THRESHOLD PARAMETER OF THE EXPECTED LOSSES

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Abstract

The objective of extreme value analysis is to quantify the probabilistic behavior of unusually large losses using only extreme values above some high threshold rather than using all of the data which gives better fit to tail distribution in comparison to traditional methods with assumption of normality.
In our case we estimate market risk using daily returns of the CROBEX index at the Zagreb Stock Exchange. Therefore, it's necessary to define the excess distribution above some threshold, i.e. Generalized Pareto Distribution (GPD) is used as much more reliable than the normal distribution due to the fact that gives the accent on the extreme values.

Parameters of GPD distribution will be estimated using maximum likelihood method (MLE). The contribution of this paper is to specify threshold which is large enough so that GPD approximation valid but low enough so that a sufficient number of observations are available for a precise fit.

Key words: *Threshold parameter, Value at Risk, Expected shortfall, Generalized Pareto Distribution*

1. INTRODUCTION

Extreme event risk is possible in every field of risk management. One of the most complicated issues for every manager is to find out the possible extreme event risk and to react in time. The objective of extreme value analysis is to quantify the probabilistic behavior of unusually large losses using only extreme values above some high threshold rather than using all of the data which gives better fit to tail distribution in comparison to traditional methods with assumption of normality. Two measures that we find most useful and reliable for describing the tail of the distribution are Value-at-Risk (VaR) and expected shortfall (ES). Value-at-risk is a high quantile of a certain distribution (mostly at 95% or 99%) and it represents the upper bound for a loss which is exceeded on a small number of events. However, regarding to some opinions, VaR is not a coherent risk measure, and it does not say anything about the potential size of a loss that exceeds it. On the other hand, expected shortfall is a coherent risk measure and answers the question about expected loss size when VaR is exceeded. In our case we estimate market risk using daily returns of the CROBEX index at the Zagreb Stock Exchange. Therefore, it’s necessary to define the excess distribution above some threshold, i.e. Generalized Pareto Distribution (GPD) is used as much more reliable than the normal distribution due to the fact that gives the accent on the extreme values. Parameters of GPD distribution will be estimated using maximum likelihood method (MLE). The contribution of this paper is to specify graphical method for selecting threshold which is large enough so that GPD approximation valid but low enough so that a sufficient number of observations are available for a precise fit.

2. LITERATURE REVIEW

Arnerić J., Jurun E., and Pivac S. (2007) have dealt with modeling volatility of Pliva stock returns on Zagreb Stock Exchange, measuring volatility reaction on market movements and the persistence of volatility. As the most appropriate model for those analyses they have chosen GARCH(p,q) model. In estimation procedure the assumption of Student's distribution was used to capture fat tails. Estimated degrees of freedom were used for precisely forecasting VaR and CVaR under non-normality assumption. The conditional expectation of continuously random variable was calculated under tail area. The authors have concluded that depending on if investor on capital market holds "long" or "short" position it's essentially important to predict possible and expected loss with appropriate probability density function.

Žiković S. and Pečarić M. (2010) have analyzed the performance of Value at Risk (VaR) models at extreme quantiles: 0.99, 0.995 and 0.999 for both long and short positions in Croatian, Zagreb stock exchange index - CROBEX. Backtesting showed that none of the usually employed VaR models correctly forecasted the risk during the ongoing global and domestic financial crisis. The only exceptions were the extreme value based models which correctly forecasted the true level of upside and downside risk. The authors have also investigated the closeness of fit of theoretical distributions to the extreme tails of CROBEX returns. Results showed that generalized Pareto distribution, which has a sound theoretical foundation, provided the best fit to both tails of CROBEX returns. The authors concluded that distribution tails differ significantly, with the right tail having a higher tail index, indicative of more extreme events.

Žiković S. and Aktan B. (2009) have investigated the relative performance of a wide array of Value at Risk (VaR) models with the daily returns of Turkish (XU100) and Croatian (CROBEX) stock index prior to and during the ongoing financial crisis. Results for Turkish XU 100 index and Croatian CROBEX index were similar in that Kupiec test showed that at high quintiles' (99% and 99.5%) only EVT models and HHS model satisfied the Basel criteria. In addition to widely used VaR models, they have also studied the behavior of conditional and unconditional extreme value theory (EVT) and hybrid historical simulation (HHS) models to generate 95%, 99% and 99.5% confidence level estimates. They concluded that during the crisis period all tested VaR model except EVT and HHS models seriously under predicted the true level of risk, with EVT models doing so at a higher cost of capital compared to HHS model.

Hsing Yu (2011) examined the relationship between the Croatian stock market index and relevant macroeconomic variables. Applying the GARCH model, the author has concluded that the Croatian stock market index is positively associated with real GDP, the M1/GDP ratio, the German stock market index and the euro area government bond yield and is negatively influenced by the ratio of the government deficit to GDP, the domestic real interest rate, the HRK/USD exchange rate, and the expected inflation rate.

Latković M. (2002) has described the most important types of risk, reasons why they appear, and has given several quantitative evaluations of risk for securities that are listed in Croatian and foreign capital markets. He has also described some econometric methods for estimation of short-term and long-term risks and has given the basic method for managing risks using the VaR method. He concluded that any quantitative method of managing risk cannot replace the experience and knowledge that has a portfolio manager.

Žiković, S. and Aktan, B. (2009) have examined the theoretical background of two nonparametric approaches to calculating VaR, historical simulation and hybrid approach developed by Boudoukh, Richardson and Whitelaw, and has examined their performance in a transitional capital market such as Republic of Croatia. The author has also evaluated and analyzed the out-of-sample forecasting accuracy of both methods on two Croatian indexes, CROBEX \u2013 the official index of Zagreb Stock Exchange and VIN - the official index of Varazdin Stock Exchange. His paper gave an overview of Value at Risk as a methodology for measuring market risk, the methodology of calculating VaR via historical simulation, a hybrid nonparametric approach to calculating VaR. Based on the performed tests it could be concluded that historical simulation should not be used for high confidence level estimates (above 95%), especially models based on shorter rolling windows. The obtained results showed that although BRW approach also has its flaws, especially when testing for temporal dependence in the tail events, it brought significant improvement to historical simulation with minimal additional computational effort. The author has concluded that BRW approach should be further studied and tested in other transitional and emerging economies, because based on obtained results it proved to be a far better alternative to historical simulation.

3. DATA AND METHODOLOGY

Since, the objective of this paper is to model extreme events on Zagreb Stock Exchange, the official stock index CROBEX is used for the analysis. Daily values are collected from www.zse.hr for period 4.1.2010. – 30.12.2011 and presented with daily returns in Figure 1.

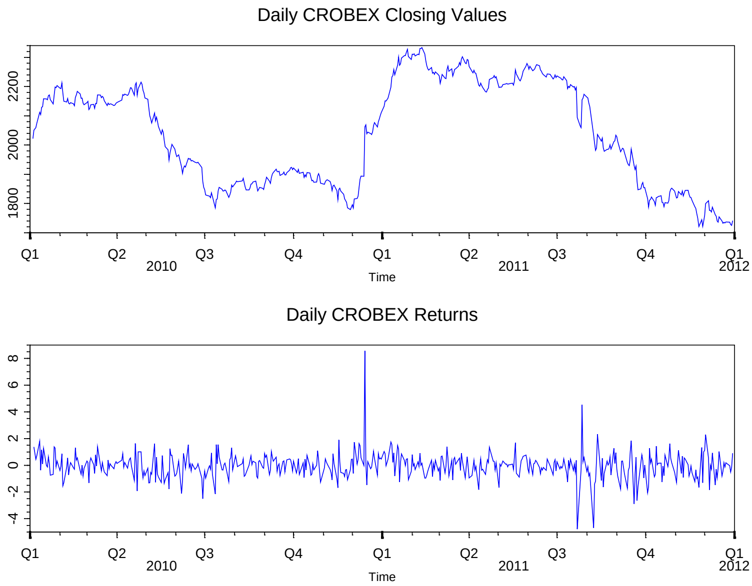

Figure 1: CROBEX Index – Daily Closing values and Returns

As shown in Figure 1 CROBEX rose and fell during selected period, with occasional sharp and large changes in daily returns. When analyzing the distribution of daily returns in Figure 2 it is obvious that normal distribution is not a good choice for sampling distribution, especially because α_4 equals 16.043 which indicates fat tails and high frequency of extreme events compared with frequency of normally distributed data.

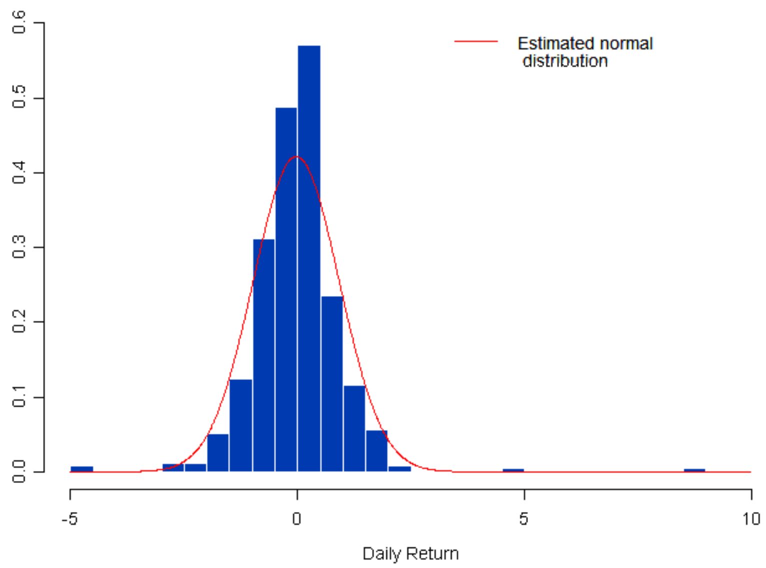

Figure 2: Histogram of Daily CROBEX Returns

Two kinds of extreme events are possible in stock market: extreme gains (positive side of the distribution) and extreme losses (negative side). In risk management more attention is assigned to the extreme losses with emphasis on how to model, measure and estimate them. In this paper extreme losses are in the main focus so we define daily return as $X_t = -\frac{\ln(CROBEX_t)}{\ln(CROBEX_{t-1})}$ with respect to convention of analyzing large losses instead of negative profits.

3.1. Risk measures

Some of the most frequent questions about risk management when analyzing stock exchange involve extreme quantile estimation. The most used risk measures are value-at-risk and expected shortfall. VaR is a high quantile of the distribution of losses *X*, formally defined as

$$VaR_q = F^{-1}(q) \tag{1}$$

where *q* is chosen as *0.95, 0.99 or 0.999* and *F* is the distribution of losses or negative returns. Expected shortfall (ES) is the expected size of a loss that exceeds VaR, defined as

$$ES_q = E\,[\,X \mid X > VaR_q\,] \tag{2}$$

with a prior chosen *q*. VaR is used for answering the question how large loss is expected in *(1-q)%* cases, and ES for estimating average loss, given that VaR is exceeded. For estimating both of the measures it is necessary to know the type of the underlying distribution *F* and to have estimated parameters. Result from the extreme value theory provides solution on selecting appropriate function F. First, we define function F_u,

$$F_u = P(X - u \le y | X > u), \text{ for } 0 \le y < x_0\text{-}u \text{ and } x_0 \le \infty. \tag{3}$$

With inverting the expression (3) the following relation is derived

$$F(x) = (1 - F(u)) F_u(x - u) + F(u) \text{ for } x > u \tag{4}$$

which is important for approximation of function F due to the following theorem, Zivot, E. and Wang, J. (2006). Theorem: For a large class of underlying distributions we can find a function $\beta(u)$ such that

$$\lim_{u \to x_0} \sup_{0 \le y < x_0 - u} \left| F_u(y) - G_{\xi,\beta(u)}(y) \right| = 0 \tag{5}$$

where $G_{\xi,\beta(u)}$ is a generalized Pareto distribution with distribution function as follows

$$G_{\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \dfrac{\xi}{\beta} x\right)^{-\frac{1}{\xi}}, & \xi \ne 0 \\ 1 - \exp\left(-\dfrac{x}{\beta}\right), & \xi = 0 \end{cases}. \tag{6}$$

Parameters of the distribution are a scale parameter β and shape parameter ξ. GPD is generalized in sense that it subsumes some other distributions depending on the value of ξ. When $\xi>0$ *then* $G_{\xi,\beta}$ is a reparametrized version of the ordinary Pareto distribution; for $\xi=0$ it corresponds to the exponential distribution and for $\xi<0$ it is known as a Pareto type II distribution. Most commonly used distribution for modeling extreme losses on financial markets is ordinary Pareto distribution ($\xi>0$) since $G_{\xi,\beta}$ is heavily tailed as well as daily returns.

As an aftereffect of the theorem, F_u is approximated with $G_{\xi,\beta}$ in relation (4) for high enough threshold u. The part $F(u)$ is estimated with historical simulation and the estimator is $(n - N_u)/n$ where n is sample size and N_u is the number of data that exceeds threshold u. The whole tail of F is estimated with maximum likelihood method because data in tail is sparse and would not provide good enough estimation. With estimated F, VaR and ES are easily estimated (details in McNeil (1999)).

The most important part and the biggest problem of the described procedure is selecting appropriate threshold u_0 because the estimation part considers data which exceed u_0. So selection of threshold needs to reconcile two opposite demands on u_0. First is to choose as high as possible u_0 due to the approximation derived from the Theorem and the second is to pick relatively low value so enough data is left for the estimation part of procedure. Some authors have suggested using a criterion like Coronel-Brizio criterion and Hernandez-Montoya procedure in Živković, Peričić (2010) and Živković, Aktan (2009). We suggest using the following theoretical results which is for the first time applied on the CROBEX data.

3.2. Mean Excess Function

When analyzing the definition of expected shortfall (2) it can be rewritten as

$$ES_q = VaR_q + E\,[X - VaR_q \mid X > VaR_q] \quad . \tag{7}$$

The second term on the right is the mean of the excess distribution over the threshold VaR_q. For the mean excess function defined as

$$e(u) = E\,[X - u \mid X > u] \quad . \tag{8}$$

It can be shown that if $X\text{-}u_0$ follows a GPD with parameter $\xi<1$ and $\beta(u_0)$ then $e(u)$ has a special form

$$e(u) = \frac{\beta(u_0) + \xi \cdot (u - u_0)}{1-\xi} = a + b \cdot u \,,\ u > u_0 \,. \tag{9}$$

The function e is a linear function of u so this theoretical result can be used to infer on selecting the appropriate threshold u_0. An upward sloping plot ($\xi>0$) indicates heavy-tailed behavior, a line with zero slope shows an exponential tail and negative ξ is a sign of thin-tailed behavior. Empirical mean excess function is defined as

$$e_n(u) = \frac{1}{n_u} \sum_{i=1}^{n_u} (x_{(i)} - u) \tag{10}$$

where $x_{(i)}$ $(i=1,\ldots,n_u)$ are the values of x_i such that $x_i>u$ for every $u>0$. The mean excess plot is a plot of $e_n(u)$ against u. From the above theoretical discussion $e_n(u)$ should be linear in u for $u>u_0$ and u_0 is the appropriate threshold.

3.3. Empirical Results

In order to select the appropriate threshold u_0, the mean excess plot is presented in figure (?). Two types of behavior can be seen, the first is horizontal and the second is with positive slope which indicates heavy tail. The interval where change takes part is [1.2, 1.3]. The threshold is chosen as with the lowest log-likelihood when data is fitted to generalized Pareto distribution. The result of procedure is u_0=1.24 with LL=-13.44 and the vertical line is added to mean excess plot so it can be seen that it separates the data satisfactorily. The chosen threshold leaves enough data (n=38, 7.6% of data) for the distribution estimation step.

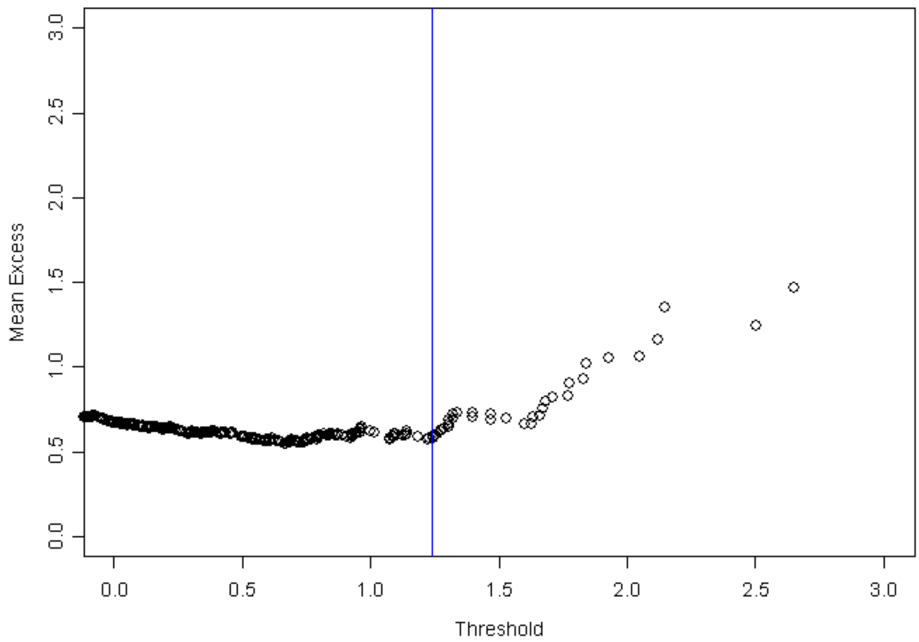

Figure 3: Sample Mean Excess Plot

The results of the estimation the function:

$$\hat{F}(x) = 1 - \left(1 + \frac{\hat{\xi}}{\hat{\beta}} x\right)^{-\frac{1}{\hat{\xi}}}, x>1.24 \tag{11}$$

are presented in Table 1.

Table 1: Estimated parameters of the GPD distribution

Parameter	Estimates	Standard Error	t-ratio
ξ	0.4331	0.2612	1.6583
β	0.3398	0.1017	3.3407

In Figure 3 tail data is compared with estimated distribution and it seems that data fit the line satisfactorily.

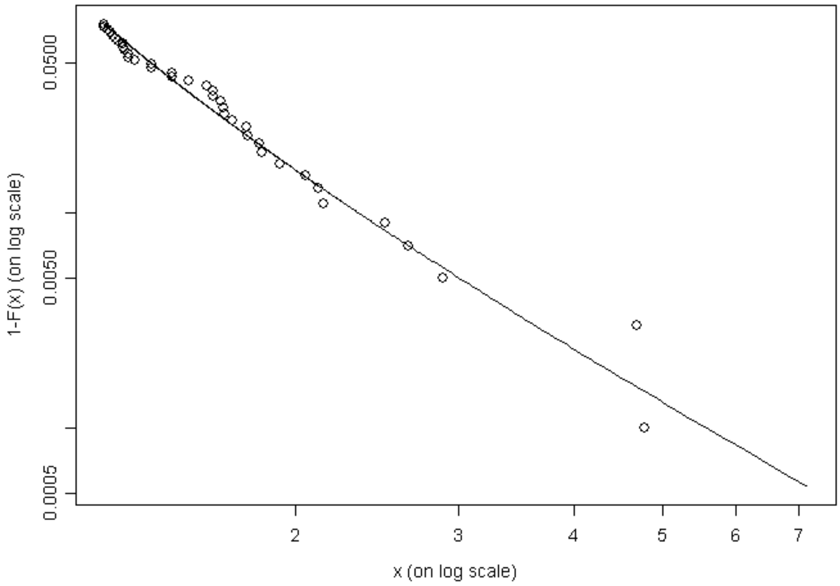

Figure 4. *Tail of Underlying Distribution*

Using estimated distribution function of negative returns, VaR and ES are calculated and presented in Table 2 when fitted to GPD and in Table 3 when fitted to normal distribution.

Table 2: Estimated VaR and ES using GPD

Quantile	**VaR**	**ES**
0.95	1.395	2.113
0.99	2.342	3.784

Table 3: Estimated VaR and ES with normality assumption

Quantile	**VaR**	**ES**
0.95	1.586	1.982
0.99	2.231	2.552

Daily returns from the first half of 2012 are used to validate estimated results and risk measures are presented in Table 4. During years 2010 and 2011 Eurozone crisis enlarged and investors all around Europe became worried. This also reflected on Zagreb Stock exchange already shaken with the extreme drop of stock values and large loss of confidence in stock market in 2009. One of the reasons for overestimating the sample value so much is a slowdown on stock exchange during 2012 when compared to 2010 and 2011.

Table 4. Sample VaR and ES for half year daily returns (2012M1-2012M6) using GPD

Quantile	VaR	ES
0.95	1.238	1.605
0.99	2.262	2.433

4. CONCLUSION

Two measures that we find most useful and reliable for describing the tail of the distribution are Value-at-Risk (VaR) and expected shortfall (ES) according to GPD distribution. Value-at-risk is a high quantile of a certain distribution (mostly at 95% or 99%) and expected shortfall is the expected loss size when the mentioned VaR is exceeded. Parameters of GPD distribution are estimated using maximum likelihood method (MLE). The contribution of this paper was to determine threshold parameter which is large enough so that GPD approximation valid but low enough so that a sufficient number of observations are available for a precise fit within mean excess function of CROBEX negative returns. Eurozone crisis enlarged in last two years of observation and investors all around Europe became worried. This also reflected on Zagreb Stock exchange already shaken with the extreme drop of stock values and large loss of confidence in stock market in 2009.

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