Preservation of efficiency and inefficiency classification in data envelopment analysis*

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Abstract. *Sufficient conditions for simultaneous efficiency preservation of all efficient Decision Making Units (DMUs) and for inefficiency preservation of all inefficient DMUs in the Additive model of Data Envelopment Analysis (DEA) under the simultaneous non-negative perturbations of all data of all DMUs are obtained. An illustrative example is provided.*

Key words: *data envelopment analysis, efficiency, additive model, preservation of efficiency and inefficiency classification, sensitivity analysis, linear programming*

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1. Introduction

Sensitivity analysis in Data Envelopment Analysis (abbreviation: DEA) for the Additive model (see Charnes *et al.* [1]) was studied by Charnes and Neralić [2] for the case of the simultaneous change of all inputs or/and of all outputs of an arbitrary efficient Decision Making Unit (abbreviation: DMU) preserving its efficiency. Sufficient conditions for preserving efficiency of DMU under these changes were obtained. Sensitivity in DEA for arbitrary perturbations (and for non-negative perturbations) of all data in the Additive model was studied by Neralić [4]. For some recent developments of sensitivity and stability analysis in DEA see Cooper *et al.* [3].

The aim of this paper is firstly to study the region of efficiency around an efficient DMU_o according to the Additive model in DEA, which is a projection of an inefficient DMU_g to the efficiency frontier, under the simultaneous non-negative perturbations of all data of all DMUs preserving efficiency of DMU_o. Based on sufficient conditions for preserving efficiency of DMU_o under these the changes

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region around DMU_o and the corresponding region around DMU_g are obtained. Considering the relationship between these regions, conditions for simultaneous preservation of efficiency of DMU_o and inefficiency of DMU_g are given. Secondly, using the result on simultaneous preservation of efficiency of DMU_o and inefficiency of DMU_g, the case of region of joint efficiency around every efficient DMU with the corresponding region around each inefficient DMU is studied under non-negative perturbations of all data of all DMUs. Sufficient conditions for preserving efficiency of all efficient DMUs and inefficiency of all inefficient DMUs simultaneously are obtained. An illustrative example is provided.

The paper is organized as follows. Some preliminaries are given in *Section 2*. Relationship between the region of efficiency around DMU_o and the corresponding region around DMU_g is studied in *Section 3*. Conditions for simultaneous preservation of efficiency of DMU_o and inefficiency of DMU_g are obtained. Region of joint efficiency around all efficient DMUs and the corresponding region around inefficient DMUs under non-negative perturbations of all data of all DMUs are also studied in *Section 3*. Conditions for simultaneous preservation of efficiency of all efficient DMUs and inefficiency of all inefficient DMUs are given in *Theorem 4*. *Section 4* contains an illustrative example. The last *Section* contains some conclusions and suggestions for further research.

2. Preliminaries

Let us suppose that there are n Decision Making Units (DMUs), each with s outputs and m inputs. We shall employ the notation Y_j, X_j for the observed vectors of outputs and inputs of the DMU_j, respectively, $j = 1, 2, \ldots, n$. The observed values are supposed to be positive numbers. Let e be the column vector of ones. We use T as a superscript to denote the transpose.

In order to see if $\text{DMU}_{j_o} = \text{DMU}_o$, with $X_{j_o} = X_o$ and $Y_{j_o} = Y_o$, is efficient according to the Additive model, the following linear programming problem should be solved

$$\min 0\lambda_1 + \cdots + 0\lambda_o + \cdots + 0\lambda_n - e^T s^+ - e^T s^-$$

subject to

$$\begin{aligned} Y_1\lambda_1 + \cdots + Y_o\lambda_o + \cdots + Y_n\lambda_n - s^+ &= Y_o \\ -X_1\lambda_1 - \cdots - X_o\lambda_o - \cdots - X_n\lambda_n - s^- &= -X_o \\ \lambda_1 + \cdots + \lambda_o + \cdots + \lambda_n &= 1 \\ \lambda_1, \ldots, \lambda_n, s^+, s^- &\geq 0. \end{aligned} \tag{1}$$

Here $\min(-e^T s^+ - e^T s^-) = -e^T s^{+*} - e^T s^{-*} = 0$ if, and only if, DMU_o is efficient (for details see Charnes *et al.* [1]).

Let us suppose that the set of efficient DMUs according to the Additive model (1) is $E = \{1, 2, \ldots, n_e\}$ and that the set of inefficient DMUs is $N = \{n_e + 1, n_e + 2, \ldots, n\}$. Let us also suppose that the set of efficient DMUs corresponding to the optimal basic λ_j^* variables of the solution of (1), including $\lambda_o^* = \lambda_q^*$, is $EB = \{j_1, j_2, \ldots, j_q, \ldots, j_h\}$, $q \leq h \leq n_e$. Without loss of generality, in order to avoid cumbersome notations, let us suppose that $EB = \{1, 2, \dots, q, \dots, h\}$. We will also use the notation $SV = \{n+1, n+2, \dots, n+s+m\}$.

Let P_j, $j = 1, 2, \dots, n+s+m$ be the columns of the coefficient matrix and let P_0 be the right-hand side vector in the linear program (1). For the efficient DMU_o there is a basic optimal solution $(\lambda^*, s^{+*}, s^{-*})$ of (1) with $\lambda_o^* = \lambda_q^* = 1, \lambda_j^* = 0, j \neq j_o = q$, $s_r^{+*} = 0, r = 1, 2, \dots, s$, $s_i^{-*} = 0, i = 1, 2, \dots, m$ and an optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 \\ -X_B & 0 & -I_B^- \\ e^T & 0 & 0 \end{bmatrix}.$$

In the optimal basis matrix B the first part

$$\begin{bmatrix} Y_B \\ -X_B \\ e^T \end{bmatrix}$$

contains the columns of the coefficient matrix of the linear programming problem (1) which correspond to the optimal basic λ_j^* variables, the second and the third part

$$\begin{bmatrix} -I_B^+ \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -I_B^- \\ 0 \end{bmatrix}$$

contain columns which correspond to optimal basic variables s_r^{+*} and s_i^{-*}, respectively. Let the inverse of matrix B be

$$B^{-1} = [b_{ij}^{-1}], \quad i, j = 1, 2, \dots, s+m+1.$$

We will use the following notation:

$$\begin{aligned} \Gamma_j &= B^{-1} P_j, \ j = 0, 1, 2, \dots, n+s+m, \\ \omega^T &= c_B^T B^{-1}, \\ z_j &= c_B^T B^{-1} P_j \\ &= \omega^T P_j, \ j = 0, 1, 2, \dots, n+s+m. \end{aligned}$$

We are interested in particular non-negative variations of all data that preserve the efficiency of an arbitrary DMU_o. Let us consider a decrease of outputs of efficient DMUs

$$\hat{y}_{rj} = y_{rj} - \alpha_r > 0, \ \alpha_r \geq 0, \ r = 1, 2, \dots, s, \ j \in E \tag{2}$$

and an increase of outputs of inefficient DMUs

$$\hat{y}_{rj} = y_{rj} + \alpha_r, \ \alpha_r \geq 0, \ r = 1, 2, \dots, s, \ j \in N. \tag{3}$$

We will also consider an increase of inputs of efficient DMUs

$$\hat{x}_{ij} = x_{ij} + \beta_i, \ \beta_i \geq 0, \ i = 1, 2, \dots, m, j \in E, \tag{4}$$

and a decrease of inputs of inefficient DMUs

$$\hat{x}_{ij} = x_{ij} - \beta_i > 0,\ \beta_i \geq 0,\ i = 1, 2, \ldots, m,\ j \in N. \tag{5}$$

Following Neralić [4] with the same notation we have

Theorem 1. *Sufficient conditions for an efficient DMU_o to preserve its efficiency after the non-negative changes of all data as in (2), (3), (4) and (5), are*

$$-\tau d(\sum_{k=1}^{h} \Gamma_{kj} - 1) \geq z_j - c_j, \quad j \in E \text{ an index of nonbasic variables}, \tag{6}$$

$$-\tau d(\sum_{k=1}^{h} \Gamma_{kj} + 1) \geq z_j - c_j, \quad j \in N \text{ an index of nonbasic variables}, \tag{7}$$

and

$$-\tau d \sum_{k=1}^{h} \Gamma_{kj} \geq z_j - c_j, \quad j \in SV \text{ an index of nonbasic variables}, \tag{8}$$

where

$$\gamma_k = \sum_{t=1}^{s} b_{kt}^{-1} \alpha_t + \sum_{t=1}^{m} b_{k,s+t}^{-1} \beta_t, k = 1, 2, \ldots, s + m + 1$$

$$p = \sum_{k=1}^{h} \gamma_k, \quad \tau = \frac{1}{1-p}, \quad d = \sum_{t=1}^{s} \omega_t \alpha_t + \sum_{t=1}^{m} \omega_{s+t} \beta_t. \tag{9}$$

For the proof and details see Neralić [4], pp. 321-322.

3. Preservation of efficiency and inefficiency classification

3.1. Let us consider an inefficient DMU_g according to the additive model with outputs Y_g and inputs X_g. Let the projection of DMU_g onto efficiency frontier be one of efficient $\text{DMU}_{j_o}, j_o \in E$, denoted as DMU_o with outputs Y_o and inputs X_o, which is used to evaluate DMU_g. (The case when the projection of DMU_g is a convex combination of efficient DMUs will not be considered here.) In that case for optimal variables $s_r^{+*}, r = 1, 2, \ldots, s$ and $s_i^{-*}, i = 1, 2, \ldots, m$ from the solution $(\lambda^*, s^{+*}, s-*)$ of linear programming problem (1) with data of DMU_g holds

$$y_{ro} = y_{rg} + s_r^{+*}, r = 1, 2, \ldots, s \tag{10}$$

and

$$x_{io} = x_{ig} - s_i^{-*}, i = 1, 2, \ldots, m \tag{11}$$

where $y_{ro}, r = 1, 2, \ldots, s$ and $x_{io}, i = 1, 2, \ldots, m$ are outputs and inputs of DMU_o respectively. For details see, for example, Charnes et al. [1], pp. 100.

Let us consider non-negative changes (2)-(5) of all data of all DMUs. Let $S_{j_o} = S_o$ be the set of solutions (α, β) of the system of inequalities (6)-(8), together with conditions (2)-(5). Let

$$R_{j_o} = R_o = \{(\hat{X}_o, \hat{Y}_o) \mid \hat{X}_o = X_o + \beta, \hat{Y}_o = Y_o - \alpha, (\alpha, \beta) \in S_o\}$$

be the region of efficiency around DMU_o (see Neralić [4], Definition 1, pp. 327) and let

$$\bar{R}^g_{j_o} = \bar{R}^g_o = \{(\hat{X}_g, \hat{Y}_g) \mid \hat{X}_g = X_g - \beta, \hat{Y}_g = Y_g + \alpha, (\alpha, \beta) \in S_{j_o}\}$$

be the region around DMU_g corresponding to the set S_o.

Theorem 2. *If the point $M' = (\alpha, \beta) = ((1/2)s^{+*}, (1/2)s^{-*})$ belongs to the set S_o, then the point $M = (X_o + (1/2)s^{-*}, Y_o - (1/2)s^{+*})$ belongs to the set $R_o \cap \bar{R}^g_o$. Moreover, in that case there is a facet F_o of S_o, such that $((1/2)s^{+*}, (1/2)s^{-*}) \in F_o$ and the points $(X_o+\beta, Y_o-\alpha), (X_g-\beta, Y_g+\alpha), (\alpha, \beta) \in F_o$ are in the corresponding facet $\hat{F}_o$ of R_o and of $\bar{R}^g_o$.*

Proof. Because of decrease (2) of outputs of DMU_o and increase (3) of outputs of DMU_g, let us consider the situation when

$$Y_o - \alpha = Y_g + \alpha.$$

According to (10), that relation holds for

$$\alpha = (1/2)(Y_o - Y_g) = (1/2)s^{+*}.$$

Similarly, because of increase (4) of inputs of DMU_o and decrease (5) of inputs of DMU_g, let us consider the situation when

$$X_o + \beta = X_g - \beta.$$

According to (11), that relation holds for

$$\beta = (1/2)(X_g - X_o) = (1/2)s^{-*}.$$

So, for $(\alpha, \beta) = ((1/2)s^{+*}, (1/2)s^{-*}) \in S_o$ we have $(\hat{X}_o, \hat{Y}_o) = (\hat{X}_g, \hat{Y}_g)$, which means that point $M = (\hat{X}_o, \hat{Y}_o) = (X_o + (1/2)s^{-*}, Y_o - (1/2)s^{+*}) = (\hat{X}_g, \hat{Y}_g) = (X_g - (1/2)s^{-*}, Y_g + (1/2)s^{+*})$ belongs to the set $R_o \cap \bar{R}^g_o$.

The point $(\alpha, \beta) = ((1/2)s^{+*}, (1/2)s^{-*}) \in S_o$ belongs to the facet F_o, which is a part of the boundary of the set S_o. Namely, if in the corresponding point M we continue to decrease outputs and increase inputs of DMU_o, with increase of outputs and decrease of inputs of DMU_g (and with corresponding changes of data for the other DMUs), DMU_o will become inefficient and DMU_g will become efficient, which means that M is on the boundary of R_o and $\bar{R}^g_o$. A similar argument holds for the other $(\alpha, \beta) \in F_o$, with corresponding points $(X_o + \beta, Y_o - \alpha)$ and $(X_g - \beta, Y_g + \alpha)$ on the facet $\hat{F}_o$, which is part of the boundary of R_o and of $\bar{R}^g_o$. □

Remark 1. *At points $(X_o + \beta, Y_o - \alpha)$ for $(\alpha, \beta) \in F_o$ efficiency of DMU_o will be preserved and at points $(X_g - \beta, Y_g + \alpha)$ for $(\alpha, \beta) \in F_o$ the status of inefficient DMU_g will be changed to efficient.*

Remark 2. *Because of conditions (2), (5), which include preservation of positivity for outputs and inputs, it can happen that $(\alpha, \beta) = ((1/2)s^{+*}, (1/2)s^{-*})$ is in the closure of the set S_o. In that case, the corresponding point $M = (X_o + (1/2)s^{-*}, Y_o - (1/2)s^{+*})$ is in the closure of the set $R_o \cap \bar{R}^g_o$.*

Theorem 3. *Let us consider the set $S'_o = S_o \setminus F_o$, with F_o being the facet in* Theorem 2. *For $(\alpha, \beta) \in S'_o$, efficiency of DMU_o and inefficiency of DMU_g will be preserved simultaneously.*

Proof. Because the facet F_o is excluded from the set S_o, it means that points on the facet $\hat{F}_o$, which is a part of the boundary of R_o and $\bar{R}_o^g$, are also excluded. According to *Theorem 2* and *Remark 1*, these are the only points in which efficiency of DMU_o will be preserved and the status of inefficient DMU_g will be changed to efficient. So, after they are excluded, for $(\alpha, \beta) \in S'_o$, efficiency of DMU_o and inefficiency of DMU_g will be preserved simultaneously. □

3.2. Let us consider non-negative changes (2) - (5) of all data for which efficiency of $\text{DMU}_j, j \in E$ is preserved. Let $R_j, j \in E$ be the region of efficiency corresponding to the set $S_j \subset \mathbb{R}^{s+m}$ of solutions (α, β) of the system of inequalities (6)-(8), together with conditions (2)-(5). Let

$$S^* = \bigcap_{j \in E} S_j.$$

If $S^* \neq \emptyset$, let R_j^* be the region of joint efficiency (see Neralić [4], Definition 2, pp. 328) around $\text{DMU}_j, j \in E$ corresponding to S^*. For $(\alpha, \beta) \in S^*$ efficiency of all efficient $\text{DMU}_j, j \in E$ under the non-negative changes (2)-(5) of all data will be preserved.

We can also consider for inefficient $\text{DMU}_g, g \in N$ the region $\bar{R}_g^*$ around DMU_g corresponding to S^*. Let $\text{DMU}_{j_o}, j_o \in E$ be the projection of DMU_g onto the efficiency frontier and let R_o^* be the region around DMU_o corresponding to S^*. As in the case of R_o and $\bar{R}_o^g$ in subsection 3. 1., we can also eliminate the facet(s) F_o^* (or closure of F_o^*, if necessary) from the set S^* with the property that, in the corresponding facet(s) $\hat{F}_o^*$ which is (are) a part of the boundary of R_o^* and $\bar{R}_o^*$, efficiency of DMU_o (and of all other $\text{DMU}_j, j \in E$) is preserved and the status of inefficient DMU_g is changed to efficient. This can be done for every inefficient $\text{DMU}_g, g \in N$ with the corresponding $\text{DMU}_{j_o}, j_o \in E$. In that way, eliminating all the necessary facets of S^*, we can get the subset Q^* of S^* with the property that for $(\alpha, \beta) \in Q^*$ efficiency of all efficient $\text{DMU}_{j_o}, j_o \in E$ and inefficiency of all inefficient $\text{DMU}_g, g \in N$ will be preserved simultaneously. So, we have the following

Theorem 4. *For $(\alpha, \beta) \in Q^* \subset S^*$ after the non-negative changes (2) - (5) of all data, efficiency of all efficient $DMU_{j_o}, j_o \in E$ and inefficiency of all inefficient $DMU_g, g \in N$ will be preserved simultaneously.*

Remark 3. *In the case when int S^* exists, we can eliminate all the facets of the set S^* and take $Q^* = \text{int } S^* \subset S^*$.*

4. Illustrative example

4.1. We will consider the following example taken from Seiford and Thrall [5], with five DMUs, one output, one input and data in *Table l.*

Output/Input \ DMU_j	1	2	3	4	5
y_{1j}	1	4	6	7	3
x_{1j}	2	3	6	9	5

Table 1. *Data for the example*

In order to see if, say, $\mathrm{DMU}_{j_o} = \mathrm{DMU}_o = \mathrm{DMU}_2$ with $Y_o = Y_2 = 4$ and $X_o = X_2 = 3$ is efficient according to (1), the following linear programming problem should be solved:

$$\min 0\lambda_1 + 0\lambda_o + 0\lambda_3 + 0\lambda_4 + 0\lambda_5 - s_1^+ - s_1^-$$

subject to

$$\begin{array}{rcl} \lambda_1 + 4\lambda_o + 6\lambda_3 + 7\lambda_4 + 3\lambda_5 - s_1^+ & = & 4 \\ -2\lambda_1 - 3\lambda_o - 6\lambda_3 - 9\lambda_4 - 5\lambda_5 - s_1^- & = & -3 \\ \lambda_1 + \lambda_o + \lambda_3 + \lambda_4 + \lambda_5 & = & 1 \\ \lambda_1, \lambda_o, \lambda_3, \lambda_4, \lambda_5, s_1^+, s_1^-, s_2^- & \geq & 0. \end{array} \tag{12}$$

The optimal solution of linear programming problem (12) is $\lambda_o^* = \lambda_2^* = 1, \lambda_1^* = \lambda_3^* = \lambda_4^* = \lambda_5^* = 0, s_1^{+*} = s_1^{-*} = 0$ and $\min(-s_1^+ - s_1^-) = -s_1^{+*} - s_1^{-*} = 0$, which means that $\mathrm{DMU}_o = \mathrm{DMU}_2$ is efficient. Optimal basic variables are $\lambda_1^*, \lambda_2^* = \lambda_o^*$ and s_1^{+*}, the optimal basis matrix is

$$B = \begin{bmatrix} 1 & 4 & -1 \\ -2 & -3 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with inverse

$$B^{-1} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & -1 & -2 \\ -1 & -3 & -5 \end{bmatrix} \tag{13}$$

and the corresponding optimum tableau in *Table 2*.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_o
λ_1	1	0	−3	−6	−2	0	−1	0
λ_2	0	1	4	7	3	0	1	1
s_1^+	0	0	7	15	7	1	3	0
$z_j - c_j$	0	0	−7	−15	−7	0	−2	0

Table 2. *Optimum tableau for* DMU_2

In order to see if DMU_5 is efficient, linear programming problem (12) should be solved with right-hand side coefficients 3 and −5 instead of 4 and −3, respectively. The optimal solution is $\lambda_2^* = 1, \lambda_1^* = \lambda_3^* = \lambda_4^* = \lambda_5^* = 0, s_1^{+*} = 1, s_1^{-*} = 2$ and $\min(-s_1^+ - s_1^-) = -1 - 2 = -3$, which means that DMU_5 is inefficient. Because of $\hat{y}_{15} = y_{15} + s_1^{+*} = 3 + 1 = 4 = y_{21}, \hat{x}_{15} = x_{15} - s_1^{-*} = 5 - 2 = 3 = x_{21}$, the projection of DMU_5 onto the efficiency frontier is an efficient DMU_2. It is easy to see that DMU_1, DMU_3 and DMU_4 are also efficient. Hence $s = 1, m = 1, n = 5$, $E = \{1,2,3,4\}, N = \{5\}$, and for DMU_2 we have $EB = \{1,2\}, q = 2, h = 2$, with $SV = \{6,7,8\}$.

4.2. Let us consider the following decrease of output of DMU_1, DMU_2, DMU_3 and DMU_4:

$$\hat{y}_{11} = 1 - \alpha > 0,\ \hat{y}_{12} = 4 - \alpha > 0,\ \hat{y}_{13} = 6 - \alpha > 0,\ \hat{y}_{14} = 7 - \alpha > 0,\ \alpha \geq 0, \tag{14}$$

and the increase of output of DMU_5:

$$\hat{y}_{15} = 3 + \alpha, \ \ \alpha \geq 0. \tag{15}$$

Let us also consider the increase of input of DMU_1, DMU_2, DMU_3 and DMU_4:

$$\hat{x}_{11} = 2 + \beta, \ \hat{x}_{12} = 3 + \beta, \ \hat{x}_{13} = 6 + \beta, \ \hat{x}_{14} = 9 + \beta, \ \beta \geq 0, \tag{16}$$

and the decrease of input of DMU_5:

$$\hat{x}_{15} = 5 - \beta > 0, \ \beta \geq 0. \tag{17}$$

We are interested in conditions that preserve efficiency of $\mathrm{DMU}_o = \mathrm{DMU}_2$ under the variations in all data (14) - (17). In the same way as in Neralić [4] it is easy to see that the set $S_o = S_2$ of solutions of the corresponding system of inequalities is the "polyhedron" $OABC$ (without facet $\overline{AB}$) in the coordinate system $\alpha O \beta$ with $O(0,0), A(1,0), B(1,5/6)$ and $C(0,7/6)$ (see *Figure 1*). After the changes of all data (14) - (17) such that the point (α, β) belongs to the "polyhedron" $OABC$ (without facet $\overline{AB}$), DMU_2 will preserve its efficiency.

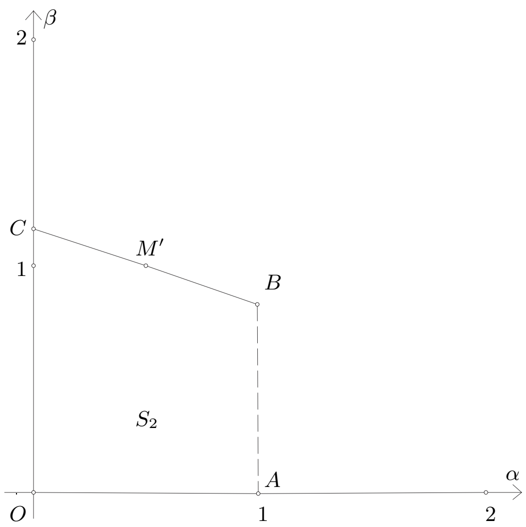

Figure 1. *Solution set* S_2

Using $S_o = S_2$ for $\mathrm{DMU}_o = \mathrm{DMU}_2$ it is easy to get the corresponding region of efficiency $R_o = R_2 = \{(\hat{X}_2, \hat{Y}_2) | \hat{X}_2 = 3 + \beta, \hat{Y}_2 = 4 - \alpha, (\alpha, \beta) \in S_2\}$ around $\mathrm{DMU}_o = \mathrm{DMU}_2$. It is the "polyhedron" $P_2A_2B_2C_2$ in the coordinate system xOy with $P_2(3,4), A_2(3,3), B_2(23/6,3), C_2(25/6,4)$ without facet $\overline{A_2B_2}$ (see *Figure 2*). For every other efficient DMU we can construct the region corresponding to the set S_2. For example, for DMU_1 this region is the "polyhedron" $P_1A_1B_1C_1$ in the coordinate system xOy with $P_1(2,1), A_1(2,0), B_1(17/6,0)$, $C_1(19/6,0)$ without facet $\overline{A_1B_1}$.

For inefficient $\mathrm{DMU}_g = \mathrm{DMU}_5$ with $Y_5 = 3, X_5 = 5$, the region around DMU_5 corresponding to the solution set $S_o = S_2$ consists of points which correspond to the points $(\alpha, \beta) \in S_2$. This region $\bar{R}_o^5 = \bar{R}_2^5 = \{(\hat{X}_5, \hat{Y}_5) | \hat{X}_5 = 5 - \beta, \hat{Y}_5 = 3 + \alpha, (\alpha, \beta) \in S_2\}$ is the "polyhedron" $P_5A_5B_5C_5$ in the coordinate system xOy with $P_5(5,3), A_5(5,4), B_5(25/6,4), C_5(23/6,3)$ without facet $\overline{A_5B_5}$ (see *Figure 2*).

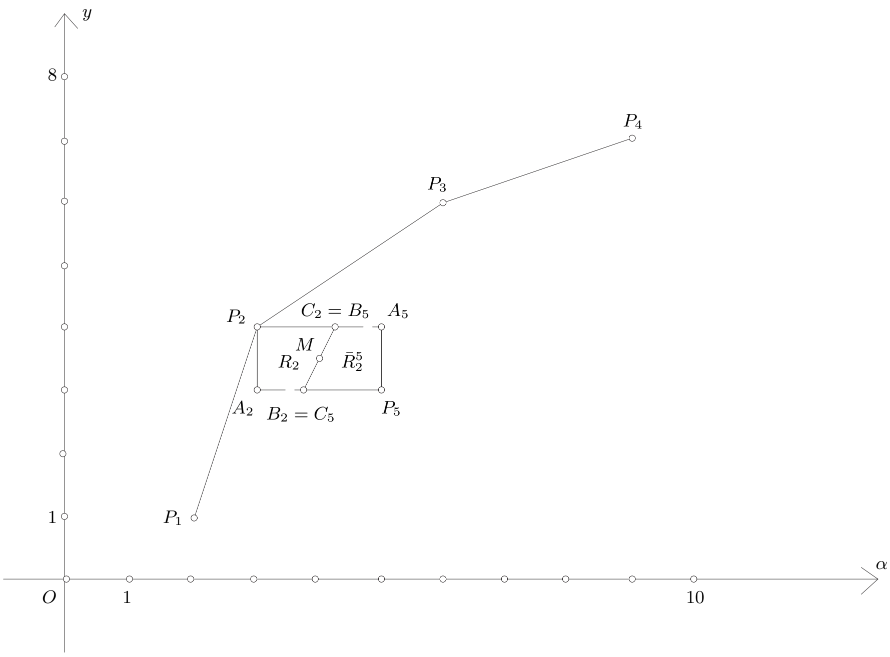

Figure 2. *Efficiency frontier with* R_2 *and* $\bar{R}_2^5$

In that case according to *Theorem 2*, because the point

$$M' = (\alpha, \beta) = ((1/2)s^{+*}, (1/2)s^{-*}) = ((1/2) \cdot 1, (1/2) \cdot 2) = ((1/2), 1)$$

belongs to the set S_2, the corresponding point

$$M = (X_o + (1/2)s^{-*}, Y_o - (1/2)s^{+*}) = (3 + 1, 4 - (1/2)) = (4, 3.5)$$

belongs to the set $R_o \cap \bar{R}_o^5 = \overline{B_2C_2}$. Also, the point $M' = ((1/2), 1)$ is on the facet $F_o = \overline{BC}$ of the set S_2, which means that points $(3+\beta, 4-\alpha), (\alpha, \beta) \in \overline{BC}$ are on the facet $\hat{F}_o = \overline{B_2C_2}$, including point $M = (4, 3.5)$, and points $(5-\beta, 3+\alpha), (\alpha, \beta) \in \overline{BC}$ are on the facet $\overline{B_5C_5} = \overline{C_2B_2}$. The facet $F_o = \overline{BC}$ is on the boundary of S_2, and the facet $\hat{F}_o = \overline{B_2C_2} = \overline{C_5B_5}$ is the corresponding boundary of R_o and $\bar{R}_o^5$. According to *Remark 1*, in the points $(3 + \beta, 4 - \alpha), (\alpha, \beta) \in \overline{BC}$ efficiency of DMU_2 will be preserved and in the points $(5 - \beta, 3 + \alpha), (\alpha, \beta) \in \overline{BC}$ the status of inefficient DMU_5 will be changed to efficient. If we consider the set $S_2' = S_2 \setminus \overline{BC}$, according to *Theorem 3* for $(\alpha, \beta) \in S_2'$ efficiency of DMU_2 and inefficiency of DMU_5 will be preserved simultaneously.

4.3. Sensitivity analysis for the other efficient DMU_1, DMU_3 and DMU_4 can be done in the same way as for DMU_2 and the sets of solutions S_1, S_3 and S_4 of the corresponding system of inequalities can be obtained. Also, we can get the corresponding regions of efficiency R_1, R_3, R_4 for DMU_1, DMU_3, DMU_4, respectively, and other regions corresponding to sets S_1, S_3 and S_4 as in the case of DMU_2. But, we can also get the set

$$S^* = S_1 \cap S_2 \cap S_3 \cap S_4.$$

It is easy to show that S^* is the "polyhedron" $ODFGH$ in the coordinate system $\alpha O \beta$ with $O(0,0), D(1,0), F(1,0.25), G(0.5,1), H(0,7/6)$ without facet $\overline{DF}$ (see *Figure 3*). For $(\alpha,\beta) \in S^*$ the efficiency of all four efficient

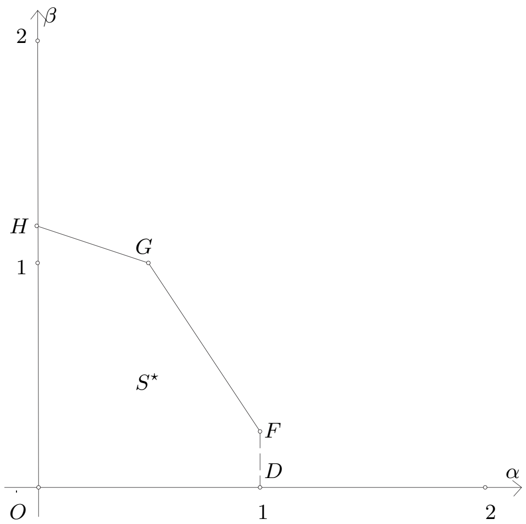

Figure 3. *Solution set S^**

DMUs will be preserved simultaneously under the corresponding changes (14) - (17) of all data of all DMUs. The region of joint efficiency $R_2^* = \{(\hat{X}_2, \hat{Y}_2) | \hat{X}_2 = 3 + \beta, \hat{Y}_2 = 4 - \alpha, (\alpha,\beta) \in S^*\}$ around DMU_2, corresponding to S^*, is the "polyhedron" $P_2 D_2 F_2 G_2 H_2$ with $P_2(3,4), D_2(3,3), F_2(3.25;3), G_2(4;3.5), H_2(25/6,4)$ without facet $\overline{D_2F_2}$ (see *Figure 4*).

For the inefficient DMU_5 the region $\bar{R}_5^* = \{(\hat{X}_5, \hat{Y}_5) | \hat{X}_5 = 5 - \beta, \hat{Y}_5 = 3 + \alpha, (\alpha,\beta) \in S^*\}$ corresponding to the set S^* is the "polyhedron" $P_5 D_5 F_5 G_5 H_5$ with $P_5(5,3), D_5(5,4), F_5(4.75;4), G_5(4;3.5), H_5(23/6,3)$ without facet $\overline{D_5F_5}$. There is a joint point $G_2 = G_5 = (4;3.5)$ of the regions R_2^* and $\bar{R}_5^*$, which corresponds to $(\alpha,\beta) = (0.5,1) \in S^*$. In that point efficiency of DMU_2 is preserved (and so is efficiency of all other efficient DMUs) but the status of inefficient DMU_5 is changed to efficient. For $(\alpha,\beta) \in \overline{FG}$, in corresponding points of $\overline{F_2G_2}$, $\overline{F_5G_5}$ efficiency of DMU_2 is preserved (with efficiency of all other efficient DMUs preserved too) and the status of inefficient DMU_5 is changed to efficient. Similar holds for $(\alpha,\beta) \in \overline{GH}$ and the corresponding points of $\overline{G_2H_2}$, $\overline{G_5H_5}$. So, in order to preserve efficiency of all efficient DMUs and preserve inefficiency of DMU_5, such points have to be excluded. It can be done considering the set $\bar{S}^* = \overline{FG} \cup \overline{GH}$. For

$(\alpha, \beta) \in Q^* = S^* \setminus \bar{S}^*$ efficiency of all efficient DMUs and inefficiency of DMU_5 will be preserved simultaneously.

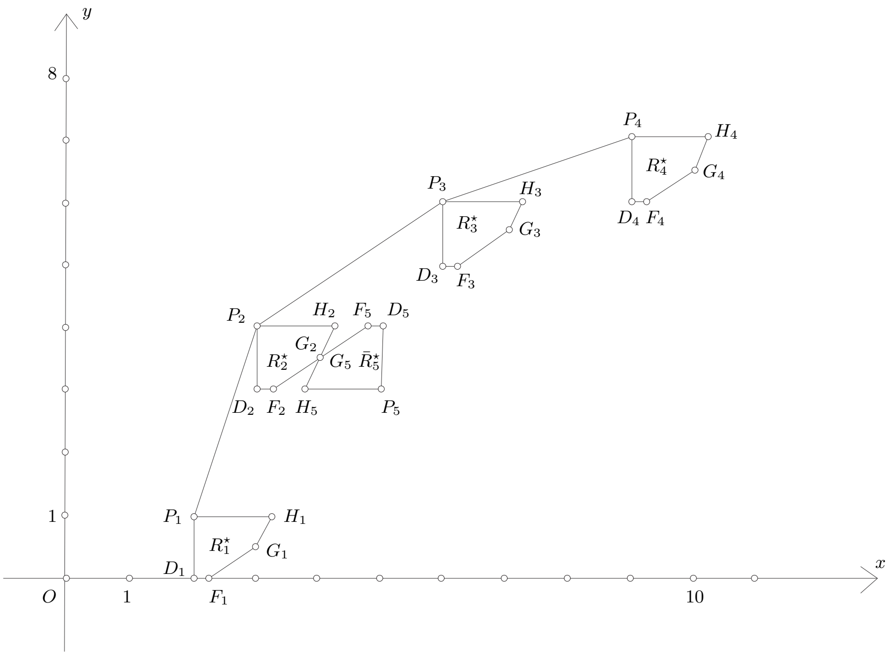

Figure 4. *Efficiency frontier with regions corresponding to* $S^\star$

5. Summary and conclusions

In this paper we consider preservation of efficiency of an efficient DMU_o, which is projection of an inefficient DMU_g onto the efficiency frontier, for the case of simultaneous non-negative perturbations of all data of all DMUs. Based on sufficient conditions for DMU_o to preserve its efficiency under these changes the region of efficiency of DMU_o and the corresponding region around DMU_g are obtained. In that way conditions for preserving efficiency of DMU_o and inefficiency of DMU_g simultaneously are given. The case of the non-negative change of all data of all DMUs, with the region of joint efficiency around all efficient DMUs and the corresponding region around each inefficient DMU, preserving efficiency of all efficient DMUs is also considered. In that case conditions for preserving efficiency of all efficient DMUs and inefficiency of all inefficient DMUs are obtained. An illustrative numerical example is provided.

Because conditions in *Theorem 1* are sufficient but not necessary, so are conditions in *Theorem 3* and *Theorem 4*. An open question is to find necessary and sufficient conditions for these cases. In order to apply the results in practice it is necessary to make an algorithm and a computer code, which is a challenge for the research in the future.

An open question is also to find (necessary and) sufficient conditions for simultaneous preservation of efficiency of efficient DMU(s) and inefficiency of inefficient DMU(s) according to the Charnes-Cooper-Rhodes (CCR) model and the Banker-Chanes-Cooper (BCC) model under the non-negative (or arbitrary) changes of outputs or/and inputs of all DMUs. The same holds for the proportionate change of all data for the Additive, CCR and BCC models. The results for these cases will be discussed elsewhere.

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