Circles in barycentric coordinates

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Abstract. *Let ABC be a fundamental triangle with the area $\triangle$. For a circle $\mathcal{K}$ the powers of vertices A, B, C with regard to $\mathcal{K}$ divided by $2\triangle$ are said to be the barycentric coordinates of $\mathcal{K}$ with respect to triangle ABC. This paper gives some theory and applications of these coordinates.*

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In [4] the most important facts about metrical relations in barycentric coordinates are proven. Some of these facts are necessary in the present paper and we enumerate them.

Let ABC be the fundamental triangle with sidelengths $a = |BC|$, $b = |CA|$, $c = |AB|$, measures A, B, C of the opposite angles and with area $\triangle$. Every (finite) point has the absolute barycentric coordinates x, y, z for which the equality $x+y+z = 1$ holds and we write $P = (x, y, z)$. For any $k \in \mathbb{R}\setminus\{0\}$ numbers $x' = kx$, $y' = ky$, $z' = kz$ are the relative barycentric coordinates of point P and we write $P = (x' : y' : z')$.

Fact 1 ([1, Cor.3]). Two points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ have the distance $|P_1P_2|$ given by

$$|P_1P_2|^2 = 2\triangle\Big[\alpha(x_1 - x_2)^2 + \beta(y_1 - y_2)^2 + \gamma(z_1 - z_2)^2\Big],$$

where $\alpha = \cot A$, $\beta = \cot B$, $\gamma = \cot C$.

Fact 2 ([1, Cor.4]). For any point $P = (x, y, z)$ we have the equalities

$$|AP|^2 = 2\triangle\Big[\alpha(1 - x)^2 + \beta y^2 + \gamma z^2\Big],$$
$$|BP|^2 = 2\triangle\Big[\alpha x^2 + \beta(1 - y)^2 + \gamma z^2\Big],$$
$$|CP|^2 = 2\triangle\Big[\alpha x^2 + \beta y^2 + \gamma(1 - z)^2\Big].$$

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Fact 3 ([1, Th.2]). For the point $P = (x, y, z)$ and any point S we have

$$|SP|^2 = x \cdot |SA|^2 + y \cdot |SB|^2 + z \cdot |SC|^2 - a^2yz - b^2zx - c^2xy.$$

Fact 4 ([1, Cor.6]). For any point $P = (x, y, z)$ and the circumcircle (O, R) of triangle ABC we have the equality $|OP|^2 = R^2 - 2\triangle\Pi$, where $2\triangle\Pi = a^2yz + b^2zx + c^2xy$.

Fact 5 ([1, Th.4]). Let points $P_i = (x_i : y_i : z_i)$ $(i = 1, 2)$ have the sums $s_i = x_i + y_i + z_i$ of coordinates. If the point $P = (x : y : z)$ satisfies the equality $P = \mu P_1 + \nu P_2$, then these three points have the ratio

$$\frac{P_1P}{P_2P} = (P_1P_2P) = -\frac{\nu}{\mu} \cdot \frac{s_2}{s_1}.$$

Any straight line $\mathcal{P}$ has the barycentric coordinates X, Y, Z determined up to proportionality and we write $\mathcal{P} = (X : Y : Z)$. The point $(x : y : z)$ lies on the line $(X : Y : Z)$ iff $xX + yY + zZ = 0$. The line at infinity is the line $\mathcal{N} = (1 : 1 : 1)$ and every point at infinity is of the form $(x : y : z)$, where $x + y + z = 0$. The line $(X : Y : Z)$ has the point at infinity $((Y - Z) : (Z - X) : (X - Y))$. The line through two points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ has the point at infinity $((x_1 - x_2) : (y_1 - y_2) : (z_1 - z_2))$.

Fact 6 ([1, Cor.16]). Two lines with the points at infinity $(x_i : y_i : z_i)$ $(i = 1, 2)$ are orthogonal iff $\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2 = 0$.

Fact 7 ([1], formulas (21) and (8)). If $\alpha = \cot A$, $\beta = \cot B$, $\gamma = \cot C$, then

$$\beta\gamma + \gamma\alpha + \alpha\beta = 1$$

and

$$a^2 = 2\triangle(\beta + \gamma), \quad b^2 = 2\triangle(\gamma + \alpha), \quad c^2 = 2\triangle(\alpha + \beta).$$

Fact 8 ([1, Th.7]). The fundamental triangle ABC has the orthocenter $H = (\beta\gamma, \gamma\alpha, \alpha\beta)$.

Fact 9 ([1, Th.8]). The oriented angle ϑ of oriented lines $\mathcal{P}_i = (X_i : Y_i : Z_i)$ $(i = 1, 2)$ is given by

$$\cot\vartheta = \frac{1}{k}(\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2),$$

where $x_i = Y_i - Z_i$, $y_i = Z_i - X_i$, $z_i = X_i - Y_i$ $(i = 1, 2)$ and

$$k = \begin{vmatrix} 1 & 1 & 1 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}.$$

Using *Fact 1* for the distance of two points $S = (x_o, y_o, z_o)$ and $P = (x, y, z)$ we obtain the equation of the circle $\mathcal{K}$ with the center S and the radius ϱ in the form

$$\alpha(x - x_o)^2 + \beta(y - y_o)^2 + \gamma(z - z_o)^2 - \frac{\varrho^2}{2\triangle} = 0. \tag{1}$$

If $P = (x, y, z)$ is any point, then $p_{P,\mathcal{K}} = |SP|^2 - \varrho^2$ is the power of point P with respect to the circle $\mathcal{K}$. Numbers

$$\lambda = \frac{1}{2\triangle} p_{A,\mathcal{K}}, \qquad \mu = \frac{1}{2\triangle} p_{B,\mathcal{K}}, \qquad \nu = \frac{1}{2\triangle} p_{C,\mathcal{K}} \tag{2}$$

are said to be the **barycentric coordinates of the circle** $\mathcal{K}$ with respect to the fundamental triangle ABC. According to *Facts 3* and *7* and the equalities $x + y + z = 1$ and (2) we obtain

$$\begin{aligned} p_{P,\mathcal{K}} &= |SP|^2 - \varrho^2 \\ &= x(|SA|^2 - \varrho^2) + y(|SB|^2 - \varrho^2) + z(|SC|^2 - \varrho^2) - a^2yz - b^2zx - c^2xy \\ &= x \cdot 2\triangle\lambda + y \cdot 2\triangle\mu + z \cdot 2\triangle\nu - 2\triangle(\beta+\gamma)yz - 2\triangle(\gamma+\alpha)zx - 2\triangle(\alpha+\beta)xy \\ &= 2\triangle\Big[(\lambda x + \mu y + \nu z)(x+y+z) - (\gamma+\alpha)yz - (\alpha+\beta)zx - (\alpha+\beta)xy\Big]. \end{aligned} \tag{3}$$

The point P lies on the circle $\mathcal{K}$ iff $p_{P,\mathcal{K}} = 0$. Therefore the following theorem holds.

Theorem 1. *The circle with the barycentric coordinates* λ, μ, ν *has the equation*

$$(\lambda x + \mu y + \nu z)(x+y+z) - (\beta+\gamma)yz - (\gamma+\alpha)zx - (\alpha+\beta)xy = 0 \tag{4}$$

and the point $P = (x,\ y,\ z)$ *has the power* $p_{P,\mathcal{K}}$ *with respect to this circle given by*

$$\frac{1}{2\triangle} p_{P,\mathcal{K}} = \lambda x + \mu y + \nu z - (\beta+\gamma)yz - (\gamma+\alpha)zx - (\alpha+\beta)xy, \tag{5}$$

i.e. $p_{P,\mathcal{K}} = 2\triangle(\lambda x + \mu y + \nu z) - a^2yz - b^2zx - c^2xy$.

Because of *Theorem 1* it follows that any circle $\mathcal{K}$ is uniquely determined by its barycentric coordinates λ, μ, ν and we write $\mathcal{K} = (\lambda,\ \mu,\ \nu)$.

The equation of the circle $\mathcal{K}$ in the form (1) gives immediately the coordinates of the center $S = (x_o,\ y_o,\ z_o)$ and the radius ϱ and the equation in the form (4) gives the barycentric coordinates of the circle $\mathcal{K} = (\lambda,\ \mu,\ \nu)$. Therefore, it is useful to know how to pass from the first equation to the second and vice versa. These passages are given by *Theorems 2* and *3*.

Theorem 2. *If the circle* $(\lambda,\ \mu,\ \nu)$ *has the center* $S = (x_o,\ y_o,\ z_o)$ *and the radius* ϱ*, then we have*

$$\begin{aligned} \lambda &= \alpha(1-x_o)^2 + \beta y_o^2 + \gamma z_o^2 - \frac{\varrho^2}{2\triangle}, \\ \mu &= \alpha x_o^2 + \beta(1-y_o)^2 + \gamma z_o^2 - \frac{\varrho^2}{2\triangle}, \\ \nu &= \alpha x_o^2 + \beta y_o^2 + \gamma(1-z_o)^2 - \frac{\varrho^2}{2\triangle}. \end{aligned} \tag{6}$$

Proof. According to *Fact 2* equalities (2) imply e.g.

$$\lambda = \frac{1}{2\triangle}(|SA|^2 - \varrho^2) = \alpha(1-x_o)^2 + \beta y_o^2 + \gamma z_o^2 - \frac{\varrho^2}{2\triangle}.$$

□

Corollary 1. *Two circles* $(\lambda,\ \mu,\ \nu)$ *and* $(\lambda',\ \mu',\ \nu')$ *with the radii* ϱ *and* ϱ' *are concentric iff*

$$\lambda-\lambda'=\mu-\mu'=\nu-\nu'=\frac{1}{2\triangle}(\varrho'^2-\varrho^2).$$

Any circle concentric with the circumcircle $(0,0,0)$ *of the triangle* ABC *has the form* $(\kappa,\ \kappa,\ \kappa)$ *for some* $\kappa\in\mathbb{R}$.

Corollary 2. *For a circle* $(\lambda,\ \mu,\ \nu)$ *with the center* $(x_o,\ y_o,\ z_o)$ *we have the equalities*

$$\lambda-\alpha+2\alpha x_o=\mu-\beta+2\beta y_o=\nu-\gamma+2\gamma z_o.$$

This *Corollary* can be used if three coordinates λ, μ, ν and one of three coordinates x_o, y_o, z_o are known, e.g. if we know that the center lies on one of the lines BC, CA, AB, or if all coordinates x_o, y_o, z_o and one of coordinates λ, μ, ν are known, e.g. if the circle passes through point A and therefore we have $\lambda=0$. If we have $x_o=1$, $y_o=z_o=0$, then $\lambda+\alpha=\mu-\beta=\nu-\gamma$, i.e. $\mu=\lambda+\alpha+\beta$, $\nu=\lambda+\gamma+\alpha$ and any circle with center A has the form $(\lambda,\ \lambda+\alpha+\beta,\ \lambda+\gamma+\alpha)$ for some $\lambda\in\mathbb{R}$. Analogously, any circle with center B has the form $(\mu+\alpha+\beta,\ \mu,\ \mu+\beta+\gamma)$ for some $\mu\in\mathbb{R}$ and any circle with center C has the form $(\nu+\gamma+\alpha,\nu+\beta+\gamma,\nu)$ for some $\nu\in\mathbb{R}$.

Theorem 3. *The center* $(x_o,\ y_o,\ z_o)$ *and the radius* ϱ *of the circle* $(\lambda,\ \mu,\ \nu)$ *are given by equalities*

$$\begin{aligned}2x_o &= -(\beta+\gamma)\lambda+\gamma\mu+\beta\nu+1-\beta\gamma,\\ 2y_o &= \gamma\lambda-(\gamma+\alpha)\mu+\alpha\nu+1-\gamma\alpha,\\ 2z_o &= \beta\lambda+\alpha\mu-(\alpha+\beta)\nu+1-\alpha\beta,\end{aligned}\tag{7}$$

$$\begin{aligned}\frac{2}{\triangle}\varrho^2 &= \alpha(\mu-\nu)^2+\beta(\nu-\lambda)^2+\gamma(\lambda-\mu)^2-2(\lambda+\mu+\nu)+2(\beta\gamma\lambda+\gamma\alpha\mu+\alpha\beta\nu)\\ &\quad+\alpha+\beta+\gamma-\alpha\beta\gamma.\end{aligned}\tag{8}$$

Proof. Because of *Theorem 2* we must prove that substitutions of x_o, y_o, z_o and ϱ^2 from (6) and (8) into the right-hand side of (5) give λ, μ, ν, respectively. But, we have at first

$$\alpha x_o^2+\beta y_o^2+\gamma z_o^2-\frac{\varrho^2}{2\triangle}=\beta\gamma\lambda+\gamma\alpha\mu+\alpha\beta\nu-\alpha\beta\gamma\tag{9}$$

because the coefficients of λ^2, $\mu\nu$, λ and 1 after these substitutions are respectively

$$\frac14\Big[\alpha(\beta+\gamma)^2+\beta\gamma^2+\gamma\beta^2-\beta-\gamma]=\frac14(\beta+\gamma)[\alpha(\beta+\gamma)+\beta\gamma-1\Big]=0,$$

$$\frac14\Big[2\alpha\beta\gamma-2\alpha\beta(\gamma+\alpha)-2\gamma\alpha(\alpha+\beta)+2\alpha\Big]=\frac{\alpha}{2}(1-\beta\gamma-\gamma\alpha-\alpha\beta)=0,$$

$$\frac{1}{4}\Big[-2\alpha(\beta+\gamma)(1-\beta\gamma)+2\beta\gamma(1-\gamma\alpha)+2\beta\gamma(1-\alpha\beta)+2-2\beta\gamma\Big]$$
$$=\frac{1}{2}(1+\beta\gamma-\gamma\alpha-\alpha\beta)=\beta\gamma,$$
$$\frac{1}{4}\Big[\alpha(1-\beta\gamma)^2+\beta(1-\gamma\alpha)^2+\gamma(1-\alpha\beta)^2-(\alpha+\beta+\gamma)+\alpha\beta\gamma\Big]$$
$$=\frac{1}{4}(\alpha\beta^2\gamma^2+\alpha^2\beta\gamma^2+\alpha^2\beta^2\gamma-5\alpha\beta\gamma)=\frac{1}{4}\alpha\beta\gamma(\beta\gamma+\gamma\alpha+\alpha\beta-5)$$
$$=-\alpha\beta\gamma,$$

where we have used the equality $\beta\gamma+\gamma\alpha+\alpha\beta=1$ from *Fact 7*, and analogously the coefficients of μ^2, ν^2, $\nu\lambda$, $\lambda\mu$, μ, ν are 0, 0, 0, 0, $\gamma\alpha$, $\alpha\beta$, respectively. After that, by (9) and (6) we get e.g.

$$\begin{aligned}\alpha x_o^2+\beta y_o^2+\gamma z_o^2-\frac{\varrho^2}{2\triangle}-2\alpha x_o+\alpha &= \beta\gamma\lambda+\gamma\alpha\mu+\alpha\beta\nu-\alpha\beta\gamma+\alpha(\beta+\gamma)\lambda\\ &\quad -\gamma\alpha\mu-\alpha\beta\nu-\alpha+\alpha\beta\gamma+\alpha\\ &= (\beta\gamma+\gamma\alpha+\alpha\beta)\lambda=\lambda.\end{aligned}$$

□

Specially, the circumcircle (0, 0, 0) of triangle ABC has the center

$$O=\left(\frac{1}{2}(1-\beta\gamma),\frac{1}{2}(1-\gamma\alpha),\ \frac{1}{2}(1-\alpha\beta)\right)$$

and the radius R given by the equality

$$\frac{2}{\triangle}R^2=\alpha+\beta+\gamma-\alpha\beta\gamma. \qquad (10)$$

¿From (8) it follows immediately: the circle (λ,μ,ν) is real iff the sum on the right side of (8) is positive.

The following theorem is very useful.

Theorem 4. *If* $P_i=(x_i,\ y_i,\ z_i)$ $(i=1,\ 2)$, *then the circle* $\mathcal{K}_{P_1P_2}=(\lambda,\ \mu,\ \nu)$ *with the diameter* P_1P_2 *is given by*

$$\begin{aligned}\lambda&=\alpha x_1x_2+\beta y_1y_2+\gamma z_1z_2+\alpha(1-x_1-x_2),\\ \mu&=\alpha x_1x_2+\beta y_1y_2+\gamma z_1z_2+\beta(1-y_1-y_2),\\ \nu&=\alpha x_1x_2+\beta y_1y_2+\gamma z_1z_2+\gamma(1-z_1-z_2).\end{aligned}$$

Proof. The segment $\overline{P_1P_2}$ has the midpoint

$$P_o=\left(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2),\frac{1}{2}(z_1+z_2)\right)$$

and *Facts 1* and *2* imply e.g.

$$\begin{aligned}\lambda &= \frac{1}{2\triangle}(|AP_o|^2 - \frac{1}{4}|P_1P_2|^2)\\ &= \alpha\left(1-\frac{x_1+x_2}{2}\right)^2 + \beta\left(\frac{y_1+y_2}{2}\right)^2 + \gamma\left(\frac{z_1+z_2}{2}\right)^2\\ &\quad -\alpha\left(\frac{x_1-x_2}{2}\right)^2 - \beta\left(\frac{y_1-y_2}{2}\right)^2 - \gamma\left(\frac{z_1-z_2}{2}\right)^2\\ &= \alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2 + \alpha(1-x_1-x_2).\end{aligned}$$

□

Corollary 3. *If $P = (x, y, z)$, then we have*

$$\begin{aligned}\mathcal{K}_{AP} &= (0,\, \alpha x + \beta(1-y),\, \alpha x + \gamma(1-z)),\\ \mathcal{K}_{BP} &= (\beta y + \alpha(1-x),\, 0,\, \beta y + \gamma(1-z)),\\ \mathcal{K}_{CP} &= (\gamma z + \alpha(1-x),\, \gamma z + \beta(1-y),\, 0).\end{aligned}$$

Corollary 4. *If $D = (0, d, d')$, $E = (e', 0, e)$, $F = (f, f', 0)$, then we have $\mathcal{K}_{AD} = (0, \beta d', \gamma d)$, $\mathcal{K}_{BE} = (\alpha e, 0, \gamma e')$, $\mathcal{K}_{CF} = (\alpha f', \beta f, 0)$. Especially, $\mathcal{K}_{BC} = (\alpha, 0, 0)$, $\mathcal{K}_{CA} = (0, \beta, 0)$, $\mathcal{K}_{AB} = (0, 0, \gamma)$.*

Corollary 5. *If $P_i = (0, y_i, z_i)$ resp. $P_i = (x_i, 0, z_i)$ resp. $P_i = (x_i, y_i, 0)$ for $i = 1, 2$, then we have*

$$\begin{aligned}\mathcal{K}_{P_1P_2} &= (\alpha + \beta y_1y_2 + \gamma z_1z_2,\, (\beta+\gamma)z_1z_2,\, (\beta+\gamma)y_1y_2),\\ \mathcal{K}_{P_1P_2} &= ((\gamma+\alpha)z_1z_2,\, \alpha x_1x_2 + \beta + \gamma z_1z_2,\, (\gamma+\alpha)x_1x_2),\\ \mathcal{K}_{P_1P_2} &= ((\alpha+\beta)y_1y_2,\, (\alpha+\beta)x_1x_2,\, \alpha x_1x_2 + \beta y_1y_2 + \gamma),\end{aligned}$$

respectively.

In the proof of *Corollary 5* we used e.g. the equalities

$$\beta y_1y_2+\gamma z_1z_2+\beta(1-y_1-y_2) = \beta(1-y_1)(1-y_2)+\gamma z_1z_2 = \beta z_1z_2+\gamma z_1z_2 = (\beta+\gamma)z_1z_2.$$

The points $B = (0, 1, 0)$ and $C = (0, 0, 1)$ have the midpoint $D = (0, \frac{1}{2}, \frac{1}{2})$ whereas the point $A = (1, 0, 0)$ and the orthocenter $H = (\beta\gamma, \gamma\alpha, \alpha\beta)$ from *Fact 8* have the midpoint $D' = (\frac{1}{2}(1+\beta\gamma), \frac{1}{2}\gamma\alpha, \frac{1}{2}\alpha\beta)$. Let us find the coordinates of the circle $\mathcal{K}_{DD'}$ using *Theorem 4*. With $x_1 = 0$, $y_1 = z_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}(1+\beta\gamma)$, $y_2 = \frac{1}{2}\gamma\alpha$, $z_2 = \frac{1}{2}\alpha\beta$ we obtain $\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2 = \frac{1}{2}\alpha\beta\gamma$ and therefore

$$\begin{aligned}\lambda &= \frac{1}{2}\alpha\beta\gamma + \alpha[1 - \frac{1}{2}(1+\beta\gamma)] = \frac{\alpha}{2},\\ \mu &= \frac{1}{2}\alpha\beta\gamma + \beta(1 - \frac{1}{2} - \frac{1}{2}\gamma\alpha) = \frac{\beta}{2}\end{aligned}$$

and analogously $\nu = \frac{\gamma}{2}$. Because of symmetry of coordinates of this circle $\mathcal{E} = (\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ it follows that this circle has diameters $\overline{DD'}$, $\overline{EE'}$, $\overline{FF'}$, where D, E, F and D', E', F' are the midpoints of segments $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and $\overline{AH}$, $\overline{BH}$, $\overline{CH}$. The obtained circle $\mathcal{E}$ is the **nine–point circle** of triangle ABC and it passes obviously through the feet $AH\cap BC$, $BH\cap CA$, $CH\cap AB$ of the altitudes AH, BH, CH of this triangle. For the center (x_o, y_o, z_o) of circle $\mathcal{E}$ by *Theorem 3* we obtain e.g.

$$2x_o = -(\beta+\gamma)\frac{\alpha}{2}+\gamma\frac{\beta}{2}+\beta\frac{\gamma}{2}+1-\beta\gamma = 1-\frac{1}{2}(\gamma\alpha+\alpha\beta) = 1-\frac{1}{2}(1-\beta\gamma) = \frac{1}{2}(1+\beta\gamma).$$

Therefore, this **nine–point center** O_9 of triangle ABC has the form

$$O_9 = \Big(\frac{1}{2}(1+\beta\gamma),\ \frac{1}{2}(1+\gamma\alpha),\ \frac{1}{2}(1+\alpha\beta)\Big).$$

This point is the midpoint of the orthocenter $H = (\beta\gamma,\ \gamma\alpha,\ \alpha\beta)$ and the circumcenter $O = (\frac{1}{2}(1-\beta\gamma),\ \frac{1}{2}(1-\gamma\alpha),\ \frac{1}{2}(1-\alpha\beta))$ of the same triangle. The homothety $(H,\ \frac{1}{2})$ maps the circumcircle (O, R) through points A, B, C onto the nine–point circle through points D', E', F' and therefore the nine–point circle has the radius $\dfrac{R}{2}$.

Now, we shall prove a statement which gives another interpretation of barycentric coordinates of a circle.

Theorem 5. *The circles* $(\lambda,\ 0,\ 0)$ *resp.* $(0,\ \mu,\ 0)$*, resp.*$(0,\ 0,\ \nu)$ *are the sets of points* P *such that*

$$\cot\angle(BP,\ CP) = \alpha-\lambda \quad \textit{resp.} \quad \cot\angle(CP,\ AP) = \beta-\mu \ \textit{resp.} \ \cot\angle(AP,\ BP) = \gamma-\nu.$$

Proof. If $P = (x : y : z)$, then we have $BP = (z : 0 : -x)$, $CP = (-y : x : 0)$,

$$k = \begin{vmatrix} 1 & 1 & 1 \\ z & 0 & -x \\ -y & x & 0 \end{vmatrix} = x(x+y+z)$$

and with $\cot\vartheta = \cot\angle(BP,\ CP) = \alpha-\lambda$ *Fact 9* implies $x(x+y+z)(\alpha-\lambda) = \alpha x^2 - \beta(z+x)y - \gamma z(x+y)$, i. e.

$$\lambda x(x+y+z) - (\beta+\gamma)yz - (\gamma+\alpha)zx - (\alpha+\beta)xy = 0. \tag{11}$$

But, this is the equation of the circle $(\lambda,\ 0,\ 0)$. □

Corollary 6. *The set of points* P *with the given oriented angle of lines* AP *and* BP *for two given points* A *and* B *is a circle through these points* A *and* B*.*

This is the famous theorem on angles inscribed in the same arc in the version for directed angles as developed by R. A. Johnson [2] and [3, numbers 16–19]; for a modern approach see [1].

With $P = (x, y, z)$ formula (11) takes the form $\lambda x = \Pi$, where

$$\Pi = (\beta+\gamma)yz + (\gamma+\alpha)zx + (\alpha+\beta)xy = \frac{1}{2\triangle}(a^2yz + b^2zx + c^2xy). \tag{12}$$

Therefore, *Theorem 5* implies two more corollaries.

Corollary 7. *For any point $P=(x,\,y,\,z)$ we have the equalities*

$$\cot\angle(BP,\,CP)=\alpha-\frac{\Pi}{x},\quad \cot\angle(CP,\,AP)=\beta-\frac{\Pi}{y},\quad \cot\angle(AP,\,BP)=\gamma-\frac{\Pi}{z},$$

where Π is given by (12).

Corollary 8. *For any point $P=(x,\,y,\,z)$ the circles BCP, CAP, ABP are the circles*

$$\left(\frac{\Pi}{x},\,0,\,0\right),\quad \left(0,\,\frac{\Pi}{y},\,0\right),\quad \left(0,\,0,\,\frac{\Pi}{z}\right),$$

respectively, where Π is given by (12).

Now, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be any two circles with centers S_1 and S_2 and radii ϱ_1 and ϱ_2. If these two circles intersect each other, then let ϑ be the angle of these circles, i.e. the directed angle of their tangents at one of their common point P. But this angle is equal to the angle of the lines S_1P and S_2P. Therefore, we have

$$\cos\vartheta=\frac{1}{2\varrho_1\varrho_2}(\varrho_1^2+\varrho_2^2-|S_1S_2|^2). \tag{13}$$

Specially, if $\vartheta=0$ or $\vartheta=\pi$, i.e. if $|S_1S_2|=|\varrho_1-\varrho_2|$ resp. $|S_1S_2|=\varrho_1+\varrho_2$, then the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ touch each other inwardly resp. outwardly. If $\vartheta=\frac{\pi}{2}$, i.e. if

$$|S_1S_2|^2=\varrho_1^2+\varrho_2^2, \tag{14}$$

then the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ are orthogonal.

Theorem 6. *Two circles $\mathcal{K}_i=(\lambda_i,\,\mu_i,\,\nu_i)$ $(i=1,2)$ are orthogonal iff*

$$\begin{aligned}&\alpha(\mu_1-\nu_1)(\mu_2-\nu_2)+\beta(\nu_1-\lambda_1)(\nu_2-\lambda_2)+\gamma(\lambda_1-\mu_1)(\lambda_2-\mu_2)\\&-(1-\beta\gamma)(\lambda_1+\lambda_2)-(1-\gamma\alpha)(\mu_1+\mu_2)-(1-\alpha\beta)(\nu_1+\nu_2)\\&+\alpha+\beta+\gamma-\alpha\beta\gamma=0.\end{aligned} \tag{15}$$

Proof. Let $S_i=(x_i,y_i,z_i)$ be the centers of $\mathcal{K}_i$ for $i=1,2$. According to *Fact 1* and *Theorem 3* we obtain

$$\begin{aligned}\frac{2}{\triangle}|S_1S_2|^2&=4\Big[\alpha(x_1-x_2)^2+\beta(y_1-y_2)^2+\gamma(z_1-z_2)^2\Big]\\&=\alpha\Big[-(\beta+\gamma)(\lambda_1-\lambda_2)+\gamma(\mu_1-\mu_2)+\beta(\nu_1-\nu_2)\Big]^2\\&\quad+\beta\Big[\gamma(\lambda_1-\lambda_2)-(\gamma+\alpha)(\mu_1-\mu_2)+\alpha(\nu_1-\nu_2)\Big]^2\\&\quad+\gamma\Big[\beta(\lambda_1-\lambda_2)+\alpha(\mu_1-\mu_2)-(\alpha+\beta)(\nu_1-\nu_2)\Big]^2,\end{aligned}$$

i.e.

$$\frac{2}{\triangle}|S_1S_2|^2=\alpha(\mu_1-\nu_1-\mu_2+\nu_2)^2+\beta(\nu_1-\lambda_1-\nu_2+\lambda_2)^2+\gamma(\lambda_1-\mu_1-\lambda_2+\mu_2)^2 \tag{16}$$

because the coefficients of e.g. $(\lambda_1-\lambda_2)^2$ and $2(\mu_1-\mu_2)(\nu_1-\nu_2)$ are

$$\alpha(\beta+\gamma)^2+\beta\gamma^2+\gamma\beta^2 = (\beta+\gamma)[\alpha(\beta+\gamma)+\beta\gamma] = \beta+\gamma,$$
$$\alpha\beta\gamma-\alpha\beta(\gamma+\alpha)-\alpha\gamma(\alpha+\beta) = -\alpha(\beta\gamma+\gamma\alpha+\alpha\beta) = -\alpha.$$

As we have e.g.

$$\alpha(\mu_1-\nu_1)^2+\alpha(\mu_2-\nu_2)^2-\alpha(\mu_1-\nu_1-\mu_2+\nu_2)^2 = 2\alpha(\mu_1-\nu_1)(\mu_2-\nu_2),$$

so by formula (8) for ϱ_1^2 and ϱ_2^2 and by (16) it follows

$$\begin{aligned}\frac{2}{\triangle}(\varrho_1^2+\varrho_2^2-|S_1S_2|^2) = {}&\alpha(\mu_1-\nu_1)(\mu_2-\nu_2)+2\beta(\nu_1-\lambda_1)(\nu_2-\lambda_2)\\ &+2\gamma(\lambda_1-\mu_1)(\lambda_2-\nu_2)-2(1-\beta\gamma)(\lambda_1+\lambda_2)\\ &-2(1-\gamma\alpha)(\mu_1+\mu_2)-2(1-\alpha\beta)(\nu_1+\nu_2)\\ &+2(\alpha+\beta+\gamma-\alpha\beta\gamma)\end{aligned}$$

and equality (14) is equivalent to equality (15). □

If $\mathcal{K}_1=\mathcal{K}=(\lambda,\ \mu,\ \nu)$ and $\mathcal{K}_2=(0,\ 0,\ 0)$, then *Theorem 6* implies the following corollary.

Corollary 9. *The circle $\mathcal{K}=(\lambda,\ \mu,\ \nu)$ is orthogonal onto the circumcircle of triangle ABC iff*

$$(1-\beta\gamma)\lambda+(1-\gamma\alpha)\mu+(1-\alpha\beta)\gamma = \alpha+\beta+\gamma-\alpha\beta\gamma. \tag{17}$$

The powers $p_{P,\mathcal{K}_i}$ of any point $P=(x,y,z)$ with respect to the circles $\mathcal{K}_i=(\lambda_i,\ \mu_i,\ \nu_i)$ $(i=1,2)$ are given by (because of *Theorem 1*)

$$\frac{1}{2\triangle}p_{P,\mathcal{K}_i} = \lambda_i x+\mu_i y+\nu_i z-\Pi \quad (i=1,2),$$

where Π is given by (12). The set of points P such that $p_{P,\mathcal{K}_1}=p_{P,\mathcal{K}_2}$ is given by the equation $(\lambda_1-\lambda_2)x+(\mu_1-\mu_2)y+(\nu_1-\nu_2)z=0$. This set is a line, the radical axis of the circles $\mathcal{K}_1$ and $\mathcal{K}_2$. We have the following theorem.

Theorem 7. *Two circles $\mathcal{K}_i=(\lambda_i,\ \mu_i,\ \nu_i)$ $(i=1,2)$ have the radical axis $\mathcal{P}_{12}=((\lambda_1-\lambda_2):(\mu_1-\mu_2):(\nu_1-\nu_2))$.*

Specially, a circle $(\lambda,\ \mu,\ \nu)$ and the circumcircle $(0,\ 0,\ 0)$ have the radical axis $(\lambda:\mu:\nu)$.

According to *Theorem 3* the point at infinity of the line S_1S_2 has the coordinates

$$\beta(\nu-\lambda)+\gamma(\mu-\lambda),\quad \gamma(\lambda-\mu)+\alpha(\nu-\mu),\quad \alpha(\mu-\nu)+\beta(\lambda-\nu),$$

where $\lambda=\lambda_1-\lambda_2$, $\mu=\mu_1-\mu_2$, $\nu=\nu_1-\nu_2$. On the other hand, from *Theorem 7* it follows that the radical axis of the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ has the point at infinity $((\mu-\nu):(\nu-\lambda):(\lambda-\nu))$. Obviously we have

$$\begin{aligned}&\alpha(\mu-\nu)\Big[\beta(\nu-\lambda)+\gamma(\mu-\lambda)\Big]+\beta(\nu-\lambda)\Big[\gamma(\lambda-\mu)+\alpha(\nu-\mu)\Big]\\ &+\gamma(\lambda-\mu)\Big[\alpha(\mu-\nu)+\beta(\lambda-\nu)\Big]=0\end{aligned}$$

and by *Fact 6* there follows the well–known fact that the radical axis of two circles is orthogonal to the join of their centers. If two circles have common points, then these points lie on their radical axis.

Theorem 8. *Three circles* $\mathcal{K}_i = (\lambda_i,\ \mu_i,\ \nu_i)$ $(i = 1, 2, 3)$ *have in pairs the radical axes* $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{23}$ *which have a common point*

$$P = \left(\begin{vmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ \nu_1 & \nu_2 & \nu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \right), \tag{18}$$

the radical center of the circles $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$.

Proof. The point P from (18) can be written in the form

$$P = \Big([\mu(\nu_1 - \nu_3) - \nu(\mu_1 - \mu_3)] : [\nu(\lambda_1 - \lambda_3) - \lambda(\nu_1 - \nu_3)] : [\lambda(\mu_1 - \mu_3) - \mu(\lambda_1 - \lambda_3)] \Big),$$

where again $\lambda = \lambda_1 - \lambda_2$, $\mu = \mu_1 - \mu_2$, $\nu = \nu_1 - \nu_2$ and by *Theorem 7* we have $\mathcal{P}_{12} = (\lambda : \mu : \nu)$. Because of

$$\lambda\Big[\mu(\nu_1 - \nu_3) - \nu(\mu_1 - \mu_3)\Big] + \mu\Big[\nu(\lambda_1 - \lambda_3) - \lambda(\nu_1 - \nu_2)\Big] + \nu\Big[\lambda(\mu_1 - \mu_3) - \mu(\lambda_1 - \lambda_3)\Big] = 0$$

point P lies on line $\mathcal{P}_{12}$. Analogously, this point lies on lines $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$. □

Two points P and P' are said to be inverse to each other with respect to the circle $(S,\ \varrho)$ if these points are collinear with the center S and if $\overrightarrow{SP} \cdot \overrightarrow{SP'} = \varrho^2$.

Theorem 9. *A point* $P(x, y, z)$ *has the inverse point*

$$P' = \left(\frac{1}{s'}(2R^2x - \alpha a^2\Pi),\ \frac{1}{s'}(2R^2y - \beta b^2\Pi),\ \frac{1}{s'}(2R^2z - \gamma c^2\Pi) \right) \tag{19}$$

with respect to the circumcircle of the triangle ABC, *where* $s' = 2R^2 - 4\triangle\Pi$ *and* Π *is given by (12).*

Proof. Owing to *Fact 7* we obtain

$$\begin{aligned} &2R^2x - \alpha a^2\Pi + 2R^2y - \beta b^2\Pi + 2R^2z - \gamma c^2\Pi \\ &= 2R^2(x + y + z) - 2\triangle\Pi[\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)] \\ &= 2R^2 - 4\triangle\Pi = s' \end{aligned}$$

and the point (19) has the sum of coordinates equal to 1. The circumcenter of the triangle ABC can be written in the form

$$O = \left(\frac{\alpha a^2}{4\triangle},\ \frac{\beta b^2}{4\triangle},\ \frac{\gamma c^2}{4\triangle} \right).$$

We have the equality $2R^2 \cdot P - s' \cdot P' = 4\triangle\Pi \cdot O$ because of e.g. $2R^2x - (2R^2x - \alpha a^2\Pi) = \Pi \cdot \alpha a^2$. Therefore *Fact 5* implies the equalities

$$(PP'O) = \frac{s'}{2R^2} = \frac{R^2 - 2\triangle\Pi}{R^2} = \frac{|OP|^2}{R^2}$$

because of the equality $R^2 - 2\triangle\Pi = |OP|^2$ implied by *Fact 4*. Hence

$$\overrightarrow{OP} = \frac{|OP|^2}{R^2}\overrightarrow{OP'}$$

and scalar multiplication by $\overrightarrow{OP}$ gives $\overrightarrow{OP} \cdot \overrightarrow{OP'} = R^2$. □

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