tag{3.11}$$

to transform the system (2.9) into the system

$$\begin{aligned} Md_x^c - d_s^c &= 0 \\ s\,d_x^c + x\,d_s^c &= \mu e - xs. \end{aligned} \tag{3.12}$$

The following inequalities are helpful in the analysis of the centering steps.

Lemma 3.7 The following inequalities hold

$$(d_x^c)^T d_s^c \le \delta^2, \qquad \left\|d_x^c d_s^c\right\|_\infty \le \delta^2, \qquad \left\|d_x^c d_s^c\right\| \le \delta^2. \tag{3.13}$$

It is straightforward to derive

$$x^c s^c = \mu(e + d_x^c d_s^c) \quad \text{and} \quad (x^c)^T s^c \le \mu(n + \delta^2). \tag{3.14}$$

Similarly as in Lemma 3.1 and Corollary 3.2, the strict feasibility of the centering steps is preserved if

$$e + d_x^c d_s^c > 0 \text{ or } \left\| d_x^c d_s^c \right\|_\infty < 1. \tag{3.15}$$

The following lemma gives an upper bound on the $\delta\left(x^c, s^c, \mu\right)$.

Lemma 3. 8 If $\delta < 1$, then $\left(x^c, s^c\right)$ is strictly feasible and

$$\delta\left(x^c, s^c, \mu\right) \leq \frac{\delta^2}{\sqrt{1-\delta^2}}. \tag{3.16}$$

Proof: The strict feasibility follows from (3.15). Next, we have

$$\begin{aligned}
\delta\left(x^c, s^c, \mu\right) &= \frac{1}{2}\left\|(v^c)^{-1} - v^c\right\| = \frac{1}{2}\left\|(v^c)^{-1}(e-(v^c)^2)\right\| \\
&= \frac{1}{2}\left\|(\sqrt{e + d_x^c d_s^c})^{-1}(e - e - d_x^c d_s^c)\right\| = \frac{1}{2}\left\|\frac{d_x^c d_s^c}{\sqrt{e + d_x^c d_s^c}}\right\| \\
&\leq \frac{1}{2}\frac{\left\|d_x^c d_s^c\right\|}{\left\|\sqrt{e + d_x^c d_s^c}\right\|} \leq \frac{1}{2}\frac{\left\|d_x^c d_s^c\right\|}{\sqrt{1 - \left\|d_x^c d_s^c\right\|_\infty}} \leq \frac{1}{2}\frac{\delta^2}{\sqrt{1-\delta^2}}.
\end{aligned}$$

The third equality is due to the definition of v^c and (3.14) while the inequalities are due to norm properties and inequalities in (3.13).

Q.E.D.

The immediate consequence of the lemma is the following corollary.

Corollary 3.9 If $\delta \leq \frac{1}{\sqrt{2}}$, then $\delta\left(x^c, s^c, \mu\right) \leq \delta^2$.

The corollary states that the centering steps converge quadratically in the $1/\sqrt{2}-$ neighborhood of the central path. This implies that the number of centering steps necessary to obtain a point in the $\tau = 1/4$ -neighborhood is no more than two. Thus, to obtain a new iterate $\left(x^+, s^+\right)$ we need one feasibility step and at most two centering steps per iteration. Hence, the following theorem:

Theorem 3.10 If $\tau = \frac{1}{4}$, $\theta = \frac{1}{12n}$, and $\mu_0 = \rho_p \rho_d$, then the Algorithm is globally convergent and requires at most

$$12n \log\left(\frac{33}{32}\frac{(x^0)^T s^0}{\varepsilon}\right)$$

iterations to obtain an ε-approximate solution of LCP (P) in (1.1).

In other words, the theorem states that the Algorithm achieves an ε-approximate solution in

$$O\left(n\log\frac{n}{\varepsilon}\right) \tag{3.17}$$

iterations. The obtained iteration bound matches the best known iteration bound for these types of methods (infeasible algorithms).

Due to the page limitation, the proofs of the lemmas in this section are mostly omitted. They can be found in [5]. An LO versions of some results that can straightforwardly be generalized to LCP can also be found in [9, 10].

4. CONCLUDING REMARKS

In this paper we have designed and analyzed IFNS-IPM presented in Figure 1. We have shown that the algorithm is globally convergent for the values of threshold and barrier parameters listed in (3.10). Furthermore, the algorithm matches the best known iteration complexity (3.17 for these types of methods.

The advantages of the method are that it does not require strictly feasible starting point (infeasible algorithm) and it uses full Newton-steps, thus, avoiding calculations of a step-size at each iteration. Furthermore, a nice feature of the method is that it simultaneously works on reducing infeasibility and achieving optimality. The disadvantage is that the method is a short-step method, i.e., because of the required choice of parameters in (3.10), iterates are forced to be close to the central path. However, our initial and limited numerical testing shows that the absence of step-size calculations provides the algorithm with some computational merit that still has to be investigated further with more extensive numerical testing.

In addition to more numerical testing, some directions for further research include generalization of the method to more general classes of LCPs such as $P_*(\kappa)$ -LCP and LCPs over symmetric cones.

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