Total domination numbers of cartesian products

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Abstract. *Let $G\Box H$ denote the cartesian product of graphs G and H. Here we determine the total domination numbers of $P_5\Box P_n$ and $P_6\Box P_n$.*

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1. Introduction

For any graph G by $V(G)$ and $E(G)$ we denote the vertex-set and the edge-set of G, respectively. For graph G subset D of the vertex-set of G is called a dominating set if every vertex $x \in V \setminus D$ is adjacent to at least one vertex of D. The domination number $\gamma(G)$ is the cardinality of the smallest dominating set.

Set D is a total dominating set if every vertex $x \in V$ is adjacent to at least one vertex of D. The total domination number $\gamma_t(G)$ is the cardinality of the smallest total dominating set.

The cartesian product of graphs G and H is a graph with $V(G\Box H) = V(G)\Box V(H)$ and $((g_1,h_1),(g_2,h_2)) \in E(G\Box H)$, if and only if either $g_1 = g_2$ and $(h_1,h_2) \in E(H)$, or $(g_1,g_2) \in E(G)$ and $h_1 = h_2$.

The study of domination numbers of products of graphs was initiated by Vizing [19]. He conjectured that the domination number of the cartesian product of two graphs is always greater than or equal to the product of the domination numbers of the two factors; a conjecture which is still unproven.

Domination numbers of cartesian products were intensively investigated (see e. g. [1], [2], [3], [6], [7], [11]).

In [3] a link is shown between the existence of tilings in Manhattan metric with $\{1\}$-bowls and minimum total dominating sets of cartesian products of paths and cycles. It is also proved that $\gamma_t(P_2\Box P_n) = 2\lfloor (n+2)/3 \rfloor$ and $\gamma_t(P_3\Box P_n) = n$, and $\gamma_t(P_4\Box P_n)$ is given without proof. Also, there is a bound for $\gamma_t(P_k\Box P_n)$ when $k,n \geq 16$.

The next observation will be used in the sequel.

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Observation 1. *Let P_n denote the path with n vertices. Then*

$$\gamma_t(P_n) = \begin{cases} 2\lfloor \frac{n}{4} \rfloor + 1, & n \equiv 1(\text{mod}4) \\ 2\lceil \frac{n}{4} \rceil, & \text{otherwise.} \end{cases}$$

Obviously, $P_1 \Box P_n = P_n$.

Observation 2. *Let $1, ..., k$ and $1, ..., n$ be the vertices of P_k and P_n, respectively. Then the vertices of $P_k \Box P_n$ are denoted by (i, j), where $i = 1, ..., k$ and $j = 1, ..., n$.*

Definition 1. *For a fixed m, $1 \leq m \leq n$, the set $(P_k)_m := P_k \Box m$ is called a column of $P_k \Box P_n$; the set $_r(P_n) := r \Box P_n$ is called a row of $P_k \Box P_n$. Any set $B = \{(P_k)_m, (P_k)_{m+1}, ..., (P_k)_{m+l}, \; |l \geq 0, \; m \geq 1, \; m + l \leq n\}$, of consecutive columns is called a block of size $k \times (l+1)$ of $P_k \Box P_n$. If another block B' ends with the column $(P_k)_{m-1}$ or begins with the column $(P_k)_{m+l+1}$, then we say that B' is adjacent to B. Block B is called internal, if it is adjacent to two other blocks. It is called external, if it is adjacent only to one block.*

2. Total domination numbers of $P_k \Box P_n$

In the sequel we give the values of $\gamma_t(P_k \Box P_n)$ for $k \in \{5, 6\}$. Here is also the proof of $\gamma_t(P_4 \Box P_n)$.

Proposition 1. *Let $n \geq 2$. Then*

$$\gamma_t(P_4 \Box P_n) = \begin{cases} 6\lfloor \frac{n}{5} \rfloor + 2, & n \equiv 0, 1(\text{mod } 5) \\ 6\lfloor \frac{n}{5} \rfloor + 4, & n \equiv 2(\text{mod } 5) \\ 6\lceil \frac{n}{5} \rceil, & n \equiv 3, 4(\text{mod } 5) \end{cases}$$

Proof. We consider the set

$$S = \Big\{ (2, 1 + 5k), (3, 1 + 5k), (1, 3 + 5k), (1, 4 + 5k), (4, 3 + 5k), (4, 4 + 5k) \\ \mid \; k = 0, 1, ..., \lfloor \frac{n}{5} \rfloor - 1 \Big\}.$$

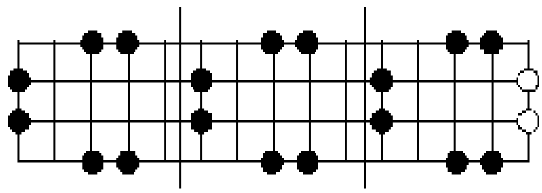

Figure 1.

a) For $n \equiv 1(\text{mod } 5)$ we consider $S_1 = S \cup \{(2, n), (3, n)\}$. This set total dominates all vertices on $P_4 \Box P_n$ and it is obviously minimal, because each vertex v is totally dominated by exactly one vertex $u \neq v$. Also, it holds $|S_1| = |S| + 2 = 6\lfloor \frac{n}{5} \rfloor + 2$.

b) For $n \equiv 2 (\text{mod } 5)$ we consider $S_2 = S \cup \{(2,n),(3,n),(2,n-1),(3,n-1)\}$. This set is a total dominating set and $|S_2| = 6\lfloor \frac{n}{5} \rfloor + 4$.
Proof of minimality: Let $n \equiv 2 (\text{mod } 5)$ and we have all vertices from S on $P_4 \Box P_n$. Then each vertex on $(P_4)_1$,...,$(P_4)_{n-3}$,$(1, n-2)$ and $(4, n-2)$ is totally dominated by exactly one vertex $u \neq v$. To totally dominate $(2, n-2)$, $(3, n-2)$, $(P_4)_{n-1}$ and $(P_4)_n$ we need at least 4 vertices. ($\gamma_t(P_2 \Box P_4) = 4$)

c) Let $n \equiv 3 (\text{mod } 5)$. Then we consider

$$S_3 = S \cup \{(2,n-2),(3,n-2),(1,n-1),(2,n-1),(3,n-1),(4,n-1)\}.$$

This set is a total dominating set and $|S_3| = 6\lfloor \frac{n}{5} \rfloor + 6 = 6\lceil \frac{n}{5} \rceil$.
Proof of minimality: As in b), if we have vertices from S on $P_4 \Box P_n$, then $(2, n-3)$, $(3, n-3)$, $(P_4)_{n-2}$, $(P_4)_{n-1}$ and $(P_4)_n$ are not totally dominated (and all other vertices are totally dominated by exactly one vertex). There is only one case when with only four vertices we can totally dominate all vertices on $(P_4)_{n-2}$,$(P_4)_{n-1}$,$(P_4)_n$. This is if $(1, n-1)$,$(2, n-1)$,$(3, n-1)$ and $(4, n-1)$ are total dominating vertices. But then vertices $(2, n-3)$ and $(3, n-3)$ are not totally dominated. To totally dominate them we need at least 2 more vertices. It follows that for each total dominating set D it holds $|D| \geq 6\lfloor \frac{n}{5} \rfloor + 6$.

d) Let $n \equiv 4 (\text{mod } 5)$. Let

$$S_4 = S \cup \{(2,n-3),(3,n-3),(1,n-1),(1,n),(4,n-1),(4,n)\}.$$

This set is a minimal total dominating set because each vertex v is totally dominated by exactly one total dominating vertex $u \neq v$. And $|S_4| = 6\lceil \frac{n}{5} \rceil$.

e) Let $n \equiv 0 (\text{mod } 5)$. Then we consider $S_0 = S \cup \{(2,n),(3,n)\}$. This set is a total dominating set and $|S_0| = 6\frac{n}{5} + 2$.
Proof of minimality: If we have all vertices from S on $P_4 \Box P_n$, then $(2, n)$ and $(3, n)$ are not totally dominated. One vertex can dominate both of these vertices, but to totally dominate them we need at least one more vertex. $\Box$

Theorem 1. *Let $n \geq 5$. Then*

$$\gamma_t(P_5 \Box P_n) = \lfloor \frac{3n+4}{2} \rfloor, \quad n \neq 6$$
$$\gamma_t(P_5 \Box P_6) = 10.$$

Proof. We give a total dominating set S of $P_5 \Box P_n$ as follows: Let $n \geq 8$. If $n = 8q$, then we can partition (split) the set of columns of $P_5 \Box P_n$ into q 5-by-8 blocks B_i, $i = 1, \cdots, q$ and dominate each such block by a set isomorphic to set $P = \{(1,3),(1,4),(1,7),(2,1),(2,7),(3,1),(3,4),(3,5),(5,2),(5,3),(5,6),(5,7)\}$ (See *Figure 2*).

(On each odd block the situation is the same, and on each even block the situation is symmetrical over axis $(3, n)$.) Then (if the last block is even) in column $(P_5)_n$ vertices $(2, n)$ and $(3, n)$ are not totally dominated, and we need at least two more total dominating vertices.

(On $P_5\Box P_5$ let $D = \{(2,1),(2,2),(4,2),(5,2),(1,4),(2,4),(3,4),(4,4),(5,4)\}$ and on $P_5\Box P_6$ let $D = \{(1,2),(2,2),(4,1),(4,2),(4,3),(2,4),(2,5),(2,6),(4,5),(5,5)\}$. On $P_5\Box P_7$ a total dominating set is P.)

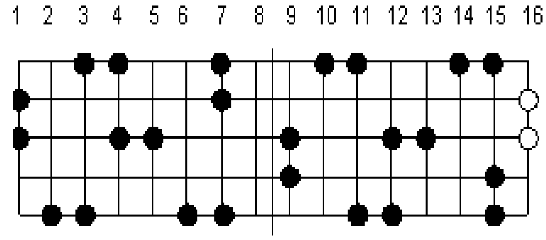

Figure 2.

If $n = 8q + l$, $1 \le l \le 7$, then in addition to the blocks B_i $i = 1, ..., q$ we totally dominate the last $5 \times l$ block $B^{(l)}_{q+1}$ by a set isomorphic to R_l $(1 \le l \le 7)$, *Figure 3.*

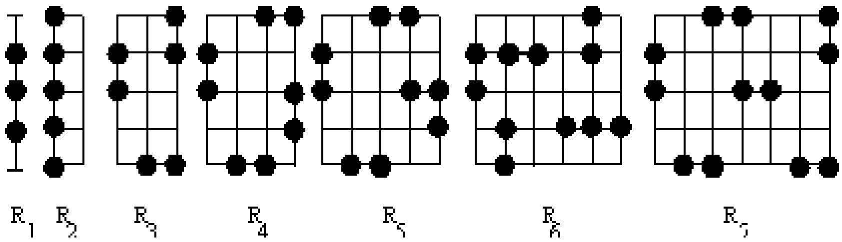

Figure 3.

To prove minimality of that set we partition graph $P_5\Box P_n$ into $5\Box 8$ blocks.

Lemma 1. *There is no total dominating set D of $P_5\Box P_n$ such that $|D\cap E| \le 11$, for each external $5\Box 8$ block E.*

Proof. Let $E = B_1$. (It contains $(P_5)_1$.) Then at most the last column on E can be totally dominated from the adjacent block. (It remains a $5\Box 7$ block.) To totally dominate a $5\Box 4$ block we need at least 8 vertices (*Proposition 1*). If in the fourth column of a $5\Box 5$ block there are five total dominating vertices, they can totally dominate all vertices from the fifth column. These five vertices totally dominate also all vertices on the thirth and fourth column. Then remaining tree vertices must totally dominate all vertices on the first two columns. But $\gamma_t(P_5\Box P_2) = 4$. It follows that we need at least one vertex more to totally dominate $5\Box 5$ block. Therefore $\gamma_t(P_5\Box P_5) = 9$. (If we have $5\Box 6$ block, by the same method it follows that we need at least 10 vertices.) From the previous it follows that if we have eight vertices on 5 columns, at least one vertex fifth column must be totally dominated from a next column.(On the second and fourth picture in the *Figure 4.* there are such cases.) Then at least one vertex on $(P_5)_5$, three vertices on $(P_5)_6$ and $(P_5)_7$ must be totally dominated from E. We need at least 4 vertices more for this ($\gamma_t(P_5\Box P_2) = 4$). It follows that to totally dominate the first seven columns we need at least 12 vertices.

If $n \equiv 0 (\mathrm{mod}\ 8)$, and $B_{(n/8)}$ is external, the result is the same, because at most the first column on $B_{(n/8)}$ can be totally dominated from the adjacent block. $\Box$

Lemma 2. *If $|D \cap E| = 12$, then at least two vertices on E must be totally dominated from the adjacent block.*

Proof. Let $E = B_1$. (From the previous proof it follows that a symmetrical case is for $B_{(n/8)}$.) From the proof of the previous Lemma it follows that if $|D \cap E| = 12$, the first 8 vertices can at most totally dominate a $5\Box 4$ bock and 4 vertices on $(P_5)_5$. Then the next 4 vertices must totally dominate remaining vertices on B_1. This is $5\Box 3$ block plus 1 vertex more. $\gamma_t(P_5 \Box P_3) = 5$, but to totally dominate one vertex more we need at least one total dominating vertex. Therefore at least two vertices on E must be totally dominated from the adjacent block. There are four cases which have only 12 vertices on B_1 (see *Figure 4*). On each of them there are no totally dominating vertices in $(P_5)_8$. □

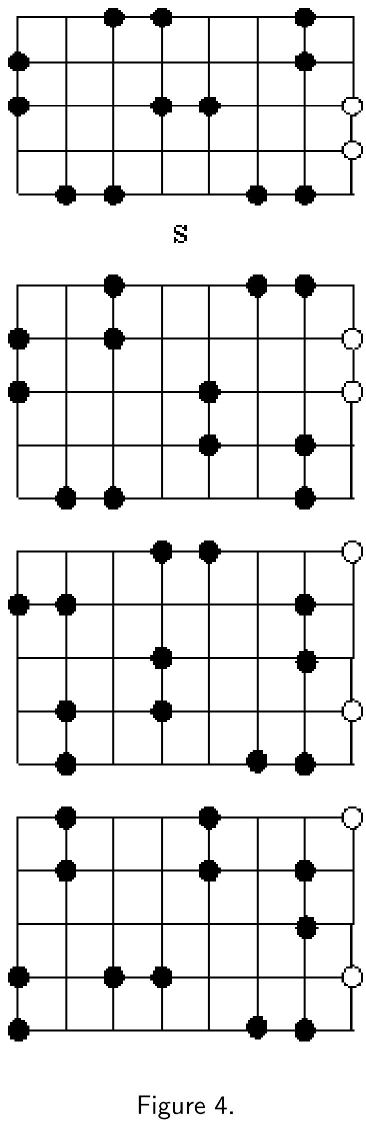

Figure 4.

Comment. If vertices of S are in E, then only the vertex $v = (2, 4)$ is dominated by 2 vertices $u, w \neq v$. For other three cases more vertices are dominated by 2 vertices.

Lemma 3. *There is no total dominating set D such that $|D \cap I| \leq 9$, for each internal block I.*

Proof. Let $I = \{(P_5)_j, ..., (P_5)_{j+7}\}$. At most the first and the last column on I can be totally dominated from adjacent blocks. It follows that at least all vertices on $5\Box 6$ block must be totally dominated by vertices from I. From the proof of *Lemma 1* it follows that we need for this at least 10 vertices.

Only for the case $\{(1, j+2), (2, j+2), (3, j+2), (4, j+2), (5, j+2), (1, j+5), (2, j+5), (3, j+5), (4, j+5), (5, j+5)\} \in D$ we have 10 vertices on I (for all other cases we have more vertices.) $\Box$

Lemma 4. *If $|D \cap B_i| = 10$ holds for some internal block B_i, then we have $|D \cap B_{i-1}| \geq 14$ and $|D \cap B_{i+1}| \geq 14$.*

Proof. From *Lemma 3* it follows that there exists only one case when there holds $|D \cap B_i| = 10$. For this case the situation on B_{i-1} and B_{i+1} is symmetrical, and we will consider only B_{i-1}. Let $B_i = \{(P_5)_j, ..., (P_5)_{j+7}\}$. Because $\{(1, j+2), (2, j+2), (3, j+2), (4, j+2), (5, j+2)\} \in D$, all vertices on $(P_5)_j$ must be totally dominated from B_{i-1}. Then $\{(1, j-1), (2, j-1), (3, j-1), (4, j-1), (5, j-1)\} \in D$. They also totally dominate all vertices on $(P_5)_{j-2}$. If B_{i-1} is internal, the vertices on the first column can be totally dominated from the adjacent block. Then at least all vertices on $(P_5)_{j-7}$,...,$(P_5)_{j-3}$ must also be totally dominated by vertices from B_{i-1}. This is a $5\Box 5$ block. From the proof of *Lemma 1* it follows that to totally dominate them we need at least 9 vertices. It follows that $|D \cap B_{i-1}| \geq 14$. $\Box$

Observation 3. *Let us have some total dominating set D with s blocks with 10 total dominating vertices. From* Lemma 3 *these blocks are internal. Then from* Lemma 4 *there are at least $s+1$ blocks with at least 14 vertices. Then such D on these $2s+1$ blocks has at least $24s+14$ total dominating vertices, and S only $24s+12$ vertices. It follows that for this case $|D| > |S|$. So, we will consider only such total dominating sets for which holds $|D \cap B_i| \geq 11$, for $i \in \{1, ..., \lfloor \frac{n}{8} \rfloor\}$.*

Lemma 5. *If $|D \cap B_i| = 11$ holds, then if B_{i-1} and B_{i+1} are internal $|D \cap B_{i-1}| \geq 12$ and $|D \cap B_{i+1}| \geq 12$ hold. If they are external then $|D \cap B_{i-1}| \geq 14$ and $|D \cap B_{i+1}| \geq 14$ hold. If B_{i-1} is internal, and B_{i+1} is external (or reverse) then holds $|D \cap B_{i-1}| \geq 12$ and $|D \cap B_{i+1}| \geq 14$ (or reverse).*

Proof. If $|D \cap B_i| = 11$ holds, from the fact that to totally dominate $5\Box 7$ block we need at least 12 vertices it follows that some vertices on $(P_5)_j$ and $(P_5)_{j+7}$ are totally dominated by vertices from adjacent blocks. From the proof of *Lemma 2* it follows that in at least one of columns $(P_5)_j$ and $(P_5)_{j+7}$ there are no total dominating vertices. Let $|D \cap (P_5)_{j+7}| = 0$. Then from *Lemma 2* also holds that at least two vertices on $(P_5)_{j+7}$ are not totally dominated by vertices from B_i.

Now we consider $(P_5)_j$. It holds $|D \cap (P_5)_j| < 3$, while if more vertices from $(P_5)_j$ are in D, all vertices on $(P_5)_j$ are totally dominated by vertices from B_i. If $|D \cap (P_5)_j| = 2$, remaining 9 total dominating vertices from B_i must dominate at least all vertices on $5\Box 5$ block and three vertices from $(P_5)_{j+1}$. But to totally dominate them we need at least 10 vertices. It follows that $|D \cap (P_5)_j| \leq 1$ holds. So, if $|D \cap B_i| = 11$ holds, and this total dominating set has one dominating vertex in the first column ($(P_5)_j$), noon total dominating vertex in the last column ($(P_5)_{j+7}$), and only two vertices on $(P_5)_{j+7}$ are totally dominated from B_{i+1}, then this set is optimal.

Such set comes if $(1, j+4), (1, j+5), (2, j+1), (2, j+2), (3, j+6), (4, j+3), (4, j+4), (4, j+6), (5, j), (5, j+1), (5, j+6) \in D$. Only one vertex on B_i is totally dominated by two vertices. Two vertices on $(P_5)_j$ must be totally dominated from B_{i-1} and only two vertices on $(P_5)_{j+7}$ must be totally dominated from B_{i+1}. Also one vertex from B_{i-1} is totally dominated.

We will consider situation on the block B_{i-1}, because here we must totally dominate one less vertex, but the proof for B_{i+1} is similar. To totally dominate two vertices on $(P_5)_j$ we need at least two vertices from $(P_5)_{j-1}$ (vertices $(1, j-1), (3, j-1)$). To totally dominate them we need at least one more vertex ($(2, j-1)$), but in this case we can totally dominate only 7 vertices on B_{i-1}. If we take one more vertex, these 4 vertices can totally dominate $(P_5)_{j-1}$, 4 vertices on$(P_5)_{j-2}$ and some vertices on $(P_5)_{j-3}$. Vertices on $(P_5)_{j-3}$,...,$(P_5)_{j-7}$ must be totally dominated by vertices from B_{i-1}. This is $5\Box 4$ block. From *Propostion 1*, to totally dominate them we need at least 8 vertices. It follows that $|D \cap B_{i-1}| \geq 12$ (if B_{i-1} is internal). If it is external we need at least two vertices more. □

Lemma 6. *If $|D \cap B_i| = 11$ and $|D \cap B_{i-1}| = 12$ hold, then $|D \cap B_{i-2}| \geq 12$ holds for B_{i-2} internal. If it is external then $|D \cap B_{i-2}| \geq 14$.*

Proof. From *Lemma 2*, and from the construction of *Lemma 5* and structure P, it follows that if $|D \cap B_i| = 11$ and $|D \cap B_{i-1}| = 12$ hold, then there is no one total dominating vertex on the first column of B_{i-1}, and at least two vertices on this column must be totally dominated by vertices from B_{i-2}. By the previous Lemma it follows the result. □

Observation 4. *By* Lemma 6 *and induction it follows that $|D \cap B_i| = 11$ and $|D \cap B_k| = 12$ $k \in \{i-1, i-2, ..., i-l\}$ hold, then $|D \cap B_{i-l-1}| \geq 12$ holds. Also it is obvious that the same result as in* Lemma 6 *holds for B_{i+1} and B_{i+2}.*

Lemma 7. *Let $|D \cap B_i| = 13$ holds. Then if B_{i-1} and B_{i+1} are internal $|D \cap B_{i-1}| \geq 11$ and $|D \cap B_{i+1}| \geq 12$ hold (or reverse). (If they are external then $|D \cap B_{i-1}| \geq 12$ and $|D \cap B_{i+1}| \geq 13$ hold.)*

Proof. Because for structure P from *Figure 2* only one vertex is totally dominated by two other vertices, and $|P| = 12$ we will take this structure on B_i plus one vertex more. We will consider the case $P \cup (5, j+7)$ (then the maximal number of vertices is totally dominated). For this case we have two total dominating vertices on the first column of B_i and all vertices on this column are totally dominated. It follows that on the last column on B_{i-1} we must not have any total dominating vertices, and it can be $|D \cap B_{i-1}| \geq 11$. On the last column on B_i there is only one total dominating vertex, and one vertex is not total dominated. By the same methods as in previous Lemmas it follows that then $|D \cap B_{i+1}| \geq 12$. If blocks are external we need at least one vertex more on each. □

$n \equiv 0 (\mathrm{mod}\ 8)$

We first assume that $n \geq 24$.

Let D be any dominating set. $|D \cap B_k| \geq 11$ holds for each block $B_k, 1 \leq k \leq \frac{n}{8}$, by *Lemma 4* and *Observation 3*. Let's assume that there are s 5×8 blocks which contain only 11 vertices of D. By *Lemma 1* these blocks are internal. Then, by *Lemma 6* and *Lemma 7*, there are at least s 5×8 blocks which contain at least 13 vertices of D. Let $B_{i_j}, 1 \leq j \leq 2s$ denote these blocks which either contain 11 or 13 vertices. Then $\mathcal{B} = \bigcup_{j=1}^{2s} B_{i_j}$ contains at least $24s$ vertices of D. By the above description of S, the set $\mathcal{B}$ contains at most $24s$ vertices of S. Hence $|D| \geq |S|$ holds for any dominating set D.

Let n=16. $|D \cap B_k| \geq 12$ holds for each block B_k k=1,2 by *Lemma 1*. If $|D \cap B_1| = 12$, at least two vertices of B_1 are dominated by vertices of B_2. Then $|D \cap B_2| \geq 14$ is obviously, and therefore $|D| \geq |S|$.

It is easy to see that $\gamma_t(P_5 \Box P_8) = 14$.

$n \equiv 1 (\text{mod } 8)$

If on each block there are 12 total dominating vertices, then from the previous case it follows that at least two vertices on $(P_5)_{n-1}$ and all vertices on $(P_5)_n$ are not totally dominated. To do that we need at least three vertices more.

$n \equiv 2 (\text{mod } 8)$

We assume that on each block $B_1,...,B_{\lfloor \frac{n}{8} \rfloor}$ we have 12 vertices. Then two vertices on $(P_5)_{n-2}$ and all vertices on $(P_5)_{n-1}$ and $(P_5)_n$ are not totally dominated. To do that we need at least 5 vertices.

For $n \equiv 3, ..., 7 (\text{mod } 8)$ the proof of minimality is the same. □

Theorem 2. *Let $n \geq 6$. Then*

$$\gamma_t(P_6 \Box P_n) = \lfloor \frac{12n+21}{7} \rfloor$$

Proof. We give a total dominating set S of $P_6 \Box P_n$ as follows: Let $n \geq 7$. If $n = 7q$ then we can partition (split) the set of columns of $P_6 \Box P_n$ into q 6-by-7 blocks B_i, $i = 1, \cdots, q$ and dominate each such block by a set isomorphic to set $P = \{(2,1), (3,1), (6,1), (6,2), (1,3), (1,4), (4,3), (4,4), (6,5), (6,6), (2,6), (3,6)\}$.

(On each odd block the situation is the same, and on each even block the situation is symmetrical (instead $(2,1)$ we have $(5,1)$, and on so on) (see *Figure 5*).

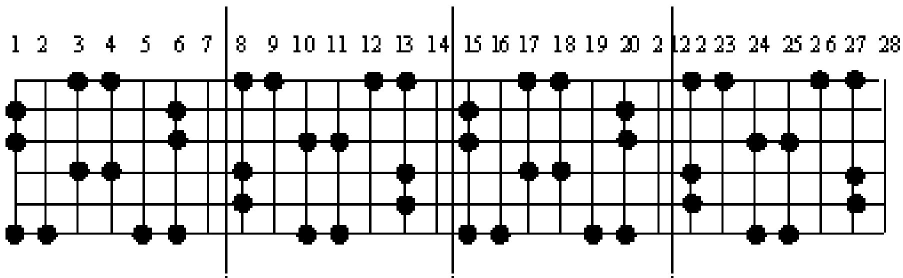

Figure 5.

The structure P on $6 \Box 7$ block is obviuosly minimal because each vertex y is totally dominated with exactly one vertex $x \in D$, $x \neq y$. The structure is optimal for the case $n \equiv 6(\text{mod } 7)$, because the all vertices on graph are totally dominated. For other n some vertices in last columns are not totally dominated when we have only vertices from P.

It follows that we must only see the situation for the cases $n \equiv k(\text{mod } 7), k \neq 6$, on the last $k+1$ columns. (We assume that all those vertices from S are on $P_6 \square P_n$, and that last $6\square 7$ block is even.)

a) $n \equiv 0(\text{mod } 7)$ In the column $(P_6)_n$ vertices $(2,n)$,$(3,n)$ and $(6,n)$ are not totally dominated. Because on $(P_6)_{n-1}$ the vertices $(1,n-1)$,$(4,n-1)$ and $(5,n-1)$ are in D, we need at least three more total dominating vertices (see *Figure 4*).

b) $n \equiv 1(\text{mod } 7)$ It follows from the fact that $\gamma_t(P_6) = 4$.

c) $n \equiv 2(\text{mod } 7)$ It is easy to see that if we have on each $6\square 7$ block structure P, and four aditional vertices in two last columns, at least $(1,n)$ and $(4,n)$ are not totally dominated. Then we need at least two more vertices (all together 6).

d) $n \equiv 3(\text{mod } 7)$ It is easy to see that if we have on each $6\square 7$ block structure P, and 6 aditional vertices in three last columns, at least $(1,n)$ and $(4,n)$ are not totally dominated. Then we need at least two more vertices (all together 8).

For the cases $n \equiv 4(\text{mod } 7)$ and $n \equiv 5(\text{mod } 7)$ same as in the previous four cases. □

References

[1] M. EL-ZAHAR, C. M. PAREEK, *Domination number of products of graphs*, Ars Combinatoria **31**(1991), 223-227.

[2] R. J. FAUDREE, R. H. SCHELP, *The domination number for the product of graphs*, Congr. Numer. **79**(1990), 29-33.

[3] S. GRAVIER, *Total domination number of grid graphs*, Discrete Applied Math. **121**(2002), 119-128.

[4] R. CHÉRIFI, S. GRAVIER, X. LAGRAULA, C. PAYAN, I. ZIGHEM, *Domination number of the cross product of paths*, Discrete Applied Math. **94**(1999), 101-139.

[5] S. GRAVIER, A. KHELLADI, *On the dominating number of cross product of graphs*, Discrete Math. **145**(1995), 273-277.

[6] M. S. JACOBSON, L. F. KINCH, *On the domination number of the products of graphs II : Trees*, J. Graph Theory **10**(1986), 97-106.

[7] M. S. JACOBSON, L. F. KINCH, *On the domination number of products of graphs I*, Ars Combinatoria **18**(1983), 33-44.

[8] C. F. DE JAENISCH, *Applications de l'analyse mathematique an jenudes echecs*, Petrograd 1862.

[9] P. K. JHA, S. KLAVŽAR, *Independance and matching in direct-product graphs*, Ars Combinatoria **50**(1998), 53-63.

[10] P. K. Jha, S. Klavžar, B. Zmazek, *Isomorphic components of Kronecker product of bipartite graphs*, Discuss. Math. Graph Theory **17**(1997), 301-309.

[11] S. Klavžar, N. Seifter, *Dominating Cartesian products of cycles*, Discrete Appl. Math. **59**(1995), 129-136.

[12] S. Klavžar, B. Zmazek, *On a Vizing-like conjecture for direct product graphs*, Discrete Math. **156**(1996), 143-246.

[13] A. Klobučar, *K-dominating sets of $P_{2k+2} \times P_n$ and $P_m \times P_n$*, Ars Combinatoria **58**(2001), 279-288.

[14] A. Klobučar, *Domination numbers of cardinal products*, Math. Slovaca **49**(1999), 387-402.

[15] A. Klobučar, *Domination numbers of cardinal products $P_6 \times P_n$*, Math. Communications **4**(1999), 241-250.

[16] A. Klobučar, N. Seifter, *K-Dominating Sets of Cardinal Products of Paths*, Ars Combinatoria **55**(2000), 33-41.

[17] M. El-Zahar, S. Gravier, A. Klobučar, *On the total domination numbers of cross products of graphs*, Journal of Combinatorial Theory series B, submitted.

[18] R. H. Lamprey, B. H. Barnes, *Products graphs and their applications,* Modelling and Simulation **5**(1974), 1119-1123.

[19] V. G. Vizing, *The cartesian product of graphs,* Vychisl. Sistemy **9**(1963), 30-43.