Classification of symmetric block design for $(71, 21, 6)$ with nonabelian group of order 21

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Abstract. *The existence of symmetric block designs for (71,21,6) was doubtful until the discovery of Z. Janko and T. van Trung of the so-called "non-human" design [3]. In this paper we have classified designs with the above parameters, admitting all possible actions of the nonabelian group of order 21 on them, which is indeed the full automorphism group of the Janko-van Trung design. We have proved that there is only one symmetric block design for (71,21,6), together with its dual, with the Frobenius group of order 21, namely the Janko-van Trung design.*

Key words: *block design, automorphism group, collineation*

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1. Introduction

The existence of a symmetric block design on 71 points (lines) was an open question for a long time. It was proved in [3] that such a design exists, having the Frobenious group of order 21 as its full automorphism group. The design constructed there was shown not to be self-dual. In our investigations we are going to prove, up to isomorphisms and duality, that there are only three possibilities for orbit structures describing the action of the nonabelian group G of order 21 on a $(71, 21, 6)$ design. Two of them are not self-dual, while one of them is self-dual.

Restricting the two mainly different orbit structures to the action of a collineation of order 7 of G, with the help of a collineation of order 3 of G, we get 14 orbit structure refinements, 8 of them being non-self-dual and 6 of them being self-dual. All these structures have been indexed and herewith we have proved the following

Theorem 1. *There are only two symmetric block designs for* $(71, 21, 6)$ *admitting an action of the nonabelian group of order* 21*. These are the "non-human" Janko - van Trung designs.*

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The method of investigation of symmetric block designs based on tactical decompositions (see for example [2], page 210) using finite group theory and computer search, developed by Z. Janko (see [4]), is very useful for constructing symmetric block designs. We shall use this method for the investigation of symmetric block designs with parameters (71,21,6), i.e. symmetric block designs consisting of 71 points and blocks, every block containing 21 points and any two different blocks intersecting at 6 points. It is well known that there are two groups of order 21, one is cyclic and the other nonabelian which is a Frobenius group. We shall denote this nonabelian group by

$$G = \langle \rho, \mu \mid \rho^7 = \mu^3 = 1, \rho^\mu \rho^2 \rangle \tag{1}$$

It is easy to see that G has one subgroup of order 7 and 7 subgroups of order 3 and that G is solvable. Without loss of generality we can represent the action of $\langle \rho \rangle$ on the 71 points of a design $\mathcal{D}$ as

$$\rho = (\infty) \;\; (I_0 \;\; I_1 \;\; I_2 \;\; I_3 \;\; I_4 \;\; I_5 \;\; I_6), \quad I = 1, 2, \ldots, 10,$$

where we have called ∞ the fixed point of $\mathcal{D}$, integers $1, 2, \ldots, 10$ represent point orbits (the so-called "big numbers") and the indices of these big numbers are integers $0, 1, \ldots, 6$. Namely, it is an easy task to prove that the automorphism ρ can fix only one point and block.

2. Orbit structures

It can be seen, using elementary facts from group and design theory, that the only possible G-orbit partition has the following lengths: 1, 7, 7, 7, 7, 7, 21, 21. Hence, using the famous computer programs by V. Ćepulić, up to isomorphisms and duality we got the following two possibilities for G-orbit structures of $\mathcal{D}$, which we shall call case A (non-self-dual) and case B (self-dual).

A	1	7	7	7	7	21	21
1	0	7	7	7	0	0	0
7	0	4	1	1	3	9	3
7	0	1	4	1	0	9	6
7	0	3	3	0	3	3	9
7	0	3	0	3	0	6	9
21	1	2	2	2	2	6	6
21	0	1	2	3	3	6	6

B	1	7	7	7	7	21	21
1	0	0	0	0	0	21	0
7	0	4	4	3	1	6	3
7	0	4	1	0	4	6	6
7	0	1	4	0	1	6	9
7	0	3	0	3	0	6	9
21	1	2	2	2	2	6	6
21	0	1	2	3	3	6	6

Restricting the above two G-orbit structures to the normal subgroup of order 7 of G, with the help of a collineation of order 3 of G, we have the following orbit structure for the subgroup $\langle \rho \rangle$ in case A:

$$
\begin{array}{c|c|cccc|ccc|ccc}
 & 1 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
\hline
1 & 0 & 7 & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 4 & 1 & 1 & 3 & 3 & 3 & 3 & 1 & 1 & 1 \\
7 & 0 & 1 & 4 & 1 & 0 & 3 & 3 & 3 & 2 & 2 & 2 \\
7 & 0 & 3 & 3 & 0 & 3 & 1 & 1 & 1 & 3 & 3 & 3 \\
7 & 0 & 3 & 0 & 3 & 0 & 2 & 2 & 2 & 3 & 3 & 3 \\
\hline
7 & 1 & 2 & 2 & 2 & 2 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\
7 & 1 & 2 & 2 & 2 & 2 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
7 & 1 & 2 & 2 & 2 & 2 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\
\hline
7 & 0 & 1 & 2 & 3 & 3 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\
7 & 0 & 1 & 2 & 3 & 3 & j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \\
7 & 0 & 1 & 2 & 3 & 3 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6
\end{array}
$$

Here the unknowns f_n, g_n, h_n, i_n, j_n, k_n, $n = 1, 2, \dots, 6$, are appearances of the "big numbers" in the corresponding blocks which can be calculated using the so-called "Hamming relations" and "Game product relations" (see for example [2]). We have obtained up to isomorphism and duality, exactly 8 orbit structures in case A and in a similar way 6 orbit structures in case B.

We list here only two big number matrices (representing the orbit structures in a slightly different, but natural way) of our 8+6=14 structures.

STRUCTURE A_6

$$
\begin{array}{ccccccccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 2 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 9 & 10 \\
1 & 2 & 2 & 2 & 2 & 3 & 5 & 5 & 5 & 6 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10 \\
1 & 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 5 & 6 & 7 & 8 & 8 & 8 & 9 & 9 & 9 & 10 & 10 & 10 \\
1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 & 10 & 10 & 10 \\
\infty & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 & 6 & 7 & 7 & 7 & 9 & 9 & 10 & 10 & 10 & 10 \\
\infty & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 6 & 7 & 8 & 8 & 9 & 9 & 9 & 9 \\
\infty & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 6 & 7 & 7 & 8 & 8 & 8 & 8 & 10 & 10 \\
1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 6 & 6 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & 10 & 10 \\
1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 8 & 9 & 9 & 10 & 10 & 10 \\
1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 7 & 7 & 7 & 7 & 8 & 8 & 9 & 9 & 9 & 10
\end{array}
$$

STRUCTURE B_1

$$
\begin{array}{ccccccccccccccccccccc}
5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 9 & 10 \\
1 & 1 & 1 & 1 & 2 & 4 & 4 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10 \\
1 & 2 & 2 & 2 & 2 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 & 10 & 10 & 10 \\
1 & 1 & 1 & 3 & 3 & 3 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 & 10 & 10 & 10 \\
\infty & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 6 & 6 & 7 & 7 & 7 & 7 & 8 & 9 & 9 & 10 & 10 & 10 \\
\infty & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 6 & 6 & 8 & 8 & 9 & 9 & 9 & 10 \\
\infty & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 6 & 7 & 7 & 8 & 8 & 8 & 8 & 10 & 10 \\
1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 & 8 & 9 & 9 \\
1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 6 & 6 & 7 & 8 & 8 & 10 & 10 & 10 & 10 \\
1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 7 & 7 & 9 & 9 & 9 & 9 & 10 & 10
\end{array}
$$

The action of a collineation of order 3 of G on the "big numbers" is given by

$$\mu = (\infty)(1)(2)(3)(4)(5\,7\,6)(8\,9\,10) \quad \text{or} \tag{2}$$
$$\mu = (\infty)(1)(2)(3)(4)(5\,7\,6)(8\,10\,9)\,, \tag{3}$$

whilst on the indices it is given by

$$\mu \,:\, x \mapsto 2x \pmod 7 \quad \text{or} \tag{4}$$
$$\mu \,:\, x \mapsto 4x \pmod 7\,. \tag{5}$$

Note that the action of μ as described above can be assumed to act on the whole point set in four different ways, namely one can take any of the actions (2) or (3) and combine it with one of the actions (4) or (5). For practical reasons we have differed only which of the actions on the index set is assumed, and therefore call the action given in (4) as case I and the one in (5) as case II.

3. Indexing of the orbit structures

Now we index our 8+6=14 refined orbit structures in cases I and II. All $14 \cdot 2 = 28$ cases have been indexed successfully via computer and we give here the results of this indexing for all these cases. We shall list here explicitly only cases $A_6(I)$ and $B_1(I)$.

For the fixed block of $\mathcal{D}$ in case $A_6(I)$ we can set

$$l_0 = \; 1_0\ 1_1\ 1_2\ 1_3\ 1_4\ 1_5\ 1_6\ 2_0\ 2_1\ 2_2\ 2_3\ 2_4\ 2_5\ 2_6\ 3_0\ 3_1\ 3_2\ 3_3\ 3_4\ 3_5\ 3_6$$

For the next μ-invariant block we can of course set

$$l_1 = \; 1_a\ 1_b\ 1_c\ 1_d\ 2_e\ 3_f\ 4_g\ 4_h\ 4_i\ 5_j\ 5_k\ 5_l\ 6_m\ 6_n\ 6_p\ 7_q\ 7_r\ 7_s\ 8_t\ 9_u\ 10_v$$

To avoid the extremely large number of combinations of indices, we shall make use of the following permutations on 71 points which keep the action of the generators of the assumed automorphism group G invariant:

$$\tau \,:\, N_i \mapsto N_{-i} \pmod 7$$
$$\mu \,:\, N_i \mapsto N_{2i} \pmod 7$$

It is not hard to see that τ inverts ρ and μ sends ρ to ρ^2. Also, τ centralizes μ.

Testing the "Hamming relations" for the four block representatives for the block orbits of length 7, we have achieved the following numbers of solutions for them:

	case A	case B
for $(l_1 + l_2)$	936 sol.	864 sol.
for $(l_1 + l_2 + l_3)$	3915 sol.	14346 sol.
for $(l_1 + l_2 + l_3 + l_4)$	12222 sol.	49305 sol.

One of the most difficult steps in solving the problem was the construction of the next block representative (for the long orbit of length 21) which is compatible with all blocks constructed so far. We spent a great deal of computer time, but at the end the result was affirmative. Luckily, from all 28 cases we saw that some of the structures have the same appearance of big numbers in cases I and II. Hence we got in fact only 12 structures to complete for indexing: $A_3(I)$, $A_3(II)$, $A_4(I)$, $A_4(II)$, $A_5(I)$, $A_6(I)$, $B_1(I)$, $B_2(II)$, $B_3(I)$, $B_4(I)$, $B_5(I)$ and $B_6(II)$.

The non self-dual case $A_6(I)$ gave one solution, which is exactly the Janko-van Trung design:

$$
\begin{array}{ccccccccccccccccc}
1_0 & 1_1 & 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 2_0 & 2_1 & 2_2 & 2_3 & 2_4 & 2_5 & 2_6 & 3_0 & 3_1 \\
 & & & & & & & & & & & 3_2 & 3_3 & 3_4 & 3_5 & 3_6 \\
1_0 & 1_1 & 1_2 & 1_4 & 2_0 & 3_0 & 4_1 & 4_2 & 4_4 & 5_0 & 5_1 & 5_5 & 6_0 & 6_4 & 6_6 & 7_0 \\
 & & & & & & & & & & & 7_2 & 7_3 & 8_0 & 9_0 & 10_0 \\
1_0 & 2_0 & 2_3 & 2_5 & 2_6 & 3_0 & 5_2 & 5_5 & 5_6 & 6_1 & 6_6 & 6_3 & 7_4 & 7_3 & 7_5 & 8_4 \\
 & & & & & & & & & & & 8_6 & 9_2 & 9_3 & 10_1 & 10_5 \\
1_1 & 1_2 & 1_4 & 2_3 & 2_5 & 2_6 & 4_3 & 4_5 & 4_6 & 5_0 & 6_0 & 7_0 & 8_3 & 8_4 & 8_5 & 9_5 \\
 & & & & & & & & & & & 9_2 & 9_6 & 10_6 & 10_1 & 10_3 \\
1_1 & 1_2 & 1_4 & 3_1 & 3_2 & 3_4 & 5_3 & 5_5 & 6_5 & 6_6 & 7_6 & 7_3 & 8_2 & 8_5 & 8_6 & 9_1 \\
 & & & & & & & & & & & 9_6 & 9_3 & 10_4 & 10_3 & 10_5 \\
\infty & 1_4 & 1_5 & 2_0 & 2_5 & 3_3 & 3_6 & 4_3 & 4_4 & 5_0 & 6_4 & 6_5 & 7_1 & 7_3 & 7_6 & 9_2 \\
 & & & & & & & & & & & 9_4 & 10_1 & 10_2 & 10_4 & 10_5 \\
\infty & 1_1 & 1_3 & 2_0 & 2_3 & 3_6 & 3_5 & 4_6 & 4_1 & 5_1 & 5_3 & 6_2 & 6_6 & 6_5 & 7_0 & 8_4 \\
 & & & & & & & & & & & 8_1 & 9_2 & 9_4 & 9_1 & 9_3 \\
\infty & 1_2 & 1_6 & 2_0 & 2_6 & 3_5 & 3_3 & 4_5 & 4_2 & 5_4 & 5_5 & 5_3 & 6_0 & 7_2 & 7_6 & 8_4 \\
 & & & & & & & & & & & 8_1 & 8_2 & 8_6 & 10_1 & 10_2 \\
1_5 & 2_2 & 2_4 & 3_0 & 3_4 & 3_6 & 4_2 & 4_3 & 4_6 & 6_0 & 6_2 & 6_5 & 6_6 & 7_5 & 7_6 & 8_0 \\
 & & & & & & & & & & & 8_4 & 8_6 & 9_0 & 10_1 & 10_3 \\
1_3 & 2_4 & 2_1 & 3_0 & 3_1 & 3_5 & 4_4 & 4_6 & 4_5 & 5_0 & 5_4 & 5_3 & 5_5 & 6_3 & 6_5 & 8_0 \\
 & & & & & & & & & & & 9_2 & 9_6 & 10_0 & 10_1 & 10_5 \\
1_6 & 2_1 & 2_2 & 3_0 & 3_2 & 3_3 & 4_1 & 4_5 & 4_3 & 5_6 & 5_3 & 7_0 & 7_1 & 7_6 & 7_3 & 8_4 \\
 & & & & & & & & & & & 8_5 & 9_0 & 9_2 & 9_3 & 10_0
\end{array}
$$

Each full block orbit can be obtained by a simple increment modulo 7 for each index in the block representative. The other cases gave no further designs, but we got solutions till the 48th block for each of them. Thus we have proved our main result.

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