ISSN 1330-3651 (Print), ISSN 1848-6339 (Online)
UDK/UDC 624.042.8.073.8:517.984

SPECTRAL METHOD IN MOVING LOAD ANALYSIS OF KIRCHHOFF-LOVE PLATES

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Original scientific paper
This paper shows an application of the spectral method in dynamics of structures for the special case of thin plate under the action of a moving load. The spectral method formulated in terms of matrix operators is first described. The application of the method to the chosen model problem of the Kirchhoff-Love plate is illustrated through examples. We then elaborate upon dynamic analysis with moving loads. Again, some illustrative examples are used to present the application of spectral matrix operators. The examples include the Dirichlet and Neumann boundary conditions for simply supported - free plate, and complex loading conditions of a moving force. The boundary conditions have been imposed using Lagrange multipliers. The proposed approach based upon the spectral matrix operators is especially suitable for dealing with strong form of the problem. This is illustrated with strong form of thin plate structural dynamics equations. Moreover, the presented approach is sufficiently general and can be easily exploited for analysis of any other structure.

Keywords: *dynamic structural analysis, matrix operators, moving load, spectral method, strong formulation, thin plates*

Spektralna metoda u analizi Kirchhoff-Love ploča

Izvorni znanstveni članak
Ovaj rad pokazuje primjenu spektralne metode u dinamičkoj analizi konstrukcija za specijalan slučaj tanke ploče pod djelovanjem pokretnog opterećenja. Najprije je opisana spektralna metoda preko matričnih operatora. Primjena ove metode na odabrani model Korchhoff-Love ploče je objašnjena kroz primjere. Zatim slijedi razrada dinamičke analize s pokretnim opterećenjima. Opet, neki slikoviti primjeri su korišteni za prezentiranje primjene spektralnih matričnih operatora. Primjeri uključuju Dirichletove i Neumannove rubne uvjete za slobodno oslonjenu-slobodnu ploču, kao i složene uvjete opterećenja pokretnom silom. Rubni uvjeti su uvedeni preko Lagrangeovih multiplikatora. Predloženi pristup zasnovan na principu spektralnih matričnih operatora je osobito pogodan za rješavanje jake formulacije problema. To je pokazano na jakoj formulaciji dinamičkih jednadžbi za tanku ploču. Predstavljeni pristup je dovoljno općenit tako da se može lako iskoristiti za analizu bilo koje druge vrste konstrukcije.

Ključne riječi: *dinamička analiza konstrukcija, jaka formulacija, matrični operatori, pokretno opterećenje, spektralna metoda, tanke ploče*

1 Introduction

Dealing with moving loads is one of very demanding tasks in structural analysis. One is facing a non-conservative problem in dynamic analysis of structures described as a system of differential equations in the case of discrete representation of structural masses, or a partial differential equation in the case of continuous mass representation.

When using a discrete model for computing the solution, the problem has to be discretized in space and time. Time discretization is usually based on finite difference technique leading to time integration scheme of the Newmark type, like in [1]. However, there are some difficulties in the calculation of accelerations resulting from moving load and a modified Newmark could be advantageous [2]. For space discretization two choices are mostly used, depending on the formulation chosen. The most popular choice is the finite element discretization that requires weak (integral) formulation of the problem [3]. The strong form (partial differential equation) is usually discretized using finite differences. Accuracy of finite differences can be improved if spectral analysis is applied to the strong form (like Fourier transforms). In spite of their good properties spectral methods are just beginning to emerge in engineering applications. There is a similar situation regarding analysis of plates under the moving load; most authors prefer to model their structures as beams [4], 5, 1]. However, there are formulations capable of dealing with 2 and 3 dimensional structures; quite a general formulation is presented in [1]. They work in convected coordinates and with weak formulation of the problem.

The approach proposed in this paper is based upon the Chebyshev spectral method [7]. The main advantage of this method against the Fourier method is that there is no need for special representation (transformation) of loading. Moreover, we show that the Chebyshev spectral method can been somewhat modified, so that it can be completely formulated using matrix operators and thus in terms of solution steps it resembles the classical stiffness matrix approach used in structural analysis.

Boundary conditions are expressed through Lagrange multipliers and can be of the Dirichlet or Neumann type, or any combination of these two, like in the plate example. This kind of approach also allows for nonholonomic conditions (e.g. boundary conditions involving velocity). Although they have not been treated in this paper, they could be important for moving load analysis [8]. One drawback of the method of Lagrange multipliers for imposing the boundary conditions is increase in size of the problem to be solved. However, this is not so pronounced, since spectral methods require quite a modest number of equations. More important drawback is possible creation of a stiff system [9], which results in certain boundary conditions, like supported and free plate in the example at the end of the paper. The resulting system is so stiff that the penalty approach that works very well for beams could not be used. However, some simple numerical tricks were enough to allow the use of standard equation solvers for the system of equations expanded with the Lagrange multipliers.

The paper is organized as follows. The second chapter is divided into several parts. The first part presents the Chebyshev spectral method using matrix operators. In the second part special attention is given to the thin plate equation. The third part deals with boundary conditions that are entirely expressed in matrix form using Lagrange multipliers and a Kronecker matrix product. The last part describes the moving load function incorporated into the Newmark integration procedure and the Chebyshev spectral method. The third chapter consists of three examples presented in three parts. All three examples deal with a plate simply supported on two opposite sides and free on the other sides. Such a plate is considered relevant since it could represent a bridge with boundary conditions that are quite general (arithmetic combination of the Dirichlet and Neumann boundary conditions) to cover all practical cases. The first part is about a static analysis of the plate, in the second one eigenvalues and eigenvectors are calculated and in the third one we perform analysis for a load moving over the plate. Verification of all the three examples is performed by comparing results for a very narrow plate with known analytic (exact) results for a beam. Concluding remarks are given at the end of the paper.

2 Spectral method formulation

2.1 Spectral matrix operators

Spectral method has been chosen to replace the series expansion used in the Fourier analysis solution of differential equations. Spectral methods offer high precision with minimal number of points used in spatial discretization.

In this work the Chebyshev polynomials are chosen for spatial interpolation of the domain of the differential equation although there are other possibilities as well. The Chebyshev polynomial is polynomial of degree N defined in points x_j according to the equation

$$p(x)=\prod_{j=0}^{N}(x-x_j),\quad x_j=\cos\left(j\cdot\frac{\pi}{N}\right),\quad j=0,1...N. \tag{1}$$

Detailed description can be found in specialized literature [10]. Here we will address some details specific to analysis of plates under the moving load.

Application of spectral method consists in solution of the strong formulation (differential equation) of structural (static or dynamic) problem. The method can be formulated using matrix differentiation operators.

$$p_x=\boldsymbol{D}_N\cdot\boldsymbol{p}, \tag{2}$$

where $\boldsymbol{p}$ is a vector of discrete data of size N, p_x is its derivative and $\boldsymbol{D}_N$ is matrix differential operator, a square matrix of size $[N\times N]$. Such matrix differential operator can be constructed for various methods (e.g. finite differences). For spectral method based on the Chebishev polynomials, entries of matrix $\boldsymbol{D}_N$, [7] are

$$\begin{aligned}&\boldsymbol{D}_{N00}=\frac{2N^2+1}{6},\quad \boldsymbol{D}_{NNN}=-\frac{2N^2+1}{6},\\&\boldsymbol{D}_{Nij}=\frac{-x_j}{2\cdot\left[1-(x_j)^2\right]},\quad j=1...(N-1),\\&\boldsymbol{D}_{Nij}=\frac{c_i}{c_j}\cdot\frac{-1^{i+j}}{x_i-x_j},\quad i\neq j,\quad j=1...N,\\&c_i=\left\|\begin{array}{l}2\text{ if } i=0\vee i=N\\1\text{ otherwise.}\end{array}\right.\end{aligned} \tag{3}$$

Spectral methods produce full differentiation matrix, which means that all points are involved in getting the result. One may argue that that is the reason for high accuracy of spectral methods.

With this matrix at hand, the solution of an ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}u=f(x), \tag{4}$$

is reduced to a solution of the linear system of equations. In this paper the solution of the system of linear equations will be formally represented by matrix inverse

$$u=\boldsymbol{D}_N^{-1}\cdot\boldsymbol{f}, \tag{5}$$

where $\boldsymbol{f}$ is vector of $f(x)$ evaluated at points x_j, (j = 0, N). Certainly, (5) is more efficiently solved using some other procedure instead of the matrix inversion.

Boundary conditions also have to be incorporated in $\boldsymbol{D}_N$. We can note the analogy where the matrix differential operator plays the role of the stiffness matrix, but it is produced using an entirely different procedure.

Normally, structural equations require higher derivatives and two dimensional modelling requires partial derivatives and modifications of matrix operators. In modelling of plates there is a need for the second, third and fourth order derivatives in the x and y directions. Higher order operators are simply produced using matrix multiplications while expansion in more than one direction can be obtained using Kronecker product of two matrices. In the case when there are N points in the x direction and M points in the y direction and numeration is consecutive in the x direction, differential operators are, respectively

$$\boldsymbol{D}_x=\boldsymbol{I}_M\oplus\boldsymbol{D}_N,\quad \boldsymbol{D}_y=\boldsymbol{D}_M\oplus\boldsymbol{I}_N, \tag{6}$$

which are the Kronecker products of unit matrices and differential operators. It is to be mentioned that the Kronecker product is not commutative. Also, if one matrix is a unit matrix, then the product simply rearranges and expands the original matrix. In our example $[N\times N]$ matrix times $[M\times M]$ matrix produces a matrix of the size $[NM\times NM]$.

2.2 Application to thin plate equation

The thin plate equation (where w stands for plate deflection) is

$$\Delta\Delta w=\frac{\partial^4}{\partial x^4}w+2\cdot\frac{\partial^2}{\partial x^2}\cdot\frac{\partial^2}{\partial y^2}w+\frac{\partial^4}{\partial y^4}w=q(x,y). \tag{7}$$

Eq. (7) is discretized using the Chebyshev polynomials with N points in the x direction and M points in the y direction (Fig. 1). Transferring into differential operator form and using matrices and products defined above, we get

$$\Delta\Delta w=\left(\boldsymbol{I}_M\oplus\boldsymbol{D}_N^4\right)+2\cdot\left(\boldsymbol{D}_M^2\oplus\boldsymbol{I}_N\right)\cdot\left(\boldsymbol{I}_M\oplus\boldsymbol{D}_N^2\right)+\left(\boldsymbol{D}_M^4\oplus\boldsymbol{I}_N\right)=Q_{xy}. \quad (8)$$

Powers of differential operators (matrices) give us higher derivatives.

The Chebyshev polynomials (and differential operators) have evaluation points that are not evenly spaced; they are denser near the boundaries. Fig. 1 shows arrangement of evaluation points on plate with normalised dimensions.

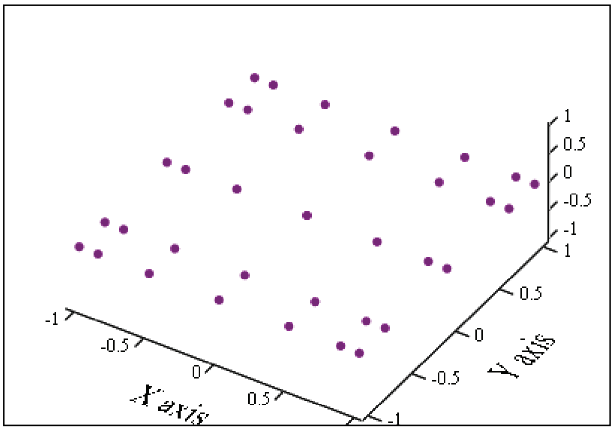

Figure 1 Square plate in normalized coordinates and discretized with the Chebyshev polynomials ($N = 6$, $M = 4$)

The arrangement of points appeared not to be a problem. Solution of a simply supported plate using spectral method with discretization $N = M = 4$ produces deflections that are accurate to 0,1 %. This has been achieved solving only 9 equations. Since results are so accurate, interpolations could be used for evaluation of extra points within the domain.

2.2.1 Boundary conditions using Lagrange multipliers

Only the simple boundary conditions could be incorporated into the matrix $\boldsymbol{D}_N$ (or $\boldsymbol{D}_M$). For the simple support condition (homogeneous Dirichlet boundary condition) that consists of removing the first and the last rows and columns from the matrix the resulting problem is reduced in size and homogenous condition is simply restored after the solution. Non-homogeneous boundary conditions could be obtained by introducing 1s on the diagonal and replacing the corresponding row and column with zeros. However, the Neumann type of boundary condition could scarcely be enforced directly into the matrix differential operator. Plate with two free ends requires more elaborated boundary conditions [11]. We require that the reaction and the moment along the free boundary y vanish

$$\frac{\partial^3}{\partial y^3}w+(2-\nu)\cdot\frac{\partial^2}{\partial y^2}\cdot\frac{\partial}{\partial y}w=0,\quad \frac{\partial^2}{\partial y^2}w+\nu\cdot\frac{\partial^2}{\partial x^2}w=0. \quad (9)$$

These boundary conditions have to be translated into differential operators

$$\left(\boldsymbol{D}_M^3\oplus\boldsymbol{I}_N\right)+(2-\nu)\cdot\left(\boldsymbol{D}_M^2\oplus\boldsymbol{I}_N\right)\cdot\left(\boldsymbol{I}_M\oplus\boldsymbol{D}_N\right)=0, \quad (10)$$

$$\left(\boldsymbol{D}_M^2\oplus\boldsymbol{I}_N\right)+\nu\cdot\left(\boldsymbol{I}_M\oplus\boldsymbol{D}_N^2\right)=0. \quad (11)$$

Differential operators from (10) and (11) act on the same points on the boundary (in each point on the boundary two conditions have to be fulfilled). That is the additional reason for the application of Lagrange multipliers in enforcing the boundary conditions. Operators in (10) and (11) have full size of the problem but they do not act on all the points of the plate. The extra points are removed through extraction of only those rows that belong to the degrees of freedom where the desired boundary condition is present. In our example that leaves us with two matrix operators of the size [$2N \times NM$]. Together with matrix operators resulting from the Dirichlet boundary conditions on the two supported sides of the plate they are assembled into constraint matrix $\boldsymbol{C}$ used in the Lagrange multiplier method

$$\boldsymbol{C}=\begin{bmatrix} I_{PM} \\ I_{PN}\cdot\left[\left(\boldsymbol{D}_M^3\oplus\boldsymbol{I}_N\right)+(2-\nu)\cdot\left(\boldsymbol{D}_M^2\oplus\boldsymbol{I}_N\right)\cdot\left(\boldsymbol{I}_M\oplus\boldsymbol{D}_N\right)\right] \\ I_{PN}\left[\left(\boldsymbol{D}_M^2\oplus\boldsymbol{I}_N\right)+\nu\cdot\left(\boldsymbol{I}_M\oplus\boldsymbol{D}_N^2\right)\right]\end{bmatrix}. \quad (12)$$

I_{PN} and I_{PM} are purging matrices for extraction of N and M points respectively and C is of size [$(2M+2N+2N)\times NM$].

2.3 Application to moving load equation

By applying the D'Alembert's principle differential equation describing two-dimensional structural behaviour under the moving load is obtained.

$$\rho\cdot\frac{\partial^2}{\partial t^2}u+\frac{1}{K}\cdot\left(\frac{\partial^4}{\partial x^4}u+2\cdot\frac{\partial^2}{\partial x^2}\cdot\frac{\partial}{\partial y^2}u+\frac{\partial^4}{\partial y^4}u\right)= \\ =p(t)\cdot\delta(x-v_x\cdot t)\cdot\delta(y-v_y\cdot t), \quad (13)$$

$u=u(x,y,t)$ is displacement in space and time, ρ is surface density, K is plate stiffness, δ is Dirac function. Load description can be simplified if we assume that it moves along one axis only. Using spectral operator (in (8)) for space discretization and assuming load is moving along one coordinate only, we can write using sub matrices

$$\begin{pmatrix}\boldsymbol{M} & 0\\ 0 & 0\end{pmatrix}\cdot\frac{\partial^2}{\partial t^2}\begin{pmatrix}u\\ \lambda\end{pmatrix}+\begin{pmatrix}\boldsymbol{\Delta\Delta} & \mathrm{C}^{\mathrm{T}}\\ \boldsymbol{C} & 0\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{u}\\ \lambda\end{pmatrix}=\begin{pmatrix}F(x,t)\\ 0\end{pmatrix}, \\ F(x,t)=p(t)\cdot\delta(x-v_x\cdot t). \quad (14)$$

$\boldsymbol{M}$ is mass matrix, 0s are zero matrices of the appropriate size, $\boldsymbol{\Delta\Delta}$ is spectral operator playing the role of the stiffness matrix, $\boldsymbol{\lambda}$ is vector of Lagrange multipliers, $\boldsymbol{u}$ is the displacement and $F(x,t)$ is loading function. The main characteristic of the moving load problem described with (14) is the right hand side. It is convenient to perform loading discretization prior to the time integration procedure (after time integration parameters, like Δt are set). After it has been discretized in space and time, solution of the dynamic equation can proceed using any time integration scheme. (Note: space discretization has to obey properties of the Dirac function δ. That results with a requirement of a constant force within one time increment). In the case of a constant moving force space and time discretization of the loading is presented in Fig. 2. It is straightforward to implement a non-constant force (like in [4])) since there are no restrictions on the forcing function $p(t)$.

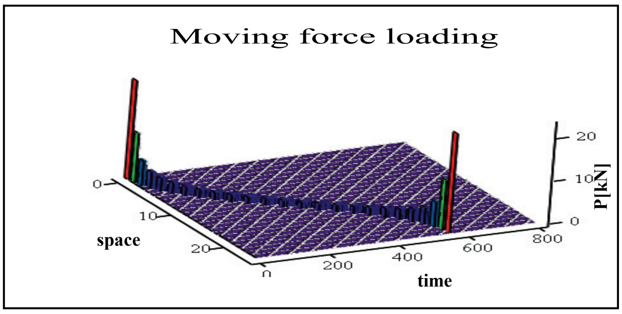

Figure 2 Space discretization of a constant moving force (in Chebishev coordinates)

Note that due to the Chebyshev polynomials used in space discretization even the constant loading does not have equal amplitude in every time increment. It is straightforward to include other forms of moving load such as force changing in time etc. After the appropriate form for the right hand side (the loading) has been found, solution procedure can be applied.

In order to apply Newmark class of integration schemes the governing equation (14) has to be rewritten in the incremental form

$$\begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 0^{-20} \end{pmatrix} \cdot \frac{\partial^2}{\partial t^2} \begin{pmatrix} \delta u \\ \delta \lambda \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Delta\Delta} & \boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{C} & 0^{-20} \end{pmatrix} \cdot \begin{pmatrix} \delta u \\ \delta \lambda \end{pmatrix} = \begin{pmatrix} \delta F \\ 0 \end{pmatrix}, \quad (15)$$

where δu is the displacement increment and δF is loading increment calculated from the discretized loading function. Mass and "stiffness" matrices could be rather stiff for some boundary conditions and introduction of 0^{-20} (very small number in place of 0s) can sometimes improve the behaviour of numerical procedures (especially eigenvalue analysis). Additionally $\boldsymbol{\Delta\Delta}$ matrix can be stabilized by introduction of rigid body modes [12]. In our case numerical procedures in MathCAD 13 performed well with 0s but those in MathCAD 11 required such a trick. In order to asses quality of the solution condition number for spectral operators has been calculated. Among many possibilities we have chosen L1 norm based on singular values of a matrix. In contrast to condition numbers based on other norms it can be evaluated for singular matrices as well. Actually, we are dealing with nearly singular matrices but due to limited calculation capabilities the practical performance can be similar to singular matrices. In Table 1 there are condition numbers for some examples. Beam is simply supported beam, plate 4 is plate simply supported on 4 sides (with different aspect ratios) and plate 2 is plate simply supported on 2 sides (which is numerically a much more difficult example due to boundary conditions on free ends).

Table 1 Sizes and condition numbers of spectral stiffness matrices for various structures

	Beam	Plate 4 1:1	Plate 4 1:5	Plate 2 1:1	Plate 2 1:5
Dirichlet	N=49 $1{,}493\times10^{10}$	N=961 $4{,}263\times10^{8}$	N=713 $5{,}025\times10^{7}$	n.a.	n.a.
Singular	N=51 $7{,}761\times10^{20}$	N=1089 $7{,}411\times10^{19}$	N=825 $1{,}362\times10^{20}$	N=1089 $3{,}797\times10^{20}$	N=825 $1{,}053\times10^{20}$
r.b.m.	N=51 $1{,}750\times10^{20}$	N=1089 $1{,}633\times10^{21}$	N=825 $4{,}480\times10^{19}$	N=1089 $9{,}66\times10^{19}$	N=825 $8{,}853\times10^{19}$
Lagrange	N=55 $8{,}804\times10^{15}$	N=1345 $1{,}226\times10^{27}$	N=1049 $1{,}813\times10^{27}$	N=1353 $7{,}234\times10^{33}$	N=1057 $2{,}926\times10^{32}$
Penalty	N=51 $1{,}235\times10^{17}$	N=1089 $2{,}889\times10^{16}$	N=825 $2{,}630\times10^{16}$	N=1089 $5{,}194\times10^{20}$	N=825 $4{,}393\times10^{20}$

In the first column of Tab. 1 there is an explanation of boundary conditions: "Dirichlet" – homogenous boundary conditions imposed through removal of points where displacement values are known to be zero, "singular" – full sized matrix without any boundary conditions, it is singular, "r.b.m." – above matrix with removal of rigid body modes, not singular anymore, "Lagrange" – r.b.m. matrix expanded with boundary conditions and Lagrange multipliers, "Penalty" – r.b.m. matrix with boundary conditions imposed through penalty number (α from 10^8 to 10^{10}).

Note: condition number alone cannot give the whole picture about matrix usability. It is evident from Table 1 that the removal of rigid body modes can worsen the condition number but at the same time matrix becomes non-singular and can be used in calculations. Also, introduction of Lagrange multipliers seems to deteriorate the performance but it is only the multiplier part of the matrix that is badly conditioned, the rest behaves well and the results are quite acceptable. Introduction of techniques that would separate Lagrange multipliers from the rest of the matrix would further confirm this statement. For example, the condition number for Plate 2 is smaller for penalty method than for the Lagrange multipliers method, but penalty procedure gives completely wrong results in that example.

3 Numerical examples

All examples are analysing the same plate: rectangular plate simply supported on two opposite sides and free on the other two; length = 10,0 m, width = 5,0 m, thickness = 0,10 m; material properties Y = 120 000 000 N/m^2, ν = 0,15. Various discretization resolutions have been applied, from N=12, M=6 to N= 24, M=12 in steps of 4 points for length and 2 for width. That resulted in 91 to 325 equations for the plate (or $(N+1)\cdot(M+1)$) and 80 to 152 equations for Lagrange multipliers (or $2(2N+2M+4)$). Also, results for very narrow plate (width = 0,25 m and ν = 0,0) have been compared with those for a beam for all three types of analysis; very good agreement has been observed but the results are not presented here.

Attention is needed in graphical presentation of results since they all come in the Chebyshev coordinates. To obtain what we are used to, they have to be mapped into regular rectangular coordinates (global coordinates, the same as used for plate geometry).

3.1 Static plate analysis

Static analysis consisted of analysis for equally spread loading and for concentrated force. Even for the lowest discretization in the first case accuracy is better than 0,1 % and in the second it is better than 1 %. Poorer performance in the case of concentrated force is due to several reasons, the most important one being that the concentrated force is not consistent loading for plate differential equation (7) and has to be spread around the point where it acts.

Fig. 3 presents deflections of the plate in global and the Chebishev coordinates. Width of the plate is chosen such that the mid point deflection is just beginning to visibly vary in the *y* direction (in order to have interesting pictures since parametric analysis is not the subject of this paper). More pronounced deflection in the perpendicular direction could be obtained by reducing the Poisson or further increasing the width.

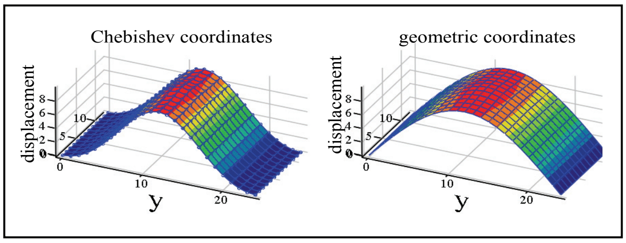

Figure 3 Deflections of plate supported at two sides

Fig. 4 demonstrates bending moments in plate in transversal and longitudinal direction and Fig. 5 shear forces for the same example.

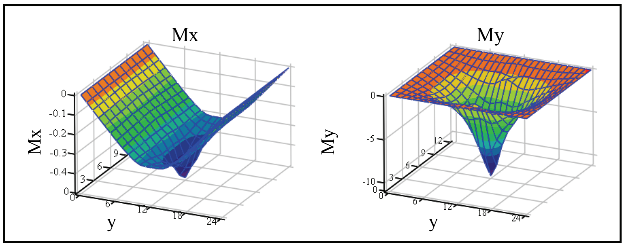

Figure 4 Moments of plate supported at two sides

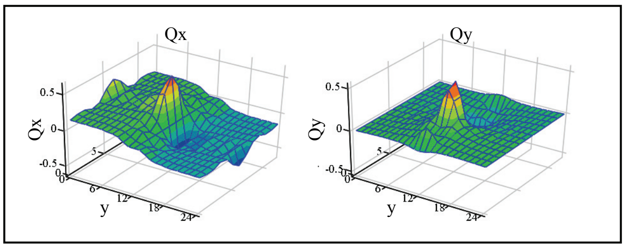

Figure 5 Shear forces of plate supported at two sides

Fig. 4 and 5 show bending moments and shear forces of plate after the transformation from the Chebishev back into the real coordinates. As the width of the plate diminishes, moments approach those of the simply supported beam.

3.2 Eigenvalue analysis

Eigenvalues and eigenvectors are obtained as a solution of the generalized eigenproblem

$$\begin{pmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \omega \cdot \begin{pmatrix} \boldsymbol{\Delta\Delta} & \boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix}. \tag{16}$$

In (16) ω is an eigenvalue. This problem is transformed into a standard eigenvalue problem by pre-multiplying with the inverse of either the expanded mass matrix or expanded spectral operator. The problem is that they are not invertible so a special technique has to be used. In this work we wanted to have all the eigenvalues (for comparison purposes) and (left) pseudo inverse of the stiffness operator has been calculated [13]. Although it is much easier to calculate pseudo inverse of the expanded mass matrix, results are much less stable. Naturally, in the solution there will be as many zero eigenvalues as there are Lagrange multipliers.

In Fig. 6 there is eigenvector of the second mode of vibration of the plate. Graphical representation of vibration modes is a good test of the solution since any instability in the eigenproblem formulation is immediately visible in the shape of the eignevector. Besides, second mode of vibration is a good indicator of the plate's behaviour regarding its width. With the second mode vibrating in the transversal direction the plate can be considered wide and with the second mode oscillating in longitudinal direction, beam-like behaviour could be expected.

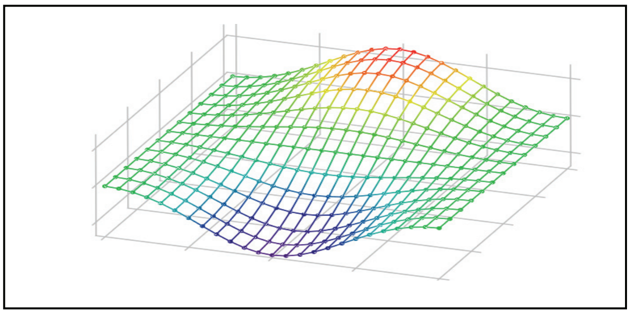

Figure 6 Second mode of vibration of the plate

3.3 Plate under moving load

This is example of the same plate under the action of the moving load. The load moves along the middle line of the plate and changes in space and time as presented in Figure 3b. Analysis in time domain is performed for Δt = 0,005 s and 800 time steps (total time of the analysis is 2 s). Analysis is extended in time for 400 cycles after the load leaves the plate (1 second).

Time analysis has been performed with a variation of the Newmark integration method (impulse acceleration method from [2]). This calculation was performed without structural damping but it can be easily added. Fig. 7 and 8 present the result of the analysis. Interpolation points between the Chebyshev nodes have been introduced for easier evaluation of results and for better graphical presentation (as in Figs. 4 and 5). Deflection in time of the middle line is compared with the exact solution that exists for beam and excellent agreement was found.

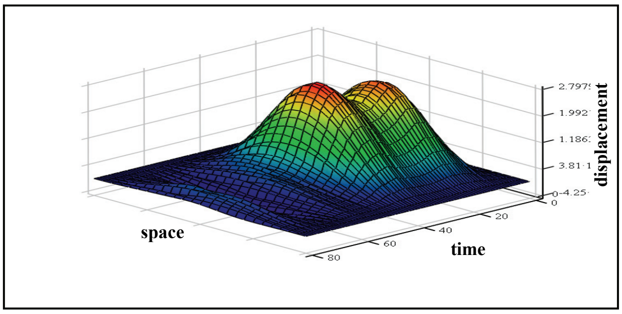

Figure 7 Displacement in time of the mid-line of plate under the moving load

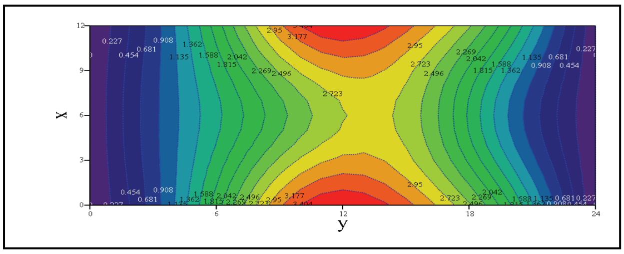

Figure 8 Displacement in time of the mid-line of plate under the moving load

It is possible to use spectral discretization in time as well as in space (as in [14]) but that brings complications similar to those that finite strips have.

4 Conclusions

In this paper discretization in space is obtained by the Chebyshev spectral method that is implemented in the form of a matrix differential operator. Resulting discretization in its form resembles the stiffness matrix and time discretization can be performed using any suitable method. The spectral differential matrix operator is fast and simple to construct; it is sufficient to construct one dimensional operator and expand it into two or more dimensions using the formalism of matrix Kronecker product. The resulting matrix is rather dense and small in size since spectral methods achieve high accuracy with a modest number of points. Boundary conditions can be treated in several ways but Lagrange multipliers offer the most general approach suitable even for the most complicated boundary conditions. Several methods have been tried out and the condition numbers of the resulting matrix operators are presented. Penalty method requires special mentioning since it has the best condition number but fails in complicated cases where the number of constraints is large (compared to the size of the stiffness operator). That remains the subject of further investigation.

The proposed procedure has been tried on a dynamic example of moving load analysis with very good results and realistic behaviour of plate under the moving load. The matrix operator formalism of the spectral method produces accurate results while retaining small size of the problem; it is very suitable for integration of all strong forms of engineering problems.

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