Generalized Veltman models with a root

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Abstract. *Provability logic is a nonstandard modal logic. Interpretability logic is an extension of provability logic. Generalized Veltman models are Kripke like semantics for interpretability logic. We consider generalized Veltman models with a root, i.e. r-validity, r-satisfiability and a consequence relation. We modify Fine's and Rautenberg's proof and prove non–compactness of interpretability logic.*

Key words: *interpretability logic, generalized Veltman semantics*

AMS subject classifications: 03F25

Received October 9, 2002 Accepted June 21, 2004

1. Introduction

The idea of treating a provability predicate as a modal operator goes back to Gödel. The same idea was taken up later by Kripke and Montague, but only in the mid-seventies was the correct choice of axioms, based on Löb's theorem, seriously considered by several logicians independently: G. Boolos, D. de Jongh, R. Magari, G. Sambin and R. Solovay. There are two key results in application of modal logic to the study of provability in arithmetic and related theories: de Jongh-Sambin fixed point theorem and Solovay's arithmetic completeness theorems.

The system GL (Gödel, Löb) is a modal propositional logic. The axioms of system GL are all tautologies, $\Box(A \to B) \to (\Box A \to \Box B)$, and $\Box(\Box A \to A) \to \Box A$. The inference rules of GL are modus ponens and necessitation $A/\Box A$.

R. Solovay 1976. proved arithmetical completeness of modal system GL. Many theories have the same provability logic - GL. It means that the provability logic GL cannot distinguish some properties, as e.g. finite axiomatizability, reflexivity, etc. Some logicians considered modal representations of other arithmetical properties, for example interpretability, Π_n-conservativity, interpolability ... Modal logics for interpretability were first studied by P. Hájek (1981) and V. Švejdar (1983). A. Visser (1990) introduced the binary modal logic IL (interpretability logic). The interpretability logic IL results from the provability logic GL, by adding the binary modal operator $\rhd$.

Roughly, the theory S interprets the theory T if there is a natural way of translating the language of S into the language of T in such a way that the translations

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of all the axioms of T become provable in S. We write $S \geq T$ if this is the case. A derived notion is that of relative interpretability over a base theory T. Let A and B be arithmetical sentences. We say that A interprets B over T if $T + A \geq T + B$. For precise definitions, see e.g. [7].

The language of the interpretability logic contains propositional letters $p_0,\ p_1, \ldots$, the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$, the unary modal operator $\Box$ and the binary modal operator $\rhd$. We use $\perp$ for false and $\top$ for true. The axioms of the interpretability logic IL are the axioms of GL and:

(J1) $\Box(A \rightarrow B) \rightarrow (A \rhd B)$

(J2) $((A \rhd B) \wedge (B \rhd C)) \rightarrow (A \rhd C)$

(J3) $((A \rhd C) \wedge (B \rhd C)) \rightarrow ((A \vee B) \rhd C)$

(J4) $(A \rhd B) \rightarrow (\Diamond A \rightarrow \Diamond B)$

(J5) $\Diamond A \rhd A$

where $\Diamond$ stands for $\neg\Box\neg$ and $\rhd$ has the same priority as $\rightarrow$. The deduction rules of IL are modus ponens and necessitation.

Arithmetical semantic of interpretability logic is based on the fact that each sufficiently strong theory S has arithmetical formulas $Pr(x)$ and $Int(x, y)$. Formula $Pr(x)$ expressing that 'x is provable in S' (i.e. formula with Gödel number x is provable in S). Formula $Int(x, y)$ expressing that '$S + x$ interprets $S + y$.' An arithmetical interpretation is a function $*$ from modal formulas into arithmetical sentences preserving Boolean connectives and satisfying

$$(\Box A)^* = Pr(\ulcorner A^* \urcorner), \qquad (A \rhd B)^* = Int(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$$

($\ulcorner A^* \urcorner$ denote Gödel number of formula A^*). A modal formula A is valid in S if $S \vdash A^*$ for each arithmetical interpretation $*$. A modal theory T is sound w.r.t. S if all its theorems are valid in S. The theory T is complete w.r.t. S if it proves exactly those formulas that are valid in S. The soundness of IL was already known and amounts to noticing that all the axioms are PA-valid and the rules of inference preserve PA-validity.

Axioms (J1)–(J3) are clear. Axiom (J4) says that relative interpretability yields relative consistency results. Axiom (J5) is the arithmetized completeness theorem: PA plus the assertion that a given theory is consistent interprets the given theory. The system IL is natural from the modal point of view, but arithmetically incomplete. For example, IL does not prove the formula W i.e. $(A \rhd B) \rightarrow (A \rhd (B \wedge \Box(\neg A)))$, which is valid in every adequate theory. Various extensions of the system IL are obtained by adding new axioms. These new axioms are called principles of interpretability.

A. Visser showed arithmetical completeness for the relation of relative interpretability over finitely axiomatized theories. A. Berarducci and V. Shavrukov independently showed arithmetical completeness for the relation of relative interpretability over theories like Peano arithmetic and Zermelo-Fraenkel set theory.

We are only interested in IL as a system of modal logic. So, we do not study arithmetical interpretations. We introduce our notation and some basic facts, following [7]. Now we define Veltman models. Then we quote de Jongh-Veltman's theorem.

Definition 1 [de Jongh and Veltman]. *An ordered quadruple*

$$\boldsymbol{W} = (W, R, \{S_w : w \in W\}, \Vdash)$$

is called the IL-model (Veltman model), if it satisfies the following conditions:

a) (W, R) *is a GL-frame, i.e. W is a nonempty set, and R is a transitive and reverse well-founded relation;*

b) For every $w \in W$ is $S_w \subseteq W[w]$, where $W[w] = \{x \in W : wRx\}$;

c) The relation S_w is reflexive and transitive, for every $w \in W$;

d) If $wRw'Rw''$ then $w'S_w w''$;

e) $\Vdash$ *is a forcing relation. We emphasize only the definition*

$$w \Vdash A \rhd B \quad \text{if and only if} \quad \forall v((wRv \,\&\, v \Vdash A) \Rightarrow \exists u(vS_w u \,\&\, u \Vdash B)).$$

Theorem 1 [de Jongh and Veltman]. *For every modal formula F we have*

$$\vdash_{IL} F \quad \text{if and only if} \quad \mathbf{W} \models F \text{ for all } IL\text{-models } \mathbf{W}.$$

The Veltman models are a basic semantics for interpretability logic. But, when we study correspondences between principles of interpretability we use other semantics. In [8] and [9] we use generalized Veltman semantics.

Definition 2 [de Jongh]. *An ordered triple $(W, R, \{S_w : w \in W\})$ is called the IL_{set}-frame, and denoted by $\boldsymbol{W}$, if we have:*

a) (W, R) *is a GL-frame;*

b) Every $w \in W$ satisfies $S_w \subseteq W[w] \times \mathcal{P}(W[w]) \backslash \{\emptyset\}$;

c) The relation S_w is quasi-reflexive for every $w \in W$, i.e. wRx implies $xS_w\{x\}$;

d) The relation S_w is quasi-transitive for every $w \in W$, i.e. if xS_wY and $(\forall y \in Y)(yS_wZ_y)$ then $xS_w(\cup_{y\in Y} Z_y)$;

e) If $wRw'Rw''$ then $w'S_w\{w''\}$;

f) If xS_wY and $Y \subseteq Z \subseteq W[w]$ then xS_wZ.

Definition 3 [de Jongh]. *An ordered quadruple $(W, R, \{S_w : w \in W\}, \Vdash)$ is called the IL_{set}-model (generalized Veltman model), and denoted by $\boldsymbol{W}$, if we have:*

(1) $(W, R, \{S_w : w \in W\})$ *is an IL_{set}-frame ;*

(2) $\Vdash$ *is the forcing relation between elements of* W *and formulas of* IL*, which satisfies the following:*

(2a) $w \Vdash \top$ *and* $w \nVdash \bot$ *are valid for every* $w \in W$*;*

(2b) $\Vdash$ *commutes with the Boolean connectives;*

(2c) $w \Vdash \Box A$ *if and only if* $\forall x(wRx \Rightarrow x \Vdash A)$*;*

(2d) $w \Vdash A \rhd B$ *if and only if*

$$\forall v((wRv \ \& \ v \Vdash A) \ \Rightarrow \ \exists V(vS_wV \ \& \ (\forall x \in V)(x \Vdash B))).$$

Definition 4. *Let* $\boldsymbol{W} = (W, R, \{S_w : w \in W\}, \Vdash)$ *be an* IL_{set}*-model.*

We say that a formula F *is* true in the model $\boldsymbol{W}$ *at a state* $w \in W$ *if we have* $w \Vdash F$*. We say that a formula* F holds in the model $\boldsymbol{W}$ *if we have* $w \Vdash F$*, for all* $w \in W$*. This fact we denote by* $\boldsymbol{W} \models F$*.*

We say that a set of formulas Γ *is* true in the model $\boldsymbol{W}$ *at a state* $w \in W$ *if we have* $w \Vdash F$*, for all* $F \in \Gamma$ *(notation:* $w \Vdash \Gamma$*). We say that a set of formulas* Γ holds in the model $\boldsymbol{W}$ *if we have* $w \Vdash \Gamma$*, for all* $w \in W$*. This fact we denote by* $\boldsymbol{W} \models \Gamma$*.*

A formula F *is* valid *if we have* $\boldsymbol{W} \models F$*, for all* IL_{set}*-model* $\boldsymbol{W}$*. A formula* F *is* satisfiable *if there is an* IL_{set}*-model* $\boldsymbol{W}$ *and some state* $w \in W$ *such that* $w \Vdash F$*.*

A set of formulas Γ *is* satisfiable *if there is an* IL_{set}*-model* $\boldsymbol{W}$ *and some state* $w \in W$ *such that* $w \Vdash \Gamma$*.*

Let Γ *be a set of formulas, and* F *a single formula. We say that* F *is* local semantic consequence of Γ *(notation:* $\Gamma \models F$*) if for all models* $\boldsymbol{W}$ *and all* $w \in W$*, if* $w \Vdash \Gamma$ *then* $w \Vdash F$*.*

It is easy to check the soundness of the system IL w.r.t. IL_{set}-models, i.e. if $\vdash_{IL} F$ then $\boldsymbol{W} \models F$, for all IL_{set}-model $\boldsymbol{W}$. In [8] we proved the completeness of the system IL w.r.t. generalized Veltman models.

2. Models with a root

In this section we consider generalized Veltman models with a root. These models are important when we consider compactness. At the beginning we would like to emphasize that we do not consider generalized Veltman models which are tree (or tree-like) models. Generalized Veltman models with a root contain a special node.

Definition 5. *Let* $\boldsymbol{W} = (W, R, \{S_w : w \in W\}, \Vdash)$ *be an* IL_{set}*-model and* $w_0 \in W$ *such that* $W[w_0] = W \backslash \{w_0\}$*. We say that the state* w_0 *is a* root *of the model* $\boldsymbol{W}$*. Then we say that* $\boldsymbol{W}$ *is a model with root, and we denote* $\boldsymbol{W}_{w_0}$*.*

We say that a formula F *is* r-true in the model $\boldsymbol{W}_{w_0}$ *if we have* $w_0 \Vdash F$*. This fact we denote by* $\boldsymbol{W} \models_r F$*. We say that a set of formulas* Γ *is* r-true in the model $\boldsymbol{W}$ *if we have* $w_0 \Vdash \Gamma$ *(notation:* $\boldsymbol{W} \models \Gamma$*).*

A formula F *is* r-valid *if we have* $w_0 \Vdash F$*, for all models* $\boldsymbol{W}$ *with a root* w_0*. We say that a formula* F *is* r-satisfiable *if there exists an* IL_{set}*-model* $\boldsymbol{W}$ *with the root such that* $\boldsymbol{W} \models_r F$*. A set of formulas* Γ *is* r-satisfiable *if there exists an* IL_{set}*-model* $\boldsymbol{W}$ *with the root such that* $\boldsymbol{W} \models_r \Gamma$*.*

Let Γ *be a set of formulas, and* F *a single formula. We say that* F *is* r-local semantic consequence of Γ *(notation:* $\Gamma \models_r F$*) if for all models* $\boldsymbol{W}$ *with a root, if* $w_0 \Vdash \Gamma$ *then* $w_0 \Vdash F$*.*

Proposition 1. *A formula F is valid if and only if F is r-valid.*

Proof. If a formula is valid then it is obviously r-valid.

Let us suppose that $\boldsymbol{W}=(W,R,\{S_w : w\in W\},\Vdash)$ is an IL_{set}-model such that $\boldsymbol{W}\not\models F$. There is a node $w_0\in W$ such that $w_0\not\Vdash F$. We define:

$$\begin{aligned}&W' = \{w_0\}\cup W[w_0]\\ &R' = R\cap W'\times W'\\ &S'_v = S_v, \quad \text{for every } v\in W'.\end{aligned}$$

Let $\Vdash'$ denote the restriction of the forcing relation $\Vdash$ on the set W'.

It is easy to check that $\boldsymbol{W}_{w_0}=(W',R',w_0,\{S'_v : v\in W'\},\Vdash')$ is an IL_{set}-model with the root w_0.

By induction on the complexity of a formula we can prove the following equivalence. For every formula B and every $v\in W'$ we have $v\Vdash' B$ if and only if $v\Vdash B$. Obviously, the last fact implies $w_0\not\Vdash' F$, i.e. $\boldsymbol{W}_{w_0}\not\models_r F$. □

By using the *Proposition 1* we get the following extension of de Jongh, Veltman theorem.

Proposition 2. *Let F be a formula. Then the following are equivalent.*

a) $\vdash_{IL} F$;

b) for each finite generalized Veltman model $\boldsymbol{W}$ we have $\boldsymbol{W}\models F$;

b) for each finite generalized Veltman model $\boldsymbol{W}$ with the root w_0 we have $\boldsymbol{W}\models_r F$.

Proposition 3. *A formula F is satisfiable if and only if F is r-satisfiable. A set of formulas Γ is satisfiable if and only if Γ is r-satisfiable.*

Proof. Let F be a satisfiable formula, and let $\boldsymbol{W}$ be an IL_{set}-model and $w_0\in W$ such that $w_0\models F$. In the same way as in the proof of the *Proposition 1* we can define the IL_{set}-model $\boldsymbol{W}_{w_0}$ with the root such that $\boldsymbol{W}_{w_0}\models_r F$. So, the formula F is r-satisfiable.

If F is an r-satisfiable formula, it is obviously that F is satisfiable formula. □

Proposition 4. *Let Γ be a set of formulas and F a formula. Then we have*

$$\Gamma\models_r F \qquad \textit{if and only if} \qquad \Gamma\models F.$$

Proof. Assume that we have $\Gamma\models_r F$. Let $\boldsymbol{W}=(W,R,\{S_v : v\in W\},\Vdash)$ be an IL_{set}-model and $w_0\in W$ such that $w_0\models\Gamma$. In the same way as in the proof of the *Proposition 1* we can define an IL_{set}-model $\boldsymbol{W}_{w_0}$ with the root w_0 such that for all $v\in W_{w_0}$ and for every formula B we have:

$$v\overset{\prime}{\Vdash} B \ \text{ if and only if } \ v\Vdash B \qquad (*)$$

Then we have $w_0\Vdash'\Gamma$. But, $\boldsymbol{W}_{w_0}$ is an IL_{set}-model with the root. So, the assumption $\Gamma\models_r F$ and the fact $\boldsymbol{W}_{w_0}\models_r\Gamma$ imply $w_0\Vdash' F$. By means of the fact $(*)$ we have $w_0\Vdash F$.

The converse is obviously true. □

3. Non-compactness of the interpretability logic w.r.t. generalized Veltman semantics

We usually use the compactness in a proof of completeness of a modal system. In this section we prove that the system IL is not compact w.r.t. generalized Veltman semantics. So, we can not use maximal consistent sets in proofs of completeness and Craig interpolation lemma for system IL. Areces, Hoogland and de Jongh in [1] use adequate sets of formulas for proving interpolation property of the system IL. We modify Fine's and Rautenberg's proof of non–compactness of the system GL (see [4]).

Proposition 5. *The interpretability logic is not compact with respect to generalized Veltman semantics, i.e. there exists a set of formulas Γ such that each finite subset of Γ is satisfiable, but the set Γ is not satisfiable.*

Proof. Let

$$\Gamma = \{\Diamond P_0,\ \Box(P_0 \to \Diamond P_1),\ \Box(P_1 \to \Diamond P_2),\ \Box(P_2 \to \Diamond P_3),\ \dots\}$$

Let Γ' be a finite subset of Γ. Let $n \in \mathbb{N}$ be the greatest number such that $\Box(P_n \to \Diamond P_{n+1}) \in \Gamma'$. We define:

$$\begin{aligned}
&W = \{0,\ 1,\ 2,\ \dots,\ n+1,\ n+2\},\\
&R = \{(i,j) : i < j,\ i,j \in W\},\\
&xS_wV \ \text{ if and only if }\ w < x,\ V \subseteq W \ \text{ and }\ (\forall y \in V)(x \leq y),\\
&i \Vdash P_{i-1}, \quad \text{for all }\ i = 1, \dots,\ n+2.
\end{aligned}$$

It is easy to check that $\boldsymbol{W}=(W, R, 0, S, \Vdash)$ is an IL_{set}-model with the root such that $W_0 \Vdash_r \Gamma'$. So, we have proved that each finite subset of Γ is r-satisfiable. The *Proposition 3* implies that each finite subset of Γ is satisfiable.

Let us suppose that the set Γ is satisfiable. By the *Proposition 3* we have that the set Γ is r-satisfiable. Then there is an IL_{set}-model with a root $\boldsymbol{W}= (W, R, w_0, \{S_w : w \in W\}, \Vdash)$ such that $\boldsymbol{W} \models_r \Gamma$, i.e. $w_0 \Vdash \Gamma$. Specially we have $w_0 \Vdash \Diamond P_0$. So, there is a state $w_1 \in W$ such that $w_0 R w_1$ and $w_1 \Vdash P_0$. The facts $w_0 \Vdash \Box(P_0 \to \Diamond P_1)$ and $w_1 \Vdash P_0$ imply $w_1 \Vdash \Diamond P_1$. The last fact implies that there exists a state $w_2 \in W$ such that $w_1 R w_2$ and $w_2 \Vdash P_1$. We can analogously continue. So, there exists a sequence of states (w_n) such that

$$w_0 R w_1 R w_2 R w_3 \dots$$

It is impossible, because the relation R is reverse well-founded. $\Box$

We would like to mention that the set Γ in the proof of the last proposition is consistent. Let us suppose that there exists a formula F such that $\Gamma \vdash_{IL} F$ and $\Gamma \vdash_{IL} \neg F$. By the definition of deduction in the system IL there is a finite subset Γ' of Γ such that $\Gamma' \vdash F$ and $\Gamma' \vdash \neg F$. We know that each finite subset of Γ is satisfiable. So, the set of formulas Γ' is satisfiable. It is easy to see that we have $\Gamma' \models F$ and $\Gamma' \models \neg F$. It is impossible.

The non–compactness of the system IL implies that there exist a set Γ and a formula F such that $\Gamma \models F$, but there is not a finite subset Γ' of Γ such that $\Gamma' \models F$.

(For example, let Γ be the set from the proof of the last proposition, and let F be the formula $P \land \neg P$).

The non–compactness of the system IL implies that the strong completeness theorem is not true for IL, i.e. there are a set Γ and a formula F such that $\Gamma \models F$, but $\Gamma \not\vdash F$. (We can use the set Γ from the proof of the last proposition, again.)

At the end we would like to emphasize that we have proved non–compactness of interpretability logic w.r.t. generalized Veltman semantics. All modal logics are compact w.r.t. Kripke models by using standard translation and compactness of first-order logic (see e.g. [2]).

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