A NOTE ON LOGARITHMIC SMOOTHING IN SEMI-INFINITE OPTIMIZATION UNDER REDUCTION APPROACH*

Francisco Guerra-Vazquez
Universidad de las Americas, Puebla, Escuela de Ciencias,
San Andres Cholula, Puebla 72820, Mexico
E-mail: francisco.guerra@udlap.mx

Jan-J. Rückmann
The University of Birmingham, School of Mathematics,
Birmingham, B152TT, United Kingdom
E-mail: J.Ruckmann@bham.ac.uk

Abstract

This note deals with a semi-infinite optimization problem which is defined by infinitely many inequality constraints. By applying a logarithmic barrier function, a family of interior point approximations of the feasible set is obtained where locally the original feasible set and its approximations are homeomorphic. Under generic assumptions on the structure of the original feasible set, strongly stable stationary points of the original problem are considered and it is shown that there is a one-to-one correspondence between the stationary points (and their stationary indices) of the original problem and those of its approximations. Corresponding convergence results, global aspects and a relationship to a standard interior-point approach are discussed.

1 Introduction

In this note we consider semi-infinite optimization problems, that is, nonlinear problems defined in a *finite*-dimensional space whose feasible sets are represented by an *infinite* number of inequality constraints. Semi-infinite programming, its theory, numerical solution methods and applications have become a very active research area within mathematical programming during the previous two decades. There is a broad range of applications where the semi-infinite model can be used; for more details we refer to [2, 7, 16, 20] as well as to the compilations [3, 17].

As a starting point of this paper, we consider a semi-infinite optimization problem (SIP) of the form

$$\text{SIP} \quad \text{minimize } f(x) \text{ s.t. } x \in M$$

with the feasible set

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$$M = \{x \in \mathbb{R}^n \mid g(x,y) \geq 0,\ y \in Y\}$$

and the index set

$$Y = \{y \in \mathbb{R}^r \mid u_l(y) = 0,\ l \in A,\ v_k(y) \leq 0,\ k \in B\},$$

where $A = \{1, \ldots, a\}$, $a < r$ and $B = \{1, \ldots, b\}$. The functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ and $u_l, v_k : \mathbb{R}^r \to \mathbb{R}$, $l \in A$, $k \in B$ are assumed to be twice continuously differentiable. The index set $Y \subset \mathbb{R}^r$ is assumed to be compact (and, in general, it is an infinite set) and, obviously, each index $\overline{y} \in Y$ represents a corresponding inequality constraint $g(\cdot, \overline{y}) \geq 0$. For a feasible point $\overline{x} \in M$, we define the set of active constraints at $\overline{x}$ as

$$Y_0(\overline{x}) = \{y \in Y \mid g(\overline{x}, y) = 0\}$$

and it is obvious that each $\overline{y} \in Y_0(\overline{x})$ is a *global minimizer* of the so-called *lower level* problem

$$LL(\overline{x}) \qquad \text{minimize } g(\overline{x}, y) \text{ s.t. } y \in Y. \tag{1-1}$$

The latter one is a finite parameter-dependent (where $\overline{x}$ is the parameter vector) problem. Note that $Y_0(\overline{x})$ can be an infinite set. A point $\overline{x} \in M$ is called a *stationary point of SIP* if $\overline{x} - Df(\overline{x}) \in N_{\overline{x}}$, where

$$N_{\overline{x}} = \left\{ \overline{x} - \sum_{j=1}^{q} \mu_j D_x g(\overline{x}, y^j) \,\middle|\, \mu_j \geq 0,\ y^j \in Y_0(\overline{x}),\ j = 1, \ldots, q,\ q < \infty \right\}$$

and the row vector $Df(\overline{x})$ denotes the gradient of f at $\overline{x}$ (the gradients, partial derivatives and Hessians $Dg(\overline{x})$, $D_x g(\overline{x}, y)$, $D_y g(\overline{x}, y)$, $D^2 f(\overline{x})$, ... are analogously defined).

In this paper we will consider a *strongly stable* stationary point $\overline{x}$ of SIP. The concept of strong stability was introduced by Kojima [15] and it refers to local existence and uniqueness of stationary points where perturbations up to second order are allowed. We will have a brief view on this concept in Section 2. Furthermore, we recall in Section 3 that under the generic assumption of the reduction approach at the point under consideration $\overline{x}$, the semi-infinite problem SIP can locally around $\overline{x}$ be transformed into a problem whose feasible set is described by *finitely* many continuously differentiable functions.

Now, the main goal of this paper is to show that, having this local finite description of SIP, we can locally apply a logarithmic smoothing approach which was introduced in [4, 10] for finite problems. There we use a logarithmic barrier function and consider a parametric family of interior point approximations of an intersection of the feasible set M with an appropriate neighbourhood of $\overline{x}$. This family is controlled by an approximation parameter $\gamma > 0$ and there exists a corresponding solution curve of stationary points converging to $\overline{x}$ as $\gamma \to 0$. We

will see that this parametric family is closely related to a standard interior-point approach.
This paper is organized as follows. Section 2 contains some auxiliary results which will be used later. In particular, we present several constraint qualifications and we recall the concept of strong stability of a stationary point. In Section 3 we discuss the reduction approach under which a semi-infinite problem can locally be described by finitely many constraints. Then, Section 4 contains the main results: we generalize an interior logarithmic smoothing approach to our semi-infinite setting. We present the existence of a solution curve and discuss topological properties and global aspects of the involved feasible sets. Finally, Section 5 presents some brief conclusions.
At the end of this section we explain some notations. If $U \subseteq \mathbb{R}^n$ is an open set and if the function $h : U \to \mathbb{R}$ is k-times continuously differentiable ($k = 1$ or $k = 2$), then we write $h \in C^k(U, \mathbb{R})$. For $\delta > 0$ and $\overline{x} \in \mathbb{R}^n$ define $B_\delta(\overline{x}) = \{x \in \mathbb{R}^n \mid \|x - \overline{x}\| \leq \delta\}$ where $\|\cdot\|$ denotes the Euclidean norm unless stated otherwise. Let $\mathbb{R}_+ = \{\alpha \in \mathbb{R} \mid \alpha > 0\}$ and let the components of $\overline{x} \in \mathbb{R}^n$ be denoted by $\overline{x}_i$, $i = 1, \ldots, n$. Finally, by an (n, m)-matrix we mean a real matrix with n rows and m columns.

2 Auxiliary results

In this section we recall several notations and auxiliary results which will be used later.

Constraint qualifications

The following two constraint qualifications are appropriate extensions (see e.g. [11]) of the well-known linear independence constraint qualification (LICQ) and Mangasarian-Fromovitz constraint qualification (MFCQ).

The *Extended Linear Independence Constraint Qualification (ELICQ)* is said to hold at $\overline{x} \in M$ if the gradients $D_x g(\overline{x}, y)$, $y \in Y_0(\overline{x})$ are linearly independent.

The *Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ)* is said to hold at $\overline{x} \in M$ if there exists a vector $\xi \in \mathbb{R}^n$ such that

$$D_x g(\overline{x}, y)\, \xi > 0, \;\; y \in Y_0(\overline{x}).$$

It is easy to see that if ELICQ holds at $\overline{x} \in M$, then EMFCQ holds at $\overline{x} \in M$ as well. Furthermore, we know from [11] that if EMFCQ holds at all $\overline{x} \in M$, then the feasible set M is a topological manifold and its boundary ∂M can be described as

$$\partial M = \left\{ x \in M \mid \min_{y \in Y_0(\overline{x})} g(x, y) = 0 \right\}.$$

We refer to [5, 20] for a broader discussion on constraint qualifications.

Throughout this paper we assume for the index set (and feasible set of the lower level problem (1-1)) Y the following

Assumption (A1). LICQ holds at all $\overline{y} \in Y$, that is, the gradients $Du_l(\overline{y})$, $l \in A$, $Dv_k(\overline{y})$, $k \in B_0(\overline{y})$ are linearly independent, where

$$B_0(\overline{y}) = \{k \in B \mid v_k(\overline{y}) = 0\}.$$

Note that Assumption (A1) is a generic condition (briefly: that is, under an appropriately induced topology, there exists a generic subset of functions u_l, v_k, $l \in A$, $k \in B$ such that Assumption (A1) is fulfilled for the corresponding set Y; for more details see [8, 11]).

Strongly stable stationary points

As mentioned in Section 1, Kojima [15] introduced the concept of a *strongly stable stationary point* for a finite optimization problem. Roughly speaking, strong stability refers to the local existence and uniqueness of a stationary point where perturbations up to second order of the describing functions (objective function, constraints) are considered and where the (locally uniquely determined) stationary point of the perturbed problem depends continuously on these perturbations. Under the linear independence constraint qualification Kojima's concept of strong stability was proven to be equivalent to Robinson's concept [18] of strong regularity (see [9] for more details).
In the following we will recall Kojima's definition of a strongly stable stationary point of a finite optimization problem. Since we will apply this definition to both the lower level problem (1-1) and the original problem SIP, we will present it for a general finite optimization problem of the form

$$\text{(P)} \quad \text{minimize} \quad h(x) \quad \text{s.t.} \quad g_j(x) \geq 0, \ j \in S = \{1, \ldots, s\}$$

where $h, g_j \in C^2(\mathbb{R}^n, \mathbb{R})$, $j \in S$. A feasible point $\overline{x} \in \mathbb{R}^n$ of (P) is called a stationary point of (P) if there exist $\overline{\lambda}_j \geq 0$, $j \in S_0(\overline{x})$ such that

$$Dh(\overline{x}) - \sum_{j \in S_0(\overline{x})} \overline{\lambda}_j Dg_j(\overline{x}) = 0 \tag{2-1}$$

where $S_0(\overline{x}) = \{j \in S \mid g_j(\overline{x}) = 0\}$. For a set $U \subseteq \mathbb{R}^n$ let

$$\text{norm}\ ((h, g_1, \ldots, g_s), U) = \max \left\{ \begin{array}{l} \sup_{x \in U} \max \{|h(x)|, \|Dh(x)\|, \|D^2 h(x)\|\}, \\ \sup_{j \in S} \sup_{x \in U} \max \{|g_j(x)|, \|Dg_j(x)\|, \|D^2 g_j(x)\|\} \end{array} \right\}$$

where $\|Q\| = \max\{\|Qx\| \mid x \in \mathbb{R}^n, \ \|x\| = 1\}$ for an (n, n)-matrix Q.

Definition 2.1 ([15]) A stationary point $\overline{x}$ of (P) is called *strongly stable* if for some $\overline{\delta} \in \mathbb{R}_+$ and each $\delta \in (0, \overline{\delta})$ there exists an $\alpha \in \mathbb{R}_+$ such that whenever $\widetilde{h}, \widetilde{g}_j \in C^2(\mathbb{R}^n, \mathbb{R})$, $j \in S$ and

$$\text{norm}\ \left(\left(\widetilde{h}, \widetilde{g}_1, \ldots, \widetilde{g}_s\right), B_{\overline{\delta}}(\overline{x})\right) \leq \alpha,$$

$B_\delta(\overline{x})$ contains a stationary point of the problem

$$\text{minimize} \quad h(x) + \widetilde{h}(x) \text{ s.t. } g_j(x) + \widetilde{g}_j(x) \geq 0, \; j \in S$$

which is unique in $B_{\overline{\delta}}(\overline{x})$.

In [19], this concept was generalized to semi-infinite problems. Furthermore, it is shown in [1] that if a stationary point $\overline{x}$ of SIP is strongly stable, then EMFCQ holds at $\overline{x}$.

Besides this topological characterization of strong stability, Kojima presented an equivalent algebraic characterization which will be recalled in the following lemma. For this let $\overline{x}$ be a stationary point of (P) and consider the following two cases.
If LICQ holds at $\overline{x}$, then the multipliers $\overline{\lambda}_j \geq 0$, $j \in S_0(\overline{x})$ in (2-1) are uniquely determined. In that case let

$$H = D^2 h(\overline{x}) - \sum_{j \in S_0(\overline{x})} \overline{\lambda}_j D^2 g_j(\overline{x}),$$

$$S_+(\overline{x}) = \left\{ j \in S_0(\overline{x}) \mid \overline{\lambda}_j > 0 \right\}$$

and for each index set $\overline{S}$ with $S_+(\overline{x}) \subseteq \overline{S} \subseteq S_0(\overline{x})$ denote the corresponding tangent space as

$$T\left(\overline{S}\right) = \left\{ x \in \mathbb{R}^n \mid Dg_j(\overline{x})\, x = 0, \; j \in \overline{S} \right\}.$$

If MFCQ holds at $\overline{x}$ (that is, there exists $\xi \in \mathbb{R}^n$ such that $Dg_j(\overline{x})\,\xi > 0$, $j \in S_0(\overline{x})$), then for any $\overline{\lambda} \in \mathbb{R}^{|S_0(\overline{x})|}$ ($|\cdot|$ denotes the cardinality) satisfying (2-1) let

$$H\left(\overline{\lambda}\right) = D^2 h(\overline{x}) - \sum_{j \in S_0(\overline{x})} \overline{\lambda}_j D^2 g_j(\overline{x})$$

and

$$T\left(\overline{\lambda}\right) = \left\{ x \in \mathbb{R}^n \mid Dg_j(\overline{x})\, x = 0, \; j \in \left\{ i \in S_0(\overline{x}) \mid \overline{\lambda}_i > 0 \right\} \right\}.$$

For the algebraic characterization of strong stability we need the following notation. Let Q be an (n, n)-matrix and $W \subset \mathbb{R}^n$ be an m-dimensional subspace of $\mathbb{R}^n$. An (n, m)-matrix K whose columns form a basis of W is called a *basis matrix of* W. By Sylvester's law, the numbers of negative, positive and zero eigenvalues of $K^\top QK$ does not depend on the particular choice of the basis matrix and, hence, we can write

$$\text{sign det } (Q|W) = \text{ sign det } \left(K^\top QK\right)$$

(where det denotes the determinant) and the number of negative (positive, zero) eigenvalues of $Q|W$ refers to the number of negative (positive, zero) eigenvalues of $K^\top QK$.

Lemma 2.1 (*algebraic characterization of strong stability* [15])
Let $\overline{x}$ be a stationary point of (P).
(i) Assume that LICQ holds at $\overline{x}$. Then, $\overline{x}$ is a strongly stable stationary point of (P) if and only if sign det $\left(H|T\left(\overline{S}\right)\right)$ is constant and not zero for all $\overline{S}$ with $S_+\left(\overline{x}\right) \subseteq \overline{S} \subseteq S_0\left(\overline{x}\right)$.
(ii) Assume that LICQ holds at $\overline{x}$ and that $\overline{x}$ is a strongly stable stationary point of (P). Then, the number of negative eigenvalues of $H|T\left(\overline{S}\right)$ is constant for all $\overline{S}$ with $S_+\left(\overline{x}\right) \subseteq \overline{S} \subseteq S_0\left(\overline{x}\right)$ and it is called the *stationary index of* $\overline{x}$ (notation: $s.i.\left(\overline{x}\right)$). In particular, it is
- $s.i.\left(\overline{x}\right) = 0$ if and only if $\overline{x}$ is a local minimizer of (P),
- $1 \leq s.i.\left(\overline{x}\right) \leq n-1$ if and only if $\overline{x}$ is a saddle point of (P),
- $s.i.\left(\overline{x}\right) = n$ if and only if $\overline{x}$ is a local maximizer of (P).

(iii) Assume that MFCQ holds at $\overline{x}$ but LICQ does not hold at $\overline{x}$. Then, $\overline{x}$ is a strongly stable stationary point of (P) if and only if all eigenvalues of $H\left(\overline{\lambda}\right)|T\left(\overline{\lambda}\right)$ are positive for all $\overline{\lambda} \in \mathbb{R}^{|S_0(\overline{x})|}$ satisfying (2-1). Set $s.i.\left(\overline{x}\right) = 0$. In particular, $\overline{x}$ is a local minimizer of (P). △

Note that neither LICQ nor the strict complementarity condition need to hold at a strongly stable stationary point. We refer again to [19] where an algebraic characterization of a strongly stable stationary point of SIP is presented.

3 The reduction approach

One of the main challenges for the design of a solution method for semi-infinite problems is that the original problem has (at least locally) to be transformed into a finite optimization problem. That might be done by discretizing the infinite index set Y or by another approximation of the original problem. In this section we recall the so-called *reduction approach*, a condition under which the original semi-infinite problem can locally be transformed equivalently into a finite problem (for more details see e.g. [6, 12, 13, 14]). We will comment at the end of this section that in a certain sense the reduction approach is a natural assumption.

Let $\overline{x} \in M$ and $\overline{y} \in Y_0\left(\overline{x}\right)$. Since $\overline{y}$ is a (global) minimizer of the lower level problem (1-1) and LICQ holds at $\overline{y}$, there exist uniquely determined multipliers $\overline{\alpha}_l$, $l \in A$ and $\overline{\beta}_k \geq 0$, $k \in B_0\left(\overline{y}\right)$ satisfying

$$D_y g\left(\overline{x}, \overline{y}\right) + \sum_{l \in A} \overline{\alpha}_l D u_l\left(\overline{y}\right) + \sum_{k \in B_0\left(\overline{y}\right)} \overline{\beta}_k D v_k\left(\overline{y}\right) = 0. \tag{3-1}$$

We introduce the following

Assumption (A2). $\overline{y} \in Y_0\left(\overline{x}\right)$ is a strongly stable stationary point of $LL\left(\overline{x}\right)$ with $\overline{\beta}_k > 0$, $k \in B_0\left(\overline{y}\right)$ in (3-1).

Let Assumption (A2) be fulfilled throughout this section ($\overline{y}$ is then called a *nondegenerate* stationary point of $LL\left(\overline{x}\right)$). Then, a moment of reflection shows that the partial Jacobian $D_{(y,\alpha,\beta)}F(x, y, \alpha, \beta)$ of the system

$$F(x,y,\alpha,\beta)=\left[\begin{array}{l} D_y g(x,y)+\sum\limits_{l\in A}\alpha_l Du_l(y)+\sum\limits_{k\in B_0(\overline{y})}\beta_k Dv_k(y) \\ u_l(y),\ l\in A \\ v_k(y),\ k\in B_0(\overline{y}) \end{array}\right]$$

is nonsingular at $(x,y,\alpha,\beta)=(\overline{x},\overline{y},\overline{\alpha},\overline{\beta})$. A straightforward application of the Implicit Function Theorem provides the existence of an open neighbourhood $V^{\overline{y}}$ of $\overline{x}$ and uniquely determined continuously differentiable functions $\widetilde{y},\widetilde{\alpha},\widetilde{\beta}$ defined on $V^{\overline{y}}$ such that

$$\left(\widetilde{y}(\overline{x}),\widetilde{\alpha}(\overline{x}),\widetilde{\beta}(\overline{x})\right)=(\overline{y},\overline{\alpha},\overline{\beta})\ \text{ and}$$

$$F\left(x,\widetilde{y}(x),\widetilde{\alpha}(x),\widetilde{\beta}(x)\right)=0 \text{ for all } x\in V^{\overline{y}}.$$

In particular, we get for the locally defined function $\widetilde{g}(x)=g(x,\widetilde{y}(x))$, $x\in V^{\overline{y}}$ that

$$D\widetilde{g}(x)=D_x g(x,y)\big|_{y=\widetilde{y}(x)}\ ,\ x\in V^{\overline{y}}$$

and we also obtain an explicit formula for $D^2\widetilde{g}(x)$ [6, 12, 13, 14] (although the function $\widetilde{y}(\cdot)$ is only implicitly known). Now, for our purpose we define the reduction approach as follows.

Definition 3.1 The *reduction approach* (briefly: RA) is said to hold at $\overline{x}\in M$ if Assumption (A2) is fulfilled for all $\overline{y}\in Y_0(\overline{x})$.

Note that (a more general variant of) the reduction approach can be formulated under the weaker condition that not all multipliers $\overline{\beta}_k\geq 0$, $k\in B_0(\overline{x})$ in (3-1) are positive (see [6, 12, 13, 14]). The above construction and the compactness of the index set Y yield the following corollary which summarizes some important properties under the reduction approach.

Corollary 3.1 Assume that RA holds at $\overline{x}$. Then
(i) The set $Y_0(\overline{x})$ is finite, say $Y_0(\overline{x})=\{y^1,\ldots,y^p\}$.
(ii) There exist an open neighbourhood V of $\overline{x}$ and uniquely determined continuously differentiable functions

$$\widetilde{y}^j: x\in V\mapsto \widetilde{y}^j(x)\in\mathbb{R}^r,\ j=1,\ldots,p$$

such that $\widetilde{y}^j(\overline{x})=y^j$ and $\widetilde{y}^j(x)$, $j=1,\ldots,p$ is a nondegenerate local minimizer of $LL(x)$ for all $x\in V$.
(iii) We have $g\left(\cdot,\widetilde{y}^j(x)\right)\in C^2(V,\mathbb{R})$, $j=1,\ldots,p$.
(iv) $M\cap V=\left\{x\in V\mid g\left(x,\widetilde{y}^j(x)\right)\geq 0,\ j=1,\ldots,p\right\}$. △

Property (iv) in Corollary 3.1 means that the feasible set M of the semi-infinite problem SIP can, locally around $\overline{x}$, be described by *finitely many* C^2-inequality constraints. In the following we will assume that RA holds at our points under consideration. By [20, 21], this assumption is natural in the sense that it is fulfilled generically for each local minimizer of the original problem SIP.

4 A logarithmic smoothing approach

Throughout this section let $\overline{x} \in M$ and assume
- that (A1) is fulfilled,
- that RA holds at $\overline{x}$ and
- that $\overline{x}$ is a strongly stable stationary point of SIP.

The latter condition implies that EMFCQ holds at $\overline{x}$. According to Corollary 3.1, let $Y_0(\overline{x}) = \{y^1, \ldots, y^p\}$ and

$$M \cap V = \{x \in V \mid g(x, \widetilde{y}^j(x)) \geq 0,\ j = 1, \ldots, p\} \tag{4-1}$$

where V and $\widetilde{y}^j$, $j = 1, \ldots, p$ are defined as in Corollary 3.1. The description (4-1) of $M \cap V$ by finitely many C^2-inequality constraints allows to generalize *locally around* $\overline{x}$ the logarithmic smoothing approach which was introduced in [4, 10] for finite problems. In this section we will briefly introduce this technique and present its main results which are local generalizations of the results in [10]. We omitted all proofs of these results in this short note since they are straightforward generalizations of the proofs in [10].

The main idea is to consider a family of interior point approximations of $M \cap V$ which depends on an approximation parameter $\gamma \in \mathbb{R}_+$ in the following way. Define the local interior-point-set as

$$M^\circ = \{x \in M \cap V \mid g(x, \widetilde{y}^j(x)) > 0,\ j = 1, \ldots, p\}$$

as well as

$$G(x) = \sum_{j=1}^{p} \ln g(x, \widetilde{y}^j(x)) \quad \text{and}$$

$$M^\gamma \cap V = \{x \in M^\circ \mid G(x) - \ln \gamma \geq 0\}. \tag{4-2}$$

Note that $G(x) - \ln \gamma \geq 0$ is equivalent to $\prod_{j=1}^{p} g(x, \widetilde{y}^j(x)) - \gamma \geq 0$.
Furthermore, let the local approximation of (P) be given as

$$P_{\overline{x}}^{\gamma} \quad \text{minimize} \ \ f(x) \ \ \text{s.t.} \ \ x \in M^\gamma \cap V. \tag{4-3}$$

This construction provides the following properties:
- The feasible set $M^\gamma \cap V$ is defined by a *single* continuously differentiable inequality constraint.
- If EMFCQ holds at all $x \in M \cap V$, then the set $\{x \in V \mid G(x) = \ln \gamma\}$ is a smooth approximation (since it is described by only one constraint) of the (in general, non-smooth) intersection of V and the boundary ∂M of M.
- There is a strong relationship between the problem $P_{\overline{x}}^{\gamma}$ in (4-3) and a standard interior-point approach. It is easy to see that the first order necessary optimality conditions of $P_{\overline{x}}^{\gamma}$,

$$Df(x) - \mu DG(x) = Df(x) - \mu D\left(\sum_{j=1}^{p} \ln g\left(x, \widetilde{y}^j(x)\right)\right) = 0, \ \mu \geq 0 \qquad \text{(4-4)}$$

are also the first order necessary optimality conditions of the interior point problem

$$\text{minimize} \ \ f(x) - \mu \sum_{j=1}^{p} \ln g\left(x, \widetilde{y}^j(x)\right) \ \text{ s.t. } \ x \in M^\circ \qquad \text{(4-5)}$$

(where $\mu \geq 0$ is a corresponding interior-point-parameter).

The following theorem contains the main result. It states that for all sufficiently small $\gamma \in \mathbb{R}_+$ the problem $P_{\overline{x}}^{\gamma}$ has a uniquely determined strongly stable stationary point which converges to $\overline{x}$ as $\gamma \to 0$.
Analogously to [15], we consider the following two cases:
• ELICQ holds at $\overline{x}$.
• EMFCQ holds at $\overline{x}$ but ELICQ does not hold at $\overline{x}$.

Theorem 4.1 *(see [10, Theorems 3.1 and 3.2] for finite problems)*
(i) Assume that ELICQ holds at $\overline{x}$. Then, there exist $\overline{\delta} \in \mathbb{R}_+$ and $\overline{\gamma} \in \mathbb{R}_+$ such that for all $\gamma \in (0, \overline{\gamma}]$ we have that
• there exists a stationary point $x(\gamma) \in B_{\overline{\delta}}(\overline{x})$ of $P_{\overline{x}}^{\gamma}$ and $x(\gamma)$ is the only stationary point of $P_{\overline{x}}^{\gamma}$ in $B_{\overline{\delta}}(\overline{x})$;
• $x(\gamma)$ is a strongly stable stationary point of $P_{\overline{x}}^{\gamma}$ and $\overline{x}$ and $x(\gamma)$ have the same stationary index (hence, $\overline{x}$ is a local minimizer or saddle point of SIP if and only if $x(\gamma)$ is a local minimizer or saddle point of $P_{\overline{x}}^{\gamma}$, respectively);
• the mapping

$$\gamma \in (0, \overline{\gamma}) \mapsto (x(\gamma), \mu(\gamma)) \in B_{\overline{\delta}}(\overline{x}) \times \mathbb{R}_+ \qquad \text{(4-6)}$$

is continuously differentiable with

$$\lim_{\gamma \to 0} (x(\gamma), \mu(\gamma)) = (\overline{x}, 0),$$

where $\mu(\gamma)$ is the uniquely determined Lagrange multiplier for $x(\gamma)$ (see (4-4)).
(ii) Assume that EMFCQ holds at $\overline{x}$ but ELICQ does not hold at $\overline{x}$. Then, there exist $\overline{\delta} \in \mathbb{R}_+$ and $\overline{\gamma} \in \mathbb{R}_+$ such that for all $\gamma \in (0, \overline{\gamma}]$ we have that
• there exists a stationary point $x(\gamma) \in B_{\overline{\delta}}(\overline{x})$ of $P_{\overline{x}}^{\gamma}$ and $x(\gamma)$ is the only stationary point of $P_{\overline{x}}^{\gamma}$ in $B_{\overline{\delta}}(\overline{x})$;
• $x(\gamma)$ is a strongly stable stationary point of $P_{\overline{x}}^{\gamma}$ with $s.i.\,(x(\gamma)) = s.i.\,(\overline{x}) = 0$ (hence, $\overline{x}$ and $x(\gamma)$ is a local minimizer of SIP and $P_{\overline{x}}^{\gamma}$, respectively);
• the mapping

$$\gamma \in (0, \overline{\gamma}) \mapsto (x(\gamma), \mu(\gamma)) \in B_{\overline{\delta}}(\overline{x}) \times \mathbb{R}_+$$

is continuous with

$$\lim_{\gamma \to 0} (x(\gamma), \mu(\gamma)) = (\overline{x}, 0),$$

where $\mu(\gamma)$ is the uniquely determined Lagrange multiplier for $x(\gamma)$. △

The following remarks refer to several properties related to the results in the previous theorem.

Remark 4.1 A moment of reflection shows that the only constraint of $P^{\gamma}_{\overline{x}}$ is active at $x(\gamma)$, that is, $G(x(\gamma)) - \ln\gamma = 0$. Furthermore, we have that $DG(x)\big|_{x=x(\gamma)}$ is non-vanishing and $\mu(\gamma) > 0$ which means that $x(\gamma)$ is a non-degenerate (strongly stable) stationary point of $P^{\gamma}_{\overline{x}}$. It is remarkable that $x(\gamma)$ and $\overline{x}$ have the same stationary index although the number of active constraints at $x(\gamma)$ and $\overline{x}$ is one and p, respectively (remember that the stationary index depends on the restriction of the Hessian of the Lagrangian to the corresponding tangent space, see Lemma 2.1).

Remark 4.2 As mentioned above, the first order necessary optimality conditions (4-4) of $P^{\gamma}_{\overline{x}}$ are also the first order necessary optimality conditions of the interior point problem (4-5). Theorem 4.1 states that the mapping in (4-6) provides a continuously differentiable solution path for this interior point approach.

The following corollary refers to some local topological properties (where, perhaps, a shrinking of the open neighbourhood V of $\overline{x}$ is necessary). This corollary is a local version of [10, Proposition 3.2].

Corollary 4.1 There exist a neighbourhood $\widetilde{V} \subset V$ of $\overline{x}$ and a $\overline{\gamma} \in \mathbb{R}_+$ such that for all $\gamma \in (0, \overline{\gamma})$ we have:
- EMFCQ holds at all $x \in M^{\gamma} \cap \widetilde{V}$ (with respect to the description (4-2)),
- $M \cap \widetilde{V}$ and $M^{\gamma} \cap \widetilde{V}$ are homeomorphic,
- $M^{\gamma} \cap \widetilde{V}$ converges to $M \cap \widetilde{V}$ in the Hausdorff metric as $\gamma \to 0$, that is,

$$\sup_{x \in M \cap \widetilde{V}} \inf_{z \in M^{\gamma} \cap \widetilde{V}} \|x - z\| \to 0 \text{ as } \gamma \to 0. \qquad \triangle$$

So far, all results in this section are related to a *local* neighbourhood of the strongly stable stationary point $\overline{x}$ of SIP. In the following corollary we consider a *global* aspect related to the whole feasible set M. We will assume that M is a compact set and that all stationary points of SIP are strongly stable. These conditions imply that there exists only a finite number of stationary points of SIP, say $\overline{x}^1, \ldots, \overline{x}^t$. This final corollary is a straightforward generalization of [10, Corollary 3.1] to our semi-infinite setting under the reduction approach.

Corollary 4.2 Assume that M is a compact set and that all stationary points of SIP are strongly stable; say, the set of stationary points of SIP is $\{\overline{x}^1, \ldots, \overline{x}^t\}$. Then, there is a $\overline{\gamma} \in \mathbb{R}_+$ such that for all $\gamma \in (0, \overline{\gamma}]$ there exist
- a set of corresponding problems $P^{\gamma}_{\overline{x}^i}$ each locally (and analogously to $P^{\gamma}_{\overline{x}}$) defined on a neighbourhood V^i of $\overline{x}^i$, $i = 1, \ldots, t$, and
- corresponding locally uniquely determined stationary points $\overline{x}^i(\gamma)$ of $P^{\gamma}_{\overline{x}^i}$, $i = 1, \ldots, t$,
- where $\overline{x}^i(\gamma)$ are strongly stable and have the same stationary index as $\overline{x}^i$, $i = 1, \ldots, t$. $\triangle$

5 Conclusions

In this note we generalized a logarithmic smoothing approach to a semi-infinite setting where the generic condition of the reduction approach is assumed at the strongly stable stationary point under consideration. This logarithmic smoothing approach was originally developed for finite optimization problems; it uses a family of logarithmic barrier functions and is closely related to a standard interior-point approach. We discussed the convergence of a solution path and presented topological properties and global aspects.

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