Blanuša double*

A. ORBANIĆ[†], T. PISANSKI[‡], M. RANDIĆ[§] AND B. SERVATIUS[¶]

Abstract. *A snark is a non-trivial cubic graph admitting no Tait coloring. We examine the structure of the two known snarks on 18 vertices, the Blanuša graph and the Blanuša double. By showing that one is of genus 1, the other of genus 2, we obtain maps on the torus and double torus which are not 4-colorable. The Blanuša graphs appear also to be a counter example for the conjecture that the orientable genus of a dot product of n Petersen graphs is $n-1$ (Tinsley and Watkins, 1985). We also prove that the 6 known snarks of order 20 are all of genus 2.*

Key words: *Blanuša snark, Blanuša double snark, dot product, embeddings*

AMS subject classifications: 05C10, 05C62

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1. Introduction

Ever since Tait [13] proved that the four-color theorem is equivalent to the statement that every planar bridgeless cubic graph is edge 3-colorable, snarks, i.e. non-trivial cubic graphs possessing no proper edge-3-coloring, have been investigated. For an explanation of the term non-trivial in the definition of a snark, see [7]. The smallest snark is a graph on 10 vertices, namely the Petersen graph. It was used by Petersen in 1898, [10], but appears already in Kempe's paper [5], see *Figure 1*. The Petersen graph appeared in the chemical literature as the graph that depicts a rearrangement of trigonal bipyramid complexes XY5 with five different ligands when axial ligands become equatorial and equatorial ligands become axial ([9], [11]).

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†Department of Mathematics, University of Ljubljana, Jadranska 19,1 000 Ljubljana, Slovenia, e-mail: `Alen.Orbanic@fmf.uni-lj.si`

‡Department of Mathematics, University of Ljubljana, Jadranska 19,1 000 Ljubljana, Slovenia, e-mail: `Tomaz.Pisanski@fmf.uni-lj.si`

§Dept. of Mathematics & Computer Sci., Drake University, Des Moines, IA 50311, U.S.A., e-mail: `mrandic@msn.com`

¶Mathematics Department, Worcester Polytechnic Institute, 100 Institute Road, Worcester, MA 01609-2280, U.S.A., e-mail: `bservat@wpi.edu`

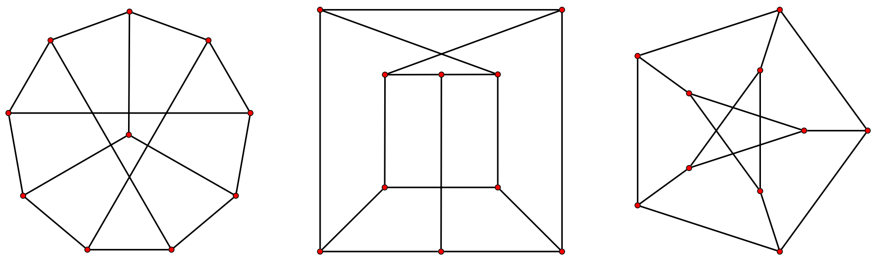


Figure 1. *Kempe's (left), Blanuša's (middle) and a standard rendering of Petersen's graph*

The second snark appearing in the literature is a graph on 18 vertices, discovered by the Croatian mathematician Danilo Blanuša, [1]. It is constructed from two copies of the Petersen graph by a construction generalized in [3] to obtain infinite families of snarks. Blanuša's construction can be applied to the Petersen graph in two different ways yielding two snarks on 18 vertices. We call the first one *the Blanuša snark* and the second one *the Blanuša double.*

Tutte conjectured that, in fact, every snark contains the Petersen graph as a minor. Robertson, Seymour and Thomas proved in [12] that Tutte's conjecture is true in general provided it is true for two special kinds of cubic graphs that are almost planar. In 2001 Robertson, Sanders, Seymour and Thomas announced a proof of Tutte's conjecture, which has not yet appeared in the literature.

An interesting and still open conjecture on snarks and their embeddings into orientable surfaces is Grünbaum's conjecture [2] which is a generalization of the 4-color theorem.

Conjecture 1 [Grünbaum [2]]. *Every embedding of a snark in an orientable surface has a cycle of length 1 or 2 (a loop or a pair of parallel edges) in the dual.*

2. The dot product

The dot product, $S_1 \,.\, S_2$, of two snarks S_1 and S_2 is defined in [3] as the graph obtained from S_1 and S_2 by removing two non-incident edges e and f from S_1 and two adjacent vertices v and w from S_2 and joining the endpoints of e and f to neighbors of u and v like in *Figure 2.* Performing this operation using the Petersen graph for both S_1 and S_2, yields, depending on the choice of the deleted edges, the Blanuša snark and the Blanuša double.

Both graphs are cubic graphs on 18 vertices and are difficult to identify. Actually in most drawings the two are hard to distinguish. In [16], for instance, the figure of the same graph appears twice. In this note we show that the genus of the Blanuša snark is 1 while the genus of its double is 2. A computer search for the number of 1-factors (Kekulé structures) shows that there are $K = 19$ Kekulé structures in the Blanuša snark and $K = 20$ in its double.

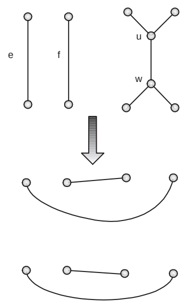


Figure 2. *The dot product*

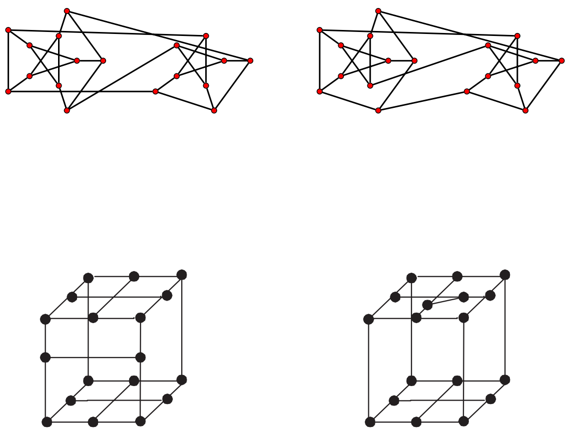


Figure 3. *The Blanuša snark and its double*

3. Genus embeddings

The simplest non-orientable surface, in which we can embed the Petersen graph is the projective plane (see *Figure 4*). The embedding in this case is pentagonal and highly symmetric. In order to exhibit a toroidal embedding, one has to provide a suitable collection of facial walks. We obtain 3 pentagonal faces, one hexagonal face and one nonagonal. We can represent this symbolically as : $5^3 6^1 9^1$. In the following tables the numbers denote the vertices and each row represents a face described as a sequence of vertices. The left half of *Figure 4* is a drawing using this combinatorial face structure and vertex labelling.

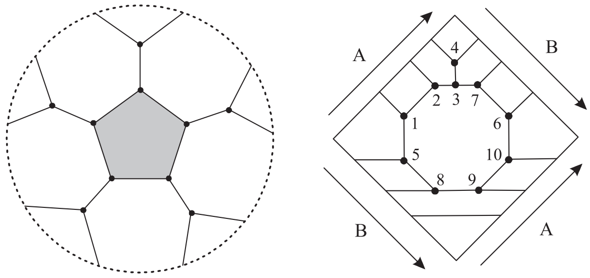


Figure 4. *The Petersen graph in the projective plane and in the torus*

The Petersen snark on the torus:

1	6	10	4	5				
2	3	4	10	9				
7	8	5	4	3				
6	7	3	2	1	5	8	9	10
1	2	9	8	7	6			

In order to prove that a graph is embeddable in surface S, we have to find a combinatorial embedding, i.e. a list of faces so that the Euler characteristic is that of S. Such a face list can be produced for the torus by the program EMBED[8] written by one of authors using the algorithm described in [4].

To prove non-embeddability of a graph in a surface we use the fact that there is a finite list of minimal forbidden minors for each surface (see [6]). For the plane the forbidden minors are K_5 and $K_{3,3}$. Unfortunately, a complete list of minimal forbidden minors for the torus is currently not known.

We again used the program EMBED to find candidates for minimal forbidden minors for torus embeddings: Starting with graph G one sequentially removes and contracts edges. If a removal or a contraction of an edge produces a toroidal graph then the operation is ignored. Otherwise the process is repeated on the obtained graph. When no removal or contraction of any edge is possible without obtaining a toroidal graph, we have obtained a candidate for a minimal forbidden minor. To document this process on the labelled graphs below, we use the following notation:

```
n : (a b c) : i j k l
```

means that the vertex labelled n in the minor was obtained by contracting all edges in a connected component on the vertices labelled a, b, c in the original graph. Vertex n is adjacent to vertices i, j, k, l in the minor. Isolated vertices of the minor are omitted.

For the snarks in the sequel it appears that the obtained candidates for forbidden minors all contain subdivisions of $K_{3,3}$ and we make use of the fact that the Kuratowski graph $K_{3,3}$ admits exactly two essentially different unlabelled embeddings into the torus, which are shown on fundamental polygons for the torus in *Figure 5*.

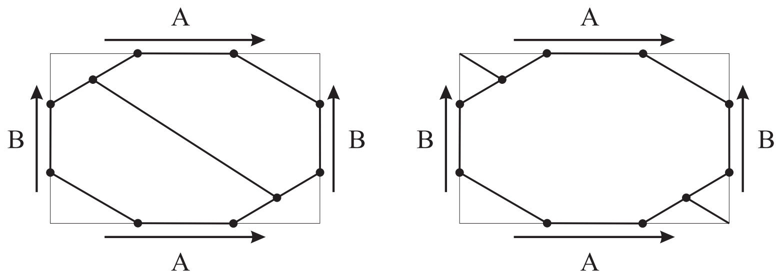


Figure 5. *Two embeddings of $K_{3,3}$ into the torus*

The first embedding in *Figure 5* is a cellular embedding, while the other is not, since the facewalk along the decagon does not correspond to a cycle of $K_{3,3}$.

Theorem 1. *The genus of the Blanuša snark is 1 while the genus of the Blanuša double is 2.*

Proof. We find embeddings of the Blanuša snark and its double by giving the collection of facial walks (i.e. combinatorial faces).

The Blanuša snark on the torus:

7	8	4	5	6			
1	2	3	4	8			
18	14	15	16	17			
11	12	13	14	18			
1	8	7	15	14	13		
5	9	10	3	2	6		
17	16	12	11	10	9		
7	6	2	1	13	12	16	15
10	11	18	17	9	5	4	3

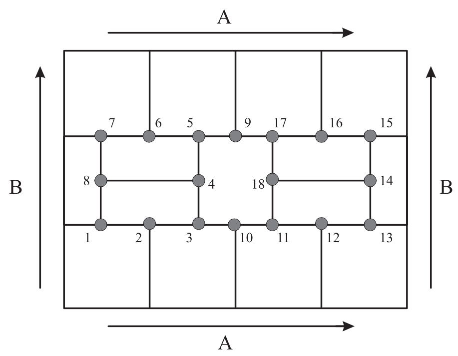


Figure 6. *The Blanuša snark on the torus*

The Blanuša double on the double torus:

9	10	8	5	7											
14	11	16	18	15	13										
1	4	8	10	3	2										
9	7	6	12	16	11										
9	11	14	17	3	10										
1	15	18	17	14	13	12	6	4							
1	2	5	8	4	6	7	5	2	3	17	18	16	12	13	15

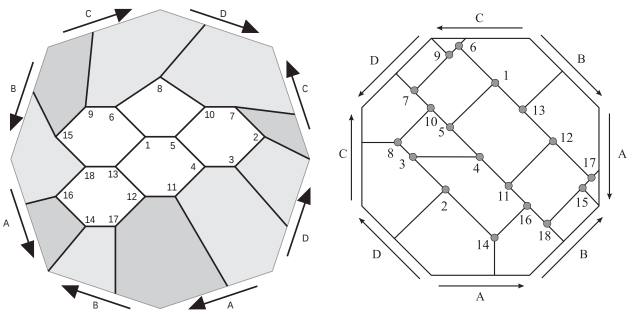
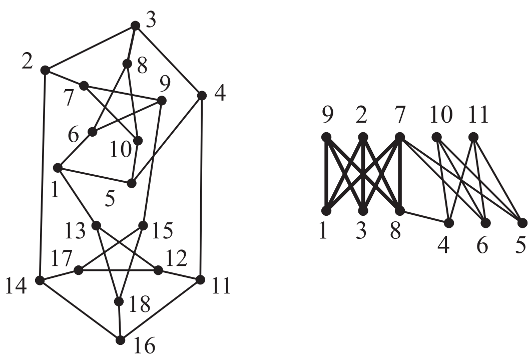


Figure 7. *Two drawings of the Blanuša double on the double torus*

For the labelled Blanuša double graph in *Figure 8* the depicted forbidden minor returned by the program EMBED was obtained as follows.



```
Vertices: 11, Edges: 5
1: (8): 7 9 2
2: (10): 3 1 8
3: (5): 9 7 2
4: (15): 10 11 8
5: (12): 7 11 10
6: (16): 7 10 11
7: (2 3 4 11): 3 8 1 5 6
8: (7 9): 2 7 9 4
9: (6 1): 8 1 3
10: (13 18): 5 4 6
11: (17 14): 4 5 6
```

Figure 8. *Blanuša double and its minimal forbidden minor for a torus*

The minor in *Figure 8* is not toroidal because vertices $4, 5, 6, 10, 11$ would have to be embedded in a face of $K_{3,3}$. For the cellular embedding that leads to a contradiction to Kuratowki's theorem right away, since the face containing the specified vertices together with the boundary would contain another subdivision of $K_{3,3}$. If the specified vertices are embedded in the singular decagon of *Figure 5*, we observe that they are attached at two consecutive vertices of the decagon, so there exists a noncontractible curve which does not separate the vertex set $4, 5, 6, 7, 8, 10, 11$, see *Figure 9*, again contradicting Kuratowski's theorem. □

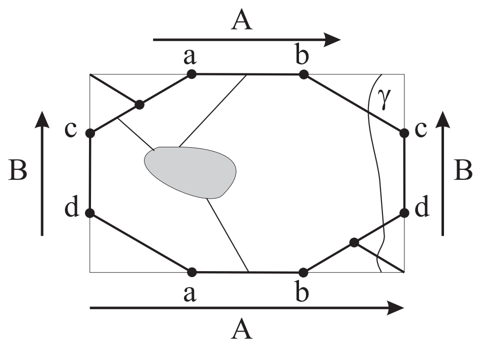


Figure 9. *Cutting a torus with a non-contractible curve γ*

Our theorem provides a counter example for the conjecture of F. C. Tinsley and J. J. Watkins. Their conjecture was that if P is a Petersen graph and P^n stands for a dot product of n Petersen graphs, then $g(P^n) = n - 1$ (g stands for the orientable genus). The Blanuša snarks are both obtained as a dot product of two Petersen snarks, but their genus is different. Another counter example is Szekeres's snark, that is obtained as a dot product of 5 Petersen graphs. From *Figure 10* one can easily find 5 disjoint subdivisions of $K_{3,3}$ which cannot be embedded into a surface of genus 4.

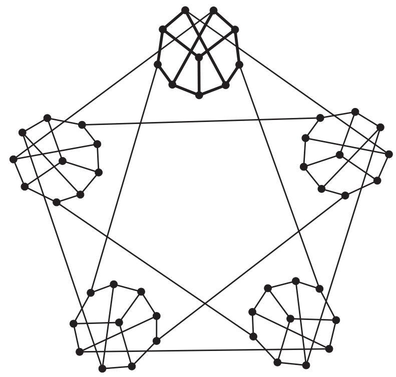


Figure 10. *The Szekeres snark with one of the 5 disjoint subgraphs $K_{3,3}$ marked bold*

4. The 6 snarks on 20 vertices

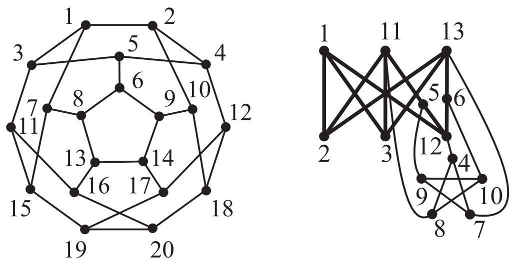
The labelled 6 snarks on 20 vertices are shown in *Figures 11* to *16*. We name the 6 snarks Sn4, Sn5,…, Sn9. Snark Sn4 is also known as the smallest *Flower snark* from the infinite family of flower snarks (see [3]).

There are exactly 6 snarks of order 20, [16].

Theorem 2. *All snarks on 20 vertices have genus 2.*

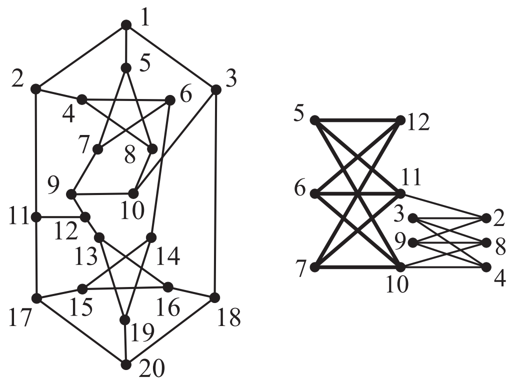
Proof. The forbidden minors for a torus of the 6 labelled snarks are shown in *Figures 11* to *16*. Using similar arguments as in the proof of *Theorem 1*, one can easily see that all 6 given minors are non-toroidal.

Embeddings of these snarks on the double torus together with the facial walks are shown in *Figures 17* to *22*. The embeddings were found by program Vega [15] using an algorithm that checks all possible combinations of local rotations. □



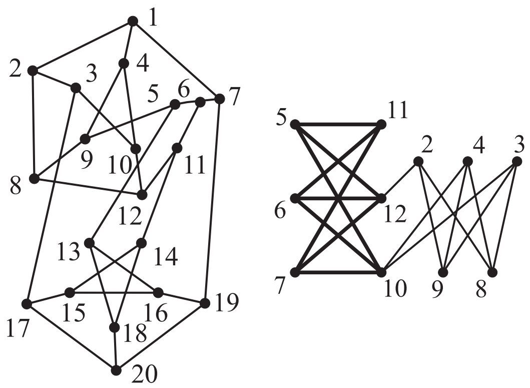

```
Sn4:
Vertices: 13, Edges: 6
1: (6): 3 12 2
2: (8): 13 11 1
3: (9): 11 13 1
4: (12): 7 12 8
5: (15): 11 12 9
6: (16): 10 13 12
7: (17): 9 4 13
8: (18): 11 10 4
9: (19): 10 5 7
10: (20): 6 8 9
11: (1 2 7 10): 5 8 3 2
12: (3 4 11 5): 6 1 4 5
13: (13 14 ): 6 7 2 3
```

Figure 11. *Sn4 and its forbidden minor*



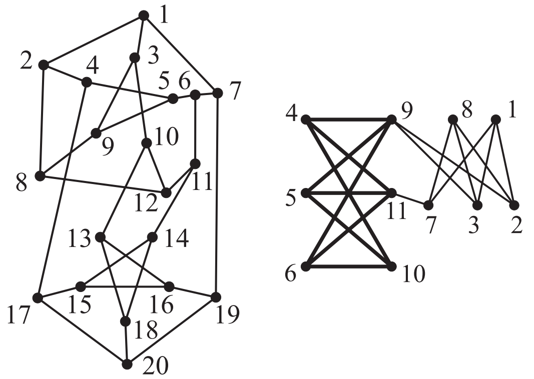

```
Sn5:
Vertices: 12, Edges: 6
2: (6): 3 9 11
3: (7): 8 2 4
4: (9): 9 3 10
5: (13): 10 12 11
6: (15): 10 11 12
7: (20): 11 10 12
8: (1 2 5): 3 10 9
9: (4 8 10): 8 4 2
10: (11 12 17): 7 8 6 5 4
11: (14 18): 5 6 7 2
12: (16 19): 7 6 5
Vertex 3 from original
graph removed
```

Figure 12. *Sn5 and its forbidden minor*



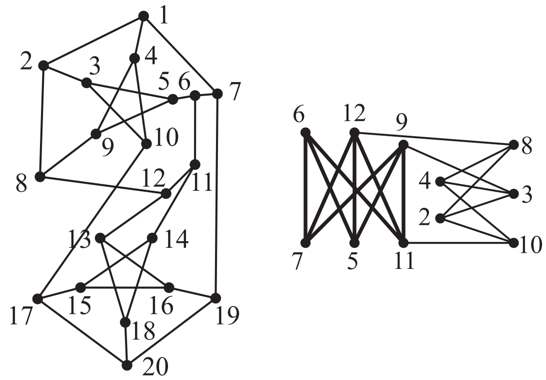

```
Sn6:
Vertices: 12, Edges: 6
2: (3): 12 8 9
3: (9): 8 10 9
4: (12): 8 9 10
5: (14): 12 11 10
6: (16): 11 12 10
7: (20): 10 11 12
8: (2 8): 3 2 4
9: (4 10): 4 3 2
10: (5 6 11 7 19): 5 6 7 3 4
11: (13 18): 5 6 7
12: (15 17): 7 6 5 2
Vertex 1 from original
graph removed
```

Figure 13. *Sn6 and its forbidden minor*



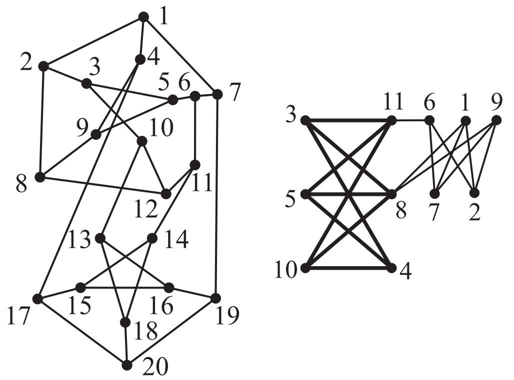
```
Sn7:
Vertices: 11, Edges: 5
1: (8): 7 2 3
2: (9): 1 9 8
3: (12): 9 1 8
4: (14): 9 11 10
5: (16): 9 10 11
6: (20): 11 10 9
7: (2 4): 1 8 11
8: (1 3 10): 3 7 2
9: (5 6 11 7 19): 4 6 5 3 2
10: (13 18): 4 6 5
11: (15 17): 5 6 4 7
```

Figure 14. *Sn7 and its forbidden minor*



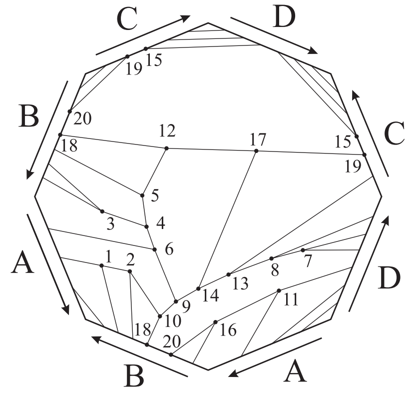
```
Sn8:
Vertices: 12, Edges: 6
2: (3): 3 10 8
3: (5): 4 9 2
4: (9): 3 8 10
5: (14): 6 9 12
6: (18): 7 5 11
7: (20): 9 6 12
8: (4 10): 12 4 2
9: (6 11 7 19): 3 11 5 7
10: (2 8 12): 11 2 4
11: (13 16): 6 12 9 10
12: (15 17): 5 11 8 7
Vertex 1 from original
graph removed
```

Figure 15. *Sn8 and its forbidden minor*



```
Sn9:
Vertices: 11, Edges: 5
1: (5): 7 8 2
2: (9): 9 1 6
3: (14): 4 8 11
4: (18): 5 3 10
5: (20): 11 8 4
6: (1 2 4): 11 7 2
7: (3 10): 1 6 9
8: (6 11 7 19): 10 3 5 1 9
9: (8 12): 7 2 8
10: (13 16): 8 11 4
11: (15 17): 5 10 3 6
```

Figure 16. *Sn9 and its forbidden minor*



List of faces:

13 16 20 19 17 14

7 8 13 14 9 10 18 20 16 11 15 19 20 18 12 17 19 15

4 6 9 14 17 12 5

3 11 16 13 8 6 4

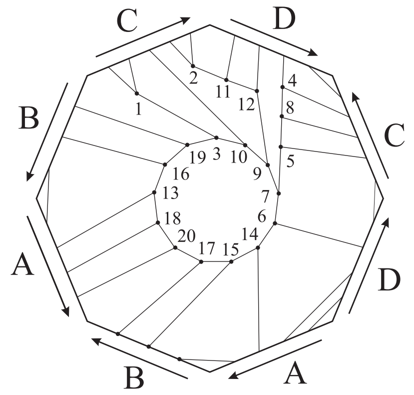
2 5 12 18 10

1 2 10 9 6 8 7

1 7 15 11 3

1 3 4 5 2

Figure 17. *Sn4 embedded into double torus*



List of faces:

14 15 16 19 20 18

11 12 13 16 15 17

4 6 14 18 13 12 9 7 5 8

3 19 16 13 18 20 17 15 14 6 7 9 10

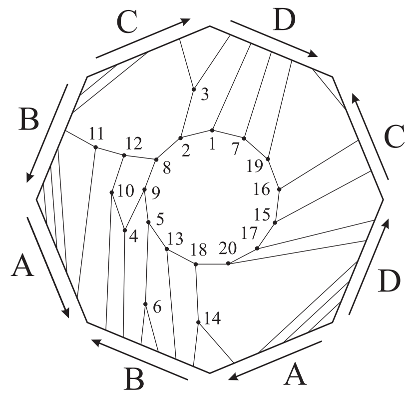
2 4 8 10 9 12 11

1 2 11 17 20 19 3

1 3 10 8 5

1 5 7 6 4 2

Figure 18. *Sn5 embedded into double torus*



List of faces:

13 16 15 14 18

6 7 19 20 18 14 11

4 9 8 12 10

3 10 12 11 14 15 17

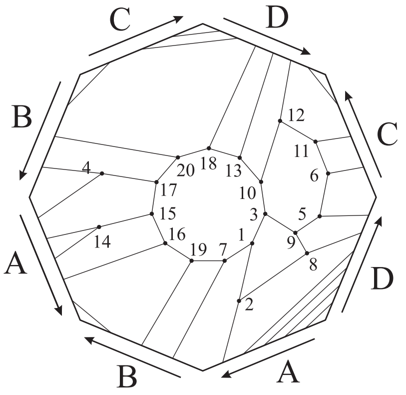
2 3 17 20 19 16 13 5 6 11 12 8

1 2 8 9 5 13 18 20 17 15 16 19 7

1 7 6 5 9 4

1 4 10 3 2

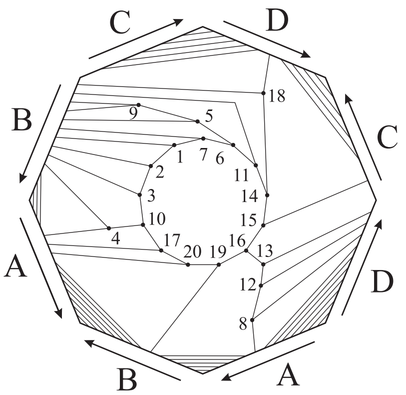
Figure 19. *Sn6 embedded into double torus*



List of faces:

13 16 15 14 18
6 7 19 20 18 14 11
4 5 9 8 12 11 14 15 17
3 9 5 6 11 12 10
2 4 17 20 19 16 13 10 12 8
1 2 8 9 3
1 3 10 13 18 20 17 15 16 19 7
1 7 6 5 4 2

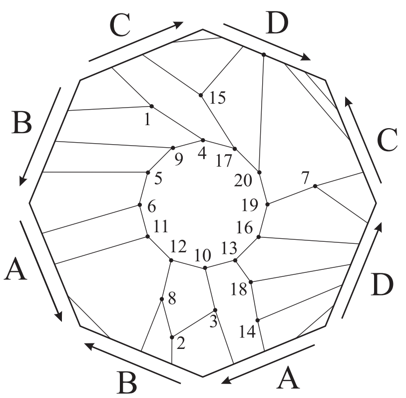
Figure 20. *Sn7 embedded into double torus*



List of faces:

13 18 20 17 15 16
11 14 18 13 12
5 6 11 12 8 9
3 5 9 4 10
2 8 12 13 16 19 7 6 5 3
1 2 3 10 17 20 19 16 15 14 11 6 7
1 7 19 20 8 14 15 17 10 4
1 4 9 8 2

Figure 21. *Sn8 embedded into double torus*



List of faces:

14 15 17 20 18
6 7 19 16 15 14 11
4 9 5 6 11 12 10 13 16 19 20 17
3 5 9 8 12 11 14 18 13 10
2 3 10 12 8
1 2 8 9 4
1 4 17 15 16 13 18 20 19 7
1 7 6 5 3 2

Figure 22. *Sn9 embedded into double torus*

Using the VEGA program the automorphisms of the 6 snarks were calculated.

Snark Sn4 (Flower snark) has 10 automorphisms and 3 vertex orbits:

$$5^2 \cdot 10 = \{\{4,7,10,16,17\},\{6,8,9,13,14\},\{1,2,11,12,19,20,18,15,5,3\}\}.$$

Snark Sn5 has 4 automorphisms and 8 vertex orbits

$$2^6 \cdot 4^2 = \{\{2,9\},\{3,6\},\{5,8\},\{11,12\},\{13,17\},\{14,19\},\{15,16,18,20\}, \{1,4,10,7\}\}.$$

Snark Sn6 has 4 automorphisms and

$$1^2 \cdot 2^3 \cdot 4^3 = \{\{5\},\{6\},\{7,11\},\{3,17\},\{9,13\},\{1,12,19,14\},\{2,10,20,15\}, \{4,8,16,18\}\}$$

Snark Sn7 has a trivial automorphism group.

Snark Sn8 has a trivial automorphism group.

Snark Sn9 has 2 automorphisms and 11 vertex orbits:

$$1^2 \cdot 2^9 = \{\{5\},\{6\},\{1,12\},\{2,8\},\{3,9\},\{4,10\},\{7,11\},\{13,17\},\{14,19\}, \{15,16\},\{18,20\}\}$$

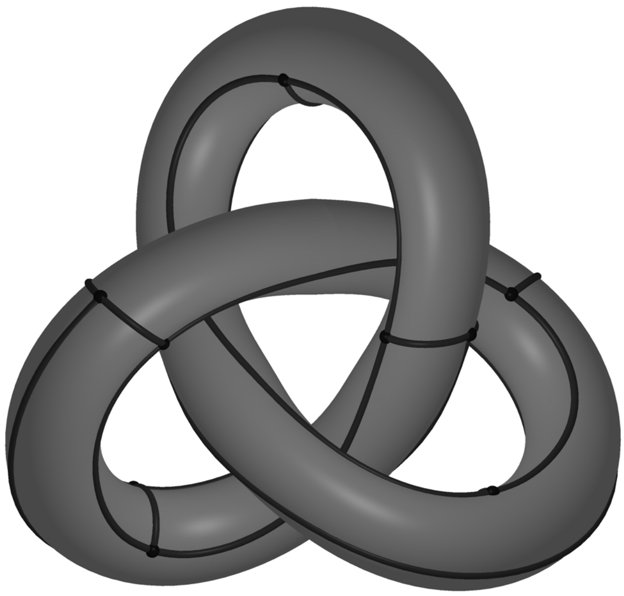


Figure 23. *"Knotted torus and Blanuša's graph on it" is the title for a computer model for a possible actual sculpture. The coordinates for the embedding of the Blanuša snark on the flat torus were produced by Tomaž Pisanski. Darko Veljan suggested that the torus should be modelled in the shape of a trefoil knot. The actual computer drawing was produced by Marko Boben.*

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