ON VULNERABILITY MEASURES OF NETWORKS[1]

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Abstract

As links and nodes of interconnection networks are exposed to failures, one of the most important features of a practical networks design is fault tolerance. Vulnerability measures of communication networks are discussed including the connectivities, fault diameters, and measures based on Hosoya-Wiener polynomial. An upper bound for the edge fault diameter of product graphs is proved.

Key words: *Persistence of profitability, Insurance industry, Croatia*

1. INTRODUCTION

In the design of large interconnection networks several factors have to be taken into account. Optimal design is important both to achieve good performance and to reduce the cost of construction and maintenance. Practical communication networks are exposed to failures of network components. Both failures of nodes and failures of connections between them happen and it is desirable that a network is robust in the sense that a limited number of failures does not break down the whole system. A lot of

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work has been done on various aspects of network fault tolerance, see for example the survey [8] and more recent papers [20,34,37]. In particular the fault diameter with faulty vertices which was first studied in [25] and the edge fault diameter has been determined for many important networks recently [11, 12, 26, 35]. In particular, the (vertex) fault diameter and the edge fault diameter of Cartesian graph products and Cartesian graph bundles were studied recently [2, 3, 4, 5]. Usually either only edge faults or only vertex faults are considered, while the case when both edges and vertices may be faulty is studied rarely. For example, [20, 34] consider Hamiltonian properties assuming a combination of vertex and edge faults. Research of mixed fault diameters was initiated in [6, 14]. An even more basic vulnerability measure is connectivity of the underlying graph, more precisely, the (vertex) connectivity, edge connectivity and, more general, mixed connectivity. Among other vulnerability measures we wish to mention here two measures that are closely related to the Wiener number [46], a well-known graphs invariant that has been extensively studied in chemical graph theory [16, 17, 47, 48]. The Wiener number is just the sum of all distances in the graph so it is clear that it is of some importance when studying communication networks. Below we will mention two examples that we find interesting because they relate seemingly so diverse topics as chemistry and communications.

The rest of the paper is organized as follows. In the next section we recall some basic definitions. In Section 3 and Section 4 we discuss some vulnerability measures. Finally, in Section 5 we study one of the measures more closely, and give a proof of a bound on edge-fault diameter of a product graph.

2. DEFINITIONS

Here we only recall some basic definitions to fix the notation, for other standard notions not defined here we adopt the usual terminology (see for example [1]). A *simple graph* $G = (V, E)$ is determined by a *vertex set* $V = V(G)$ and a set $E = E(G)$ of (unordered) pairs of vertices, called the set of *edges*. As usual, we will use the short notation uv for edge $\{u, v\}$. For an edge $e = uv$ we call u and v its *endpoints*. Weighted graphs can be used in chemical graph theory to model molecules with heteroatoms [10, 23, 24, 27, 33, 42] but also can obviously be used elsewhere, i.e. when modeling communication networks. A *weighted graph* $G=(V, E,w, \lambda)$ is a combinatorial object, which has, besides the set $V = V(G)$ of vertices and a set $E = E(G)$ edges, two *weighting functions*, w and λ. Usually, $w : V(G) \rightarrow \mathrm{IR}^+$ assigns positive real numbers (weights) to vertices and $\lambda : E(G) \rightarrow \mathrm{IR}^+$ assigns positive real numbers (lengths) to edges. It is often convenient to consider the union of *elements* of a graph, $S(G)=V(G) \cup E(G)$. Given $X \subseteq S(G)$ then $S(G) \setminus X$ is a subset of elements of G. However, note that in general $S(G) \setminus X$ may not induce a graph. As we need notation for subgraphs with some missing (faulty) elements, we will formally define $G \setminus X$, the subgraph of G after deletion of

X, as follows: Let $X \subseteq S(G)$, and $X = X_E \cup X_V$, where $X_E \subseteq E(G)$ and $X_V \subseteq V(G)$. Then $G \setminus X$ is the subgraph of $(V(G), E(G) \setminus X_E)$ induced on vertex set $V(G) \setminus X_V$. A *walk* between x and y is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ where $x = v_0$, $y = v_k$, and $e_i = v_{i-1}v_i$ for each i. The *length* of the walk W is the sum of the lengths of its edges, $\ell(W)=\sum_{i=1}^{k} \lambda(v_{i-1}v_i)$. In the special case when all edges have weight 1, $\ell(W)$ is the number of edges in W. A walk with all vertices distinct is called a *path*, and the vertices v_0 and v_k are called the *endpoints* of the path. A path P in G, defined by a sequence $x = v_0, e_1, v_1, e_2, v_2,\ldots, v_{k-1}, e_k, v_k = y$ can alternatively be seen as a subgraph of G with $V(P) = \{v_0,v_1,v_2,\ldots,v_k\}$ and $E(P) = \{e_1,e_2,\ldots,e_k\}$. Note that the reverse sequence gives rise to the same subgraph. Hence we use P for a path either from x to y or from y to x. The *distance* $d_G(u, v)$, or simpler $d(u, v)$, between vertices u and v in graph G is the length of a shortest walk between u and v. If there is no such path, we write $d(u, v)= \infty$. The *diameter* of graph G is the maximal distance in G, $D(G) = \max_{u,v\in V(G)} d_G(u, v)$.

3. VULNERABILITY MEASURES

Below we recall some vulnerability measures. The list is not meant to be complete; the choice is guided by our previous work and the wish to mention some open problems and avenues for future research thus supporting the talk at the conference KOI'12.

3.1. Connectivity

A graph is *connected* if there is a path between each pair of vertices, and is *disconnected* otherwise. The *connectivity* (or *vertex connectivity*) of a connected graph G, $\kappa(G)$, is the minimum cardinality over all vertex-separating sets in G. As the complete graph K_n has no vertex-separating sets, we define $\kappa(K_n)= n - 1$. We say that G is *k-connected* (or *k-vertex connected*) for any $k \le \kappa(G)$. The *edge connectivity* of a connected graph G, $\lambda(G)$, is the minimum cardinality over all edge-separating sets in G. A graph G is said to be *k-edge connected* for any $k \le \lambda(G)$. In other words, the edge connectivity $\lambda(G)$ of a connected graph G is the smallest number of edges whose removal disconnects G, and the (vertex) connectivity $\kappa(G)$ of a connected graph G (other than a complete graph) is the smallest number of vertices whose removal disconnects G. It is well-known that (see, for example, [1], page 224) $\kappa(G) \le \lambda(G) \le \delta_G$, where δ_G is the smallest vertex degree of G. Thus if a graph G is k-connected, then it is also k-edge connected. The reverse does not hold in general. For a later reference recall that by Menger's theorems (see, for example, [1], pages 230, 234) we know that in a k-connected graph G there are at least k vertex disjoint paths between any two vertices in G, and if G is k-edge connected then there are at least k edge disjoint paths between any two vertices in G. *Mixed connectivity* is defined as follows [14].

Definition 1 *Let G be any connected graph. A graph G is (p, q)+connected, if G remains connected after removal of any p vertices and any q edges.*

We wish to remark that the mixed connectivity studied here is closely related to connectivity pairs as defined in [7], see also [6, 13]. Briefly speaking, a connectivity pair of a graph is an ordered pair (k, ℓ) of two integers such that there is some set of k vertices and ℓ edges whose removal disconnects the graph and there is no set of $k - 1$ vertices and ℓ edges or of k vertices and $\ell - 1$ edges with this property. Clearly (k, ℓ) is a connectivity pair of G exactly when: (1) G is $(k - 1, \ell)$+connected, (2) G is $(k, \ell - 1)$+connected, and (3) G is not (k, ℓ)+connected. In fact, as shown in [13], (2) implies (1), so (k, ℓ) is a connectivity pair exactly when (2) and (3) hold.

From the definition we easily observe that any connected graph G is $(0,0)$+connected, $(p, 0)$+connected for any $p < \kappa(G)$ and $(0,q)$+connected for any $q < \lambda(G)$. In our notation $(i,0)$+connected is the same as $(i+1)$-connected, i.e. the graph remains connected after removal of any i vertices. Similarly, $(0, j)$+connected means $(j+1)$-edge connected, i.e. the graph remains connected after removal of any j edges.

Clearly, if G is a (p, q)+connected graph, then G is (p',q')+connected for any $p' \leq p$ and any $q' \leq q$. Furthermore, for any connected graph G with $k < \kappa(G)$ faulty vertices, at least k edges are not working. Roughly speaking, a graph G remains connected if any faulty vertex in G is replaced with a faulty edge. It is known [13] that if a graph G is (p, q)+connected and $p > 0$, then G is $(p-1,q+1)$+connected. Hence for $p > 0$ we have a chain of implications: (p,q)+connected $\rightarrow$ $(p-1,q+1)$+connected $\rightarrow \ldots \rightarrow$ $(1, p+q-1)$+connected $\rightarrow$ $(0,p+q)$+connected, which generalizes the well-known proposition that any k-connected graph is also k-edge connected. Therefore, a graph G is (p, q)+connected if and only if $p < \kappa(G)$ and $p+q < \lambda(G)$. For any (p,q)+connected graph we have $p+q < \lambda(G) \leq \delta_G$, thus each vertex of a (p, q)+connected graph has at least $p + q + 1$ neighbors, and hence (p, q)+connected graph has at least $p + q + 2$ vertices.

If for a graph G $\kappa(G) = \lambda(G) = k$, then G is (i, j)+connected exactly when $i + j < k$. However, if $2 \leq \kappa(G) < \lambda(G)$, the question whether G is (i, j)+connected for $1 \leq i < \kappa(G) \leq i + j < \lambda(G)$ is not trivial. It is interesting to note that in general the knowledge of $\kappa(G)$ and $\lambda(G)$ is not enough to decide whether G is (i, j)+connected.

Mixed connectivity is a generalization of vertex and edge connectivity: a graph G is $(p, 0)$+connected for all $p < \kappa(G)$ and is not $(p, 0)$+connected for $p \geq \kappa(G)$. Furthermore, G is $(0,q)$+connected for all $q < \lambda(G)$ and is not $(0,q)$+connected for $q \geq \lambda(G)$. In particular, any graph G is $(\kappa(G) - 1, 0)$+connected and $(0, \lambda(G) - 1)$+connected.

Problem [13]. It is well known that both (vertex) connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$ can be computed efficiently. It is an open question whether it is possible to efficiently decide whether a graph G is (a, b)+connected.

3.2. Edge, vertex, and mixed fault-diameters

Definition 2 *Let G be a k-edge connected graph and $0 \leq a < k$. The a-edge fault-diameter of G is*

$$D_a^E(G) = \max\{D(G \setminus X) \mid X \subseteq E(G), |X| = a\}.$$

Definition 3 *Let* G *be a k-connected graph and $0 \leq a < k$. The a-fault diameter (or a-vertex fault-diameter) of G is*

$$D_a^V(G) = \max\{D(G \setminus X) \mid X \subseteq V(G), |X| = a\}.$$

Note that $D^E_a(G)$ is the largest diameter among diameters of subgraphs of G with a edges deleted, and $D^V_a(G)$ is the largest diameter over all subgraphs of G with a vertices deleted. In particular, $D^E_0(G) = D^V_0(G) = D(G)$, the diameter of G.

For $a \geq \kappa(G)$, the a-vertex fault-diameter of a graph G does not exist, and for $b \geq \lambda(G)$, the b-edge fault-diameter of a graph G does not exist. We write $D^E_a(G) = \infty$, $D^V_b(G) = \infty$ as some of the graphs are not edge-connected or vertex-connected,

respectively.

Remark 4 *It is easy to see that for any connected graph G the inequalities below hold.*

1. $D(G) = D^E_0(G) \leq D^E_1(G) \leq D^E_2(G) \leq \ldots \leq D^E_{\lambda(G)-1}(G) < \infty$.
2. $D(G) = D^V_0(G) \leq D^V_1(G) \leq D^V_2(G) \leq \ldots \leq D^E_{\kappa(G)-1}(G) < \infty$.

Note that, intuitively, one may expect $D^E_a(G) \leq D^V_a(G)$ because deleting a vertices in a connected graph always means that at least a edges were deleted. However, this is not the case. Namely, there are examples of graphs where $D^E_a(G) = D^V_a(G)+1, D^E_a(G) = D^V_a(G)$, *and* $D^E_a(G) < D^V_a(G)$ showing that the bound of Theorem 5 is tight.

Theorem 5 *[6] Let G be a k-connected graph and $0 < a < k \leq \kappa(G)$. Then*

$$D^E_a(G) \leq D^V_a(G) + 1.$$

Definition 6 *Let G be a (p, q)+connected graph. The (p, q)-mixed fault-diameter of G is*

$$D^M_{(p,q)}(G) = \max\{D(G \setminus X) \mid X = X_E \cup X_V,\ X_E \subseteq E(G),\ X_V \subseteq V(G),\ |X_V| = p,\ |X_E| = q\}$$

Note that by Definition 6 the endpoints of edges of set X_E can be in X_V . In this case we actually get a subgraph of G with a vertices and less than b edges deleted, but it is not difficult to see that the diameter of such subgraph is smaller or equal to the diameter of some subgraph of G where exactly a vertices and exactly b edges are deleted. So the condition that the endpoints of edges of set X_E are not in X_V is not necessary to be included in Definition 6.

Remark 7 *The mixed fault-diameter $D^M_{(p,q)}(G)$ is the largest diameter among diameter of subgraphs of G with q edges and p vertices deleted, hence $D^M_{(0,0)}(G) = D(G)$, $D^M_{(0,a)}(G) = D^E_a(G)$ and $D^M_{(a,0)}(G) = D^V_a(G)$.*

Remark 8 *Let $H^V_a = \{G \setminus X \mid X \subseteq V(G), |X| = a\}$ and $H^E_b = \{G \setminus X \mid X \subseteq E(G), |X| = b\}$. It is easy to see that*

1. $\max \{D^E_b(H) \mid H \in H^V_a\} = D^M_{(a,b)}(G)$,

2. $\max \{D^V_b(H) \mid H \in H^E_b\} = D^M_{(a,b)}(G)$.

For more relations among the fault-diameters see [6, 14]. Later we will discuss in more detail the bounds for fault diameters of product graphs.

4. VULNERABILITY MEASURES BASED ON THE HOSOYA-WIENER POLYNOMIAL

The Wiener number (or, Wiener index) $W(G)$ of a graph G with vertex set $\{v_1, v_2, ..., v_n\}$ is defined as the sum of distances between all pairs of vertices of G,

$$W(G) = \frac{1}{2}\sum_{i=1,\, j=1}^{n} d(v_i, v_j).$$

In more than 60 year after H. Wiener discovered remarkable correlation between the value $W(G)$ of the molecular graph G and some chemical properties of the molecule [46], the Wiener number and related graph invariants very extensively studied [16,17,47,48]. A closely related notion is the Hosoya-Wiener polynomial of a graph G which is defined as

$$H(x) = H(G,x) = \sum_{u,v \in V(G)} x^{d(u,v)}.$$

This definition, which is used for example in [18], slightly differs from the definition used by Hosoya [19] (see also [32]):

$$\hat{H}(x) = \hat{H}(G,x) = \sum_{u,v \in V(G); u \neq v} x^{d(u,v)}.$$

Obviously, $H(\lambda; x) = \hat{H}(\lambda; x) + |V(G)|$.

Perhaps the most interesting property of the Hosoya-Wiener polynomial is that its derivative at 1 equals the Wiener number. It is not difficult to prove the statement for weighted graphs.

Lemma 9 *[43]* $W(G) = H'(G, 1)$.

Below we recall two vulnerability measures that appear in studies of communication networks.

4.1. Reliability Wiener index

In [31] the *reliability Wiener index* is defined as follows. For two vertices $i, j \in V$ we denote the set of all directed paths from i to j with $P_{\vec{ij}}$ and with

$$F_{\vec{ij}} = \max_{P \in P_{\vec{ij}}} \{w(P)\},$$

the weight of the most reliable path from i to j. We can say that $F_{\vec{ij}}$ is the reliability of (i, j). Define

$$R_1^+(i) = \sum_{j=1}^{n} F_{\vec{ij}} \quad \text{the out-reliability of vertex } i,$$

$$R_1^-(i) = \sum_{j=1}^{n} F_{\vec{ji}} \quad \text{the in-reliability of vertex } i,$$

$$W_{R_1^+}(G) = \sum_{v \in V(G)} R_1^+(v) \quad \text{the out-reliability Wiener index of } G,$$

$$W_{R_1^-}(G) = \sum_{v \in V(G)} R_1^-(v) \quad \text{the in-reliability Wiener index of } G.$$

Obviously, in the case of a graph G, $R_1^- = R_1^+$ and $W_{R_1^-}(G) = W_{R_1^+}(G)$, so we can define the reliability Wiener index by

$$W_{R_1}(G) = \frac{1}{2}\sum_{v\in V(G)} R_1(v),$$

Where $R_1 := R_1^- = R_1^+$.

The problem of finding $F_{\vec{ij}}$ can be solved as follows:

$$R_1^+(i) = \sum_{j=1}^{n} F_{\vec{ij}} = \sum_{j=1}^{n} \exp(-l_{\vec{ij}}), \text{ where } l_{\vec{ij}} = \min_{P\in P_{\vec{ij}}}\{w'(P)\}, w'(P) = -\sum_{i=1}^{k} \ln(w(v_i, v_{i+1})).$$

The most reliable path from i to j in $G = (V, E, w)$ can be calculated by Dijkstra's algorithm on a weighted digraph $G' = (V, E, w')$ where $w': E \to IR^+$ is defined by $w'(i, j) = -\ln(w(i, j))$. The reliability of (i, j) in $G = (V, E, w)$ is

$$F_{\vec{ij}} = \exp(-l_{\vec{ij}}).$$

By Floyd's (or Dijkstra's) algorithm we can calculate $l_{\vec{ij}}$ on a weighted digraph $G' = (V, E, w')$.

Instead of using only one most reliable walk, one can take into consideration all walks between a pair of vertices. Denote by $W_k(\vec{ij})$ the set of all walks of length k from i to j and by $\mu_k(\vec{ij})$ the reliability of all walks of length k from i to j:

$$\mu_k(\vec{ij}) = \sum_{P\in W_k(\vec{ij})} w(P).$$

For a strongly connected weighted digraph G and a vertex i in G we can define

$$R_*^+(i) = \sum_{j=1, j\neq i}^{n} \sum_{k=0}^{\infty} \mu_k(\vec{ij}), \quad \text{the out-reliability of } i$$

and the index R_* of G:

$$R_*(G) = \sum_{v\in V(G)} R_*^+(v).$$

A sufficient condition for the convergence of the series $\sum_{k=0}^{\infty} \mu_k(\vec{ij})$ giving rise to a method for computing $R_*(G)$ is

Theorem 10 *[31] Let* $\mathbf{A} = (a_{ij})$ *be the adjacency matrix of a graph G, which entries are the weights of the edges. If* $\max_{i,j}\{a_{ij}\} \leq \frac{1}{n}$, *then*

$$R_*(G) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} ((\mathbf{I}-\mathbf{A})^{-1})_{ij}.$$

As can be seen, this index is easy to compute, but it is applicable only to networks with small weights.

While both definitions can be efficiently computed, the models have some drawbacks. In the first case, one considers only one path and ignores the possibility that there may be many paths that have the same reliability. Second definition takes into account all walks which means including the walks that visit the same vertex several times. Obviously, variants where systems of disjoint paths are considered may provide a better reliability measure. On the other hand, it is likely that the problems related will be computationally more challenging.

4.2. Residual closeness

The residual closeness (see [28] and the references there)

$$C = \sum_i C(i) = \sum_i \sum_{j \neq i} 2^{-d(i,j)},$$

has been proposed as a new concept of graph vulnerability. However, it is just the value of the Hosoya-Wiener polynomial at 1/2:

$$\hat{H}(1/2) = \sum_{u \neq v} \left(\frac{1}{2}\right)^{d(u,v)} = \sum_{u \neq v} 2^{-d(u,v)} = C.$$

Not surprisingly, the connection between Hosoya-Wiener polynomial and residual closeness is largely unknown.

Both examples show that the Hosoya-Wiener polynomial, a well known concept from chemical graphs theory, may be very useful when considering the properties of communications networks. Let us only mention that the polynomial is not computationally hard to compute, there are even linear time algorithms for special classes of graphs [44].

5. UPPER BOUNDS FOR FAULT-DIAMETERS OF PRODUCT GRAPH

The concept of fault diameter of Cartesian product graphs was first described in [25], but the upper bound was wrong, as shown by Xu, Xu and Hou who corrected the mistake [35]. Roughly speaking the upper bound is the sum of fault diameters of the factors plus one, and in addition, one more faulty vertex in the product is allowed. The same bound was later proved for graph bundles [2] and generalized to product graphs [36]. The formal definition of product graph is given below. In our notation, the most general version of the result reads

Theorem 11 *[36] Let B and F be k_F-connected, k_B-connected graphs and $0 \leq a < k_F$, $0 \leq b < k_B$. Then the vertex fault-diameter of $G = B * F$, is $D^V_{a+b+1}(G) \leq D^V_a(F) + D^V_b(B) + 1$.*

Generalized bounds of this type were proved for Cartesian product of more than two factors [4].

Analogous result for the edge fault-diameter will be proved in Section 5 while the case of Cartesian product of more than two factors has been solved in [3].

Theorem 12 *Let F and B be k_F-edge connected and k_B-edge connected graphs respectively, $0 \leq a < k_F$, $0 \leq b < k_B$. Then*

$$D^E_{a+b+1}(B * F) \leq D^E_a(F) + D^E_b(B) + 1.$$

It is conjectured that analogous bounds for mixed fault-diameters exist; the proof however seems to be more involved. A partial result appears in [14, 15].

5.1. Product graphs

Let B and F be graphs. The *product graph* $B * F$ is constructed by taking a copy of $F(v) \cong F$ for each vertex v of B and connecting two copies corresponding to adjacent vertices by a perfect matching φ_e. Formally, $\varphi : e = uv \in E(B) \to \varphi_e$, where φ_e is a bijection $V(F(u)) \to V(F(v))$. Here $F(v)$ stands for the copy of F associated with $v \in V(B)$. Obviously, such matchings assigned to the edge $e = uv$ give rise to two bijections: φ_{uv} is a bijection $V(F(u)) \to V(F(v))$ and φ_{vu} a bijection $V(F(v)) \to V(F(u))$, such that $\varphi_{vu} = \varphi_{uv}^{-1}$.

Remark. As the product is determined by two graphs B, F and φ, an assignment of bijections between copies of F to edges of B, the product graph should be denoted by

$$G = B *^{\varphi} F$$

but we will write as usual $G = B * F$, because there is no danger of confusion as we always work with a given product graph.

The product graphs $B * F$ were first defined in [8]. The concept generalizes several graph constructions as special cases. For example, if all bijections between adjacent copies of F are isomorphisms, the resulting graph is known as Cartesian graph bundle [5, 29, 30], and if all the isomorphisms are identities, the resulting graph is the Cartesian product $B \square F$ [21]. Another family of graphs that often appears in the literature are permutation graphs [9]; in terms of product graphs, for arbitrary G, the product $K_2 * G$ is a permutation graph.

In the Section 5 we will borrow some notions that are usual in the study of graph bundles (see, for example [5]). We define the mapping $p: G \rightarrow B$ that maps graph elements of G to graph elements of B, i.e. $p: V(G) \cup E(G) \rightarrow V(B) \cup E(B)$. In particular, here we also assume that the vertices of G are mapped to vertices of B and the edges of G are mapped either to vertices or to edges of B. We say an edge $e \in E(G)$ is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*. The mapping p will also be called the *projection* (of the product $G = B * F$ to its base B). For each $x \in V(G)$ its copy of F is denoted by F_x, formally, $F_x = p^{-1}(\{p(x)\})$. Recalling the notation $F(u)$, when referring to the copy of F which corresponds to the vertex $u \in V(B)$, we have $F(u)= p^{-1}(\{u\})$. Note that $F_x = F(p(x))$.

Let G be a graph, $x, y \in V(G)$ be distinct vertices, P be a path from x to y in G, and $z \in V(P) \setminus \{x, y\}$. We will use $x \xrightarrow{P} z$ to denote the subpath $\widetilde{P} \subseteq P$ from x to z. If z is adjacent to x in P, we will simply write $x \rightarrow z$. Given a graph G and $X \subseteq E(G)$, we say that a path P from a vertex x to a vertex y *avoids* X in G, if $E(P) \cap X = \emptyset$.

Let $G = B * F$, $u, v \in V(B)$ be distinct vertices of B, P be a path from u to v in B, and $x \in F(u)$. Then $\widetilde{P}(x)$ is the path from $x \in F(u)$ to a vertex in $F(v)$, such that $p(\widetilde{P}(x))= P$ and $\ell(\widetilde{P}(x)) = \ell(P)$. We call $\widetilde{P}(x)$ the *lift* of the path P to the vertex $x \in V(G)$. Let $u, v \in V(B)$ be distinct vertices, and P be a path from u to v in B. Then it is easy to see that 1. and 2. below hold.

1. If P_1 and P_2 are lifts of P to the same vertex $x \in V(G)$, then $P_1 = P_2$.

2. Let $x, x' \in F(u)$. Then $\widetilde{P}(x)$ and $\widetilde{P}(x')$ have different endpoints in $F(v)$ and are disjoint (edge and vertex disjoint) if and only if $x \neq x'$ (i.e. the lifts avoid each other).

It may be interesting to note that while it is well-known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) [21], there may be many different graph bundle representations of the same graph [40]. Note that in some cases finding a representation of G as a graph bundle can be found in polynomial time [22, 38, 39, 40, 41, 45]. For example, one of the easy classes are the Cartesian graph bundles over triangle-free base [22]. However, we are not aware of any attempts to design a recognition algorithm for product graphs.

Problem. What is the complexity of product graph recognition?

5.2. An upper bound on edge fault-diameter of product graph

In this section we prove a theorem that gives an upper bound for the edge fault-diameter of product graph $G = B * F$ in terms of its factors B and F. We will use the following technical lemma in the proof of Theorem 12.

Lemma 13 *Let $G = Q * F$ be the product of a path Q with vertices $V(Q) = \{v_0, v_1, ..., v_k\}$ and a graph F with $D^E_a(F) < \infty$. Let s and t be vertices of G with coordinates $s = (s_1 = v_0, s_2)$ and $t = (t_1 = v_k, t_2)$ and let $X \subseteq E(G)$ be a set of edges with $|X| \leq a + 1$. Then $d_{G \setminus X}(s, t) \leq D^E_a(F) + \ell(Q) + 1$.*

Proof. Let Q be a path, $V(Q) = \{v_0, v_1, ..., v_k\}$, $G = Q * F$, $k_F \geq a + 1$. Let s $\in F(v_0)$ and $t \in F(v_k)$ be vertices of G, and let $X \subseteq E(G)$ be a set of edges with $|X| \leq a + 1$. We distinguish two cases.

First, if $|F(v_k) \cap X| = a + 1$ then $|F(v_0) \cap X| = 0$ and $\tilde{Q}(t)$ avoids X. Hence there is a path R from s to the endpoint of the path $\tilde{Q}(t)$ within fibre $F(v_0)$ of length $\ell(R) \leq D^E_0(F) \leq D^E_a(F)$, and $\ell(\tilde{Q}(t)) = \ell(Q)$. Therefore $d_{G \setminus X}(s, t) \leq D^E_a(F) + \ell(Q) + 1$.

Second, assume $|F(v_k) \cap X| \leq a$. As F is at least $(a + 1)$-edge connected, there are at least $a + 1$ neighbors of s in $F(v_0)$. Denote the neighbors by u_i, $i = 1, 2, ..., a+1$. Among the $a+2$ edge-disjoint paths from s to vertices u'_i $(i = 1, 2, \cdots, a+2)$ in $F(v_k)$, constructed as

$$s \longrightarrow u_i \xrightarrow{\tilde{Q}(u_i)} u'_i$$

of length $1 + \ell(Q)$, and the path

$$s \xrightarrow{\tilde{Q}(s)} s' = u'_{a+2}$$

of length $\ell(Q)$, at least one avoids X. Without loss of generality assume that

$$P_1 : s \longrightarrow u_1 \xrightarrow{\tilde{Q}(u_1)} u'_1$$

avoids X. As $|F(v_k) \cap X| \leq a$, there is a path R in $F(v_k)$ avoiding X from u'_1 to t of length $\ell(R) \leq D^E_a(F)$. Therefore there is a path

$$P : s \xrightarrow{P_1} u'_1 \xrightarrow{R} t$$

from s to t of length $\ell(P) \leq 1 + \ell(Q) + D^E_a(F)$, and hence $d_{G \setminus X}(s, t) \leq D^E_a(F) + \ell(Q) + 1$. □

Now we give the proof of Theorem 12.

Proof. Let $k = a + b + 2$ and denote $G = B * F$. As $\lambda(G) \geq \lambda(F) + \lambda(B) \geq k_F + k_B \geq a + 1 + b + 1 > a + b + 1$, $D^E_{k-1}(G)$ is well-defined. Let δ_F be the minimum degree of F and δ_B be the minimum degree of B. Recall that $\delta_F \geq \lambda(F) > a$ and $\delta_B \geq \lambda(B) > b$. Let $X \subseteq E(G)$ be such that $|X| = k - 1 = a + b + 1$, and $x, y \in V(G)$ be two distinct vertices. We shall construct a path P from x to y in $G \setminus X$, with length $\ell(P) \leq D^E{}_a(F) + D^E{}_b(B) + 1$.

As before, let $p: G \rightarrow B$ be the projection from $G = B * F$ to B, so $p(X) \subseteq V(B) \cup E(B)$. Denote the set of degenerate edges in X by X_D, and the set of nondegenerate edges by X_N, $X = X_D \cup X_N$, $p(X_D) \subseteq V(B)$ and $p(X_N) \subseteq E(B)$. Let $|X_D| = a_0$ and $|X_N| = b_0$. Then $a_0 + b_0 = a + b + 1$.

1. First assume that x and y are in distinct copies of F, i.e. $p(x) \neq p(y)$. We now distinguish two cases.

Case A. If $b_0 < b$, then there is a path Q between $p(x)$ and $p(y)$ in B that avoids $p(X_N)$ of length $\ell(Q) \leq D^E{}_{b0}(B) \leq D^E{}_b(B)$. Let $y' \in F_x$ be the endpoint of the path $\tilde{Q}(y)$. If $|F_x \cap X_D| \leq a$, then there is a path R from x to y within F_x that avoids X_D of length $\ell(R) \leq D^E{}_a(F)$. Therefore there is a path P from x to y in $G \setminus X$

$$P : y \xrightarrow{\tilde{Q}(y)} y' \xrightarrow{R} x$$

of length $\ell(P) \leq D^E{}_b(B) + D^E{}_a(F)$.

If $|F_x \cap X_D| \geq a + 1$, then $|(G \setminus F_x) \cap X_D| \leq a_0 - (a + 1) = b - b_0$, so outside F_x we have at most $b - b_0$ degenerate edges of X. As B is $(b + 1)$-edge connected, and $b_0 < b$, there are at least $b + 1 - b_0$ neighbors of $p(x)$, such that the edges from $p(x)$ to these neighbors avoid $p(X_N)$. As there are more such neighbors than degenerate edges of X outside of F_x $(b + 1 - b_0 > b - b_0)$, there is a neighbor u of $p(x)$ in B, such that $|F(u) \cap X_D| = 0$ and $e = \{p(x), u\} \notin p(X_N)$. As $b_0 < b$, there is a path Q from u to $p(y)$ in B that avoids $p(X_N)$ and has length $\ell(Q) \leq D^E{}_{b0}(B) \leq D^E{}_b(B)$. Let $u' \in F(u)$ be the endpoint of the path $\tilde{Q}(y)$. As $|F(u) \cap X_D| = 0$, there is a path R from $\varphi_e(x)$ to u' within $F(u)$ of length $\ell(R) \leq D^E{}_{a0}(F) \leq D^E{}_a(F)$. Therefore there is a path P from y to x in $G \setminus X$

$$P : y \xrightarrow{\tilde{Q}(y)} u' \xrightarrow{R} \varphi_e(x) \longrightarrow x$$

of length $\ell(P) \leq D^E{}_b(B) + D^E{}_a(F) + 1$. Note that if $y \in F(u)$, P has a length $\ell(P) \leq D^E{}_a(F) + 1$.

Case B. Let $b_0 \geq b$. First we choose b edges in X_N, $\{e_1, e_2, ..., e_b\} \subseteq X_N$. Then there is a path Q from $p(x)$ to $p(y)$ in B that avoids $p(\{e_1, e_2, ..., e_b\})$, with length $\ell(Q) \leq D^E{}_b(B)$. Therefore the subgraph $p^{-1}(Q)$, which is isomorphic to $Q * F$, intersects X in at most $b_0 - b$ nondegenerate edges and at most a_0 degenerate edges of X, i.e. $|p^{-1}(Q) \cap X| \leq b_0 - b + a_0 = a + 1$. By Lemma 13, there is a path P from x to y with length $\ell(P) \leq D^E{}_a(F) + \ell(Q) + 1 \leq D^E{}_a(F) + D^E{}_b(B) + 1$.

2. To complete the proof, we have to consider the case where x and y are in the same fibre, i.e. $p(x) = p(y)$, and $F_x = F_y$. If $|F_x \cap X_D| \leq a$ then there is a path of length at most $D^E{}_a(F)$ within the fibre. If $a + 1 \leq |F_x \cap X_D|$, then $|X_N| = b_0 \leq b$. In this case, as before, there is a neighbor u of $p(x)$ in B, such that $|F(u) \cap X_D| = 0$ and $e = \{p(x), u\} \notin p(X_N)$. We may construct a path P as

$$P : x \longrightarrow \varphi_e(x) \xrightarrow{R} \varphi_e(y) \longrightarrow y,$$

and $\ell(P) \leq 1 + D^E{}_0(F) + 1 \leq D^E{}_b(B) + D^E{}_a(F) + 1$. □

The next simple example shows that the bound in Theorem 12 is tight.

Example 14 *Let $G = P_2 \square P_2$, see Figure 1. G is a graph bundle with fiber $F = P_2$ over the base graph $B = P_2$. Then for $a = b = 0$ we have $D^E{}_{a+b+1}(G) = 3$, and $D^E{}_b(B) + D^E{}_a(F) + 1 = 1 + 1 + 1 = 3$.*

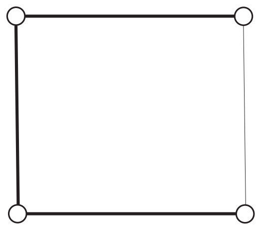

Figure 1: $G = P_2 \square P_2$ with one faulty link.

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