Mann-Ishikawa iterations and Mann-Ishikawa iterations with errors are equivalent models

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Abstract. *Mann-Ishikawa iterations and Mann-Ishikawa iterations with errors are equivalent models for several classes of operators.*

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1. Preliminaries

Introduced in [4], Mann iteration is a viable method to approximate the fixed point of an operator, when Banach principle is not functional. Let X be a Banach space, let $T : X \to X$ be a map. Let $x_1 \in X$. *Mann iteration* is given by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n. \tag{1}$$

The sequence $(\alpha_n)_n \subset (0, 1)$ is convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Ishikawa introduced later in [2] the following iteration,

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n = 1, 2, \ldots . \end{aligned} \tag{2}$$

Sequences $(\alpha_n)_n, (\beta_n)_n \subset (0, 1)$ are convergent such that

$$\lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 0, \text{ and} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

In [2] the conditions on the above sequences were $0 < \alpha_n \leq \beta_n < 1$. A better condition, introduced in [7], is $0 < \alpha_n \ , \beta_n < 1$. Now, letting $\beta_n = 0, \forall n \in N$ from Ishikawa iteration (2), we get Mann iteration (1). Let us consider the following iteration, see [3]:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n + e_n. \tag{3}$$

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Errors $(e_n)_n \subset X$ satisfy $\sum_{n=1}^{\infty} \|e_n\| < \infty$. This iteration is known as *Mann iteration with errors.* In [3] *Ishikawa iteration with errors* is defined as

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T v_n + p_n, \\ v_n &= (1-\beta_n)u_n + \beta_n T u_n + q_n, \quad n = 1, 2, \dots . \end{aligned} \tag{4}$$

Errors $(p_n)_n, (q_n)_n$ and $(e_n)_n \subset X$ satisfy

$$\sum_{n=1}^{\infty} \|p_n\| < \infty, \ \lim_{n\to\infty} \|q_n\| = 0, \ \sum_{n=1}^{\infty} \|e_n\| < \infty, \tag{5}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are the same as those from (1)and (2). When $e_n = 0$, respectively $p_n = q_n = 0, \forall n \in N$ then we deal with Mann and Ishikawa iteration.

In [8] it was proven that for several classes of Lipschitzian operators, Mann and Ishikawa iteration methods are equivalent. We will prove further that Mann and Ishikawa iterations are equivalent models with Mann and Ishikawa iterations with errors. Thus the study of convergence of the above iterations is reduced to the study of Mann iteration, which is more convenient to be used.

Let us denote the identity map by I.

Definition 1. *Let X be a real Banach space. A map $T : X \to X$ is called strongly pseudocontractive if there exists $k \in (0,1)$ such that we have*

$$\|x-y\| \le \|x - y + r[(I-T-kI)x - (I-T-kI)y]\|, \tag{6}$$

for all $x, y \in X$, and $r > 0$.

The following lemma can be found in [3].

Lemma 1 [[3]]. *Let $(a_n)_n$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \le (1-\lambda_n)a_n + \sigma_n + w_n, \tag{7}$$

where $\lambda_n \in (0,1)$, $w_n \ge 0$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\sum_{n=1}^{\infty} w_n < \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main result

Let us denote $F(T) = \{x^* : Tx^* = x^*\}$. We are able now to give the following result:

Theorem 1. *Let X be a Banach space and let $T : X \to X$ be a Lipschitzian with $L \ge 1$, strongly pseudocontractive map. If $u_1 = x_1 \in X$, let $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, suppose that for iteration* (4) *the errors satisfy* (5); *then the following two assertions are equivalent:*

(i) Ishikawa iteration (2) *converges to $x^* \in F(T)$,*

(ii) Ishikawa iteration with errors (4) *converges to the same $x^* \in F(T)$.*

Proof. Corollary 1 from [1] assures that $F(T) \neq \emptyset$; strong pseudocontractivity assures the uniqueness of the fixed point.

Supposing Ishikawa iteration with errors (4) converges and taking $p_n = q_n = 0, \forall n \in N$, we get the convergence of (2). We will prove that the convergence of Ishikawa iteration (2) implies the convergence of Ishikawa iteration with errors (4). The proof is similar to the proof of Theorem 4 from [8]. We have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} + \\ &\quad -(2-k)\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n(Tx_{n+1} - Ty_n) \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} + \\ &\quad -(2-k)\alpha_n[(1-\alpha_n)x_n + \alpha_n T y_n] + \alpha_n x_n + \alpha_n(Tx_{n+1} - Ty_n) \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} + \\ &\quad -(1-k)\alpha_n x_n + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n). \end{aligned} \tag{8}$$

Also

$$\begin{aligned} u_n &= u_{n+1} + \alpha_n u_n - \alpha_n T v_n - p_n \\ &= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} + \\ &\quad -(2-k)\alpha_n u_{n+1} + \alpha_n u_n + \alpha_n(Tu_{n+1} - Tv_n) - p_n \\ &= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} + \\ &\quad -(2-k)\alpha_n[(1-\alpha_n)u_n + \alpha_n T v_n + p_n] + \\ &\quad +\alpha_n u_n + \alpha_n(Tu_{n+1} - Tv_n) - p_n \\ &= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} + \\ &\quad -(1-k)\alpha_n u_n + (2-k)\alpha_n^2(u_n - Tv_n) + \alpha_n(Tu_{n+1} - Tv_n) \\ &\quad -p_n - (2-k)\alpha_n p_n \\ &= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} + \\ &\quad -(1-k)\alpha_n u_n + (2-k)\alpha_n^2(u_n - Tv_n) + \alpha_n(Tu_{n+1} - Tv_n) \\ &\quad -p_n(1 + (2-k)\alpha_n). \end{aligned} \tag{9}$$

From (8) and (9) we get

$$\begin{aligned} x_n - u_n = &\ (1+\alpha_n)(x_{n+1} - u_{n+1}) \\ &+\alpha_n\left((I - T - kI)x_{n+1} - (I - T - kI)u_{n+1}\right) \\ &-(1-k)\alpha_n(x_n - u_n) + (2-k)\alpha_n^2(x_n - u_n - Ty_n + Tv_n) \\ &+\alpha_n(Tx_{n+1} - Tu_{n+1} - Ty_n + Tv_n) + p_n(1 + (2-k)\alpha_n). \end{aligned} \tag{10}$$

Taking $(1+\alpha_n)(x_{n+1} - u_{n+1}) + \alpha_n\left((I - T - kI)x_{n+1} - (I - T - kI)u_{n+1}\right)$ in norm we have

$$\begin{aligned} &\|(1+\alpha_n)(x_{n+1} - u_{n+1}) + \alpha_n\left((I - T - kI)x_{n+1} - (I - T - kI)u_{n+1}\right)\| \\ &= (1+\alpha_n)\left\|(x_{n+1} - u_{n+1}) + \frac{\alpha_n}{1+\alpha_n}\ \left((I - T - kI)x_{n+1} - (I - T - kI)u_{n+1}\right)\right\|; \end{aligned}$$

and using (6) with $x := x_{n+1}$ and $y := u_{n+1}$, we obtain

$$\begin{aligned} & \left\| (1+\alpha_n)(x_{n+1}-u_{n+1}) + \alpha_n \left((I-T-kI)x_{n+1} - (I-T-kI)u_{n+1} \right) \right\| \\ & \geq (1+\alpha_n) \left\| x_{n+1} - u_{n+1} \right\| . \end{aligned} \tag{11}$$

Taking the norm in (10) and then using (11), we get

$$\begin{aligned} & \|x_n - u_n\| \\ & \geq \left\| (1+\alpha_n)(x_{n+1}-u_{n+1}) + \alpha_n \left((I-T-kI)x_{n+1} - (I-T-kI)u_{n+1} \right) \right\| \\ & \quad -(1-k)\alpha_n \|x_n - u_n\| - (2-k)\alpha_n^2 \|x_n - u_n - Ty_n + Tv_n\| \\ & \quad -\alpha_n \|Tx_{n+1} - Tu_{n+1} - Ty_n + Tv_n\| - \|p_n\| (1 + (2-k)\alpha_n) \\ & \geq (1+\alpha_n) \|x_{n+1} - u_{n+1}\| - (1-k)\alpha_n \|x_n - u_n\| - (2-k)\alpha_n^2 \\ & \quad \|x_n - u_n - Ty_n + Tv_n\| - \alpha_n \|Tx_{n+1} - Tu_{n+1} - Ty_n + Tv_n\| \\ & \quad - \|p_n\| (1 + (2-k)\alpha_n). \end{aligned}$$

We obtain

$$\begin{aligned} & (1+\alpha_n) \ \|x_{n+1} - u_{n+1}\| \\ & \leq (1 + (1-k)\alpha_n) \|x_n - u_n\| + (2-k)\alpha_n^2 \|x_n - u_n - Ty_n + Tv_n\| \\ & \quad +\alpha_n \|Tx_{n+1} - Tu_{n+1} - Ty_n + Tv_n\| + \|p_n\| (1 + (2-k)\alpha_n) \\ & \leq (\ 1 + (1-k) \ \alpha_n \ ) \ \|x_n - u_n\| + (2-k) \ \alpha_n^2 \ \|u_n - Tv_n\| \\ & \quad + (2-k) \ \alpha_n^2 \ \|x_n - Ty_n\| + \alpha_n \ \|Tx_{n+1} - Ty_n\| \\ & \quad +\alpha_n \ \|Tu_{n+1} - Tv_n\| + \|p_n\| \ (1 + (2-k) \ \alpha_n) . \end{aligned} \tag{12}$$

We aim to evaluate $\|u_n - Tv_n\|$ and $\|Tu_{n+1} - Tv_n\|$:

$$\begin{aligned} \|u_n - Tv_n\| & \leq \|u_n - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tv_n\| \\ & \leq \|x_n - u_n\| + \|x_n - Tx_n\| + L \ \|x_n - v_n\| . \end{aligned}$$

$$\begin{aligned} & \|x_n - v_n\| \\ & = \|(1-\beta_n)(x_n - u_n) + \beta_n(x_n - Tu_n) - q_n\| \\ & \leq (1-\beta_n) \|x_n - u_n\| + \beta_n \|x_n - Tu_n\| + \|q_n\| \\ & \leq (1-\beta_n) \|x_n - u_n\| + \beta_n \big[\|Tx_n - Tu_n\| + \|x_n - Tx_n\| \big] + \|q_n\| \\ & \leq (1-\beta_n) \|x_n - u_n\| + \beta_n L \|x_n - u_n\| + \beta_n \|x_n - Tx_n\| + \|q_n\| \\ & = (1 - \beta_n + \beta_n L) \|x_n - u_n\| + \beta_n \|x_n - Tx_n\| + \|q_n\| \\ & \leq L \ \|x_n - u_n\| + \beta_n \ \|x_n - Tx_n\| + \|q_n\| , \end{aligned} \tag{13}$$

because $1 \leq L \Rightarrow 1 - \beta_n + \beta_n L \leq L$.

We have

$$\begin{aligned} & \|u_n - Tv_n\| \\ & \leq \|x_n - u_n\| + \|x_n - Tx_n\| + L\big(L \|x_n - u_n\| + \beta_n \|x_n - Tx_n\| + \|q_n\| \big) \\ & \leq (1 + L^2) \ \|x_n - u_n\| + (1 + L\beta_n) \ \|x_n - Tx_n\| + L \ \|q_n\| . \end{aligned}$$

Now, $\|Tu_{n+1} - Tv_n\|$ satisfies

$$\begin{aligned}\|Tu_{n+1} - Tv_n\| &\leq L\,\|u_{n+1} - v_n\| = L\,\|(1-\alpha_n)u_n + \alpha_n Tv_n - v_n + p_n\|\\ &\leq L\,[\,(1-\alpha_n)\ \|u_n - v_n\| + \alpha_n\ \|Tv_n - v_n\| +\ \|p_n\|\,]\end{aligned}$$

Using (13) we evaluate:

$$\begin{aligned}\|Tv_n - v_n\| &\leq \|Tv_n - Tx_n\| + \|Tx_n - x_n\| + \|x_n - v_n\|\\ &\leq (1+L)\,\|x_n - v_n\| + \|Tx_n - x_n\|\\ &\leq (1+L)[L\,\|x_n - u_n\| + \beta_n\,\|Tx_n - x_n\| + \|q_n\|] + \|Tx_n - x_n\|\\ &= (1+L)\,L\ \|x_n - u_n\| + [\,(1+L)\,\beta_n + 1\,]\ \|Tx_n - x_n\|\\ &\quad +(1+L)\ \|q_n\|\end{aligned}$$

and

$$\begin{aligned}\|u_n - v_n\| &= \|u_n - (1-\beta_n)u_n - \beta_n Tu_n - q_n\| = \beta_n\,\|u_n - Tu_n\| + \|q_n\|\\ &\leq \beta_n[\|u_n - x_n\| + \|Tx_n - x_n\| + \|Tu_n - Tx_n\|] + \|q_n\|\\ &\leq \beta_n\ (\,(1+L)\ \|x_n - u_n\| +\ \|Tx_n - x_n\|\,)\ +\ \|q_n\|\,.\end{aligned}$$

One obtains

$$\begin{aligned}&\|Tu_{n+1} - Tv_n\|\\ &\leq L[\,(1-\alpha_n)\ \|u_n - v_n\| + \alpha_n\ \|Tv_n - v_n\| +\ \|p_n\|\,]\\ &\leq L\,\Big\{\,(1-\alpha_n)\ (\,\beta_n\ (\,(1+L)\ \|x_n - u_n\| +\ \|Tx_n - x_n\|\,)\ + \|q_n\|\,)\\ &\quad +\alpha_n\ (\,(1+L)\,L\ \|x_n - u_n\| +\ [\,(1+L)\beta_n + 1\,]\ \|Tx_n - x_n\|\\ &\quad +(1+L)\ \|q_n\|\,)\ +\ \|p_n\|\ \Big\}\\ &= (1-\alpha_n)\,\beta_n\,(1+L)\,L\ \|x_n - u_n\| + L\,(1-\alpha_n)\,\beta_n\ \|Tx_n - x_n\|\\ &\quad +L\,(1-\alpha_n)\ \|q_n\| + \alpha_n\,(1+L)\,L^2\ \|x_n - u_n\|\\ &\quad +\alpha_n\,L\,[\,(1+L)\beta_n + 1\,]\ \|Tx_n - x_n\| + \alpha_n\,L\,(1+L)\ \|q_n\| +\ L\ \|p_n\|\\ &= \big(\,L\,(1-\alpha_n)\,\beta_n\,(1+L)\ +\ \alpha_n\,(1+L)\,L^2\,\big)\ \|x_n - u_n\|\\ &\quad +(\,\beta_n\,L\,(1-\alpha_n) +\ \alpha_n\,L\,[\,(1+L)\,\beta_n + 1\,]\,)\ \|Tx_n - x_n\|\\ &\quad +(\,(1+L)\,\alpha_n + (1-\alpha_n)\,)\ L\ \|q_n\| + L\ \|p_n\|\,.\end{aligned}$$

Also, we have

$$\begin{aligned}\|u_n - Tv_n\| &\leq \|x_n - u_n\| + \|x_n - Tx_n\| + L\,\|x_n - v_n\|\\ &\leq \|x_n - u_n\| + \|x_n - Tx_n\|\\ &\quad +L\big[L\,\|x_n - u_n\| + \beta_n\,\|x_n - Tx_n\| + \|q_n\|\,\big]\\ &= (1+L^2)\ \|x_n - u_n\| + (1+\beta_n\,L)\ \|x_n - Tx_n\| +\ L\ \|q_n\|\,.\end{aligned}$$

Taking (12) with the above evaluations for $\|u_n - Tv_n\|$, $\|Tu_{n+1} - Tv_n\|$, and using the following inequalities $(1+\alpha_n)^{-1} \leq 1-\alpha_n+\alpha_n^2$, $(1+\alpha_n)^{-1} \leq 1$, we get

$$\begin{aligned}
(1+&\alpha_n)\|x_{n+1}-u_{n+1}\| \\
&\leq (1+(1-k)\,\alpha_n)\ \|x_n-u_n\| \\
&\quad +(2-k)\alpha_n^2\left((1+L^2)\|x_n-u_n\|+(1+\beta_n L)\|x_n-Tx_n\|+L\|q_n\|\right) \\
&\quad +(2-k)\ \alpha_n^2\ \|x_n-Ty_n\|+\alpha_n\ \|Tx_{n+1}-Ty_n\|+ \\
&\quad +\alpha_n\ \left(L\,(1-\alpha_n)\,\beta_n\,(1+L)+\alpha_n\,(1+L)\,L^2\right)\ \|x_n-u_n\| \\
&\quad +\alpha_n\ (\ \beta_n\ L\,(1-\alpha_n)+\alpha_n\ L\ [\ (1+L)\beta_n+1\ ]\ )\ \|x_n-Tx_n\| \\
&\quad +\alpha_n\ (\ (1+L)\ \alpha_n\ +(1-\alpha_n)\ )\ L\ \|q_n\| \\
&\quad +\alpha_n\ L\ \|p_n\|+\|p_n\|\ (1+(2-k)\ \alpha_n) \\
&\leq \{\ (1+(1-k)\ \alpha_n)\ +(2-k)\ \alpha_n^2\ \left(1+L^2\right) \\
&\quad +\alpha_n\ L\ (1+L)\ (\ (1-\alpha_n)\ \beta_n+\alpha_n\ L)\ \}\ \|x_n-u_n\| \\
&\quad +\{\ (2-k)\ \alpha_n^2\ (1+\beta_n\ L)+ \\
&\quad +\alpha_n\ [\ \beta_n\ L\ (1-\alpha_n)+\alpha_n\ L\ [\ (1+L)\beta_n+1\ ]\ ]\ \}\ \|x_n-Tx_n\| \\
&\quad +(2-k)\ \alpha_n^2\ \|x_n-Ty_n\|+L\ \alpha_n\ \|x_{n+1}-y_n\|+ \\
&\quad +\alpha_n L\ (\ (1-\alpha_n)+(1+L)\ \alpha_n+\ (2-k)\ \alpha_n)\ \|q_n\| \\
&\quad +(1+(2-k)\ \alpha_n+\alpha_n\ L)\ \|p_n\|\,.
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| \leq\ &\{\ (1+(1-k)\ \alpha_n)\ \left(1-\alpha_n+\ \alpha_n^2\right)\ +\ (2-k)\ \alpha_n^2\ \left(1+L^2\right) \\
&+\alpha_n\ L\ (1+L)\ ((1-\alpha_n)\ \beta_n+\alpha_n\ L)\ \}\ \|x_n-u_n\| \\
&+\{\ (2-k)\ \alpha_n^2\ (1+\beta_n\ L) \\
&+\alpha_n\ [\ \beta_n\ L\ (1-\alpha_n)+\ \alpha_n L\ [\ (1+L)\ \beta_n+1]\ ]\ \}\ \|x_n-Tx_n\| \\
&+(2-k)\ \alpha_n^2\ \|x_n-Ty_n\|+L\ \alpha_n\ \|x_{n+1}-y_n\| \\
&+\alpha_n\ L\ (\ (1-\alpha_n)\ +\ (1+L)\ \alpha_n+\ (2-k)\ \alpha_n\ )\ \|q_n\| \\
&+\ (\ 1+(2-k)\ \alpha_n+\alpha_n\ L\ )\ \|p_n\|\,.
\end{aligned}$$

That is

$$a_{n+1} \leq \gamma_n\ a_n+\sigma_n+w_n,$$

where

$$\begin{aligned}
a_n &:= \|x_n-u_n\|\,, \\
\gamma_n &:= [\ 1+\ (1-k)\ \alpha_n\ ]\ (1-\ \alpha_n+\ \alpha_n^2)\ +(2-k)\ (1+L^2)\ \alpha_n^2 \\
&\qquad +\alpha_n\ L\ (1+L)\ [\ \beta_n\ (1-\alpha_n)\ +\ L\ \alpha_n\ ], \\
w_n &:= \|p_n\|\ (1+(2-k)\ \alpha_n+\alpha_n\ L)\,, \\
\sigma_n &:= \alpha_n\ \{\ \{\ (2-k)\ \alpha_n\ (1+\beta_n\ L) \\
&\qquad +[\ \beta_n\ L\ (1-\alpha_n)\ +\ \alpha_n\ L\ [\ (1+L)\ \beta_n+1\ ]\ ]\ \}\ \|x_n-Tx_n\| \\
&\qquad +(2-k)\ \alpha_n\ \|x_n-Ty_n\|+L\ \|x_{n+1}-y_n\| \\
&\qquad +L\ \|q_n\|\ (\ (1-\alpha_n)\ +(3\ +L-k\ )\ \alpha_n\ )\ \}.
\end{aligned}$$

Remark that γ_n is the same as in formula (27) from [8]. The same motivation as in [8] leads us to

$$\gamma_n \leq 1 - k^2\alpha_n, \text{ from a sufficient large } n.$$

We get relation (7) with $\lambda_n := k^2\alpha_n$

$$a_{n+1} \leq (1-\lambda_n)\, a_n + \sigma_n + w_n.$$

Using (5) and using that Ishikawa iteration (2) converges i.e. $\lim_{n\to\infty} x_n = x^*$, (more precisely using $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0,\ \lim_{n\to\infty} \|x_n - Tx_n\| = 0$), it is easy to see that $\sigma_n = o(\lambda_n)$, and $\sum_{n=1}^{\infty} w_n < \infty$. All the assumptions from *Lemma 1* are satisfied, hence we have $\lim_{n\to\infty} a_n = 0$. That is

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{14}$$

We suppose that $\lim_{n\to\infty} x_n = x^*$. Relation (14) and the following inequality

$$\|u_n - x^*\| \leq \|x_n - u_n\| + \|x_n - x^*\| \to 0, (n \to \infty),$$

lead us to $\lim_{n\to\infty} u_n = x^*$. □

If we consider $\beta_n = 0$, in (2) and (4), then we have the following result

Theorem 2. *Let X be a Banach space and $T : X \to X$ be a Lipschitzian with $L \geq 1$, strongly pseudocontractive map. If $u_1 = x_1 \in X$, let $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, suppose that the errors satisfy* (5), *then the following two assertions are equivalent:*

(i) Mann iteration (1) *converges to $x^* \in F(T)$,*

(ii) Mann iteration with errors (3) *converges to the same $x^* \in F(T)$.*

The following result is from [8].

Theorem 3 [[8]]. *Let K be a closed convex (not necessarily bounded) subset of an arbitrary Banach space X and let T be a Lipschitzian pseudocontractive selfmap of K. Let us consider Mann iteration and Ishikawa iteration with the same initial point and with the conditions $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $x^* \in F(T)$. Then the following conditions are equivalent:*

(i) Mann iteration (1) *converges to $x^* \in F(T)$,*

(ii) Ishikawa iteration (2) *converges to $x^* \in F(T)$.*

Take $K := X$ in the above result. *Theorem 1*, *Theorem 2* and *Theorem 3* lead us to the conclusion:

Corollary 1. *In the same assumptions as in* Theorem 1 *we have the equivalence between the convergences of* (1), (2), (3) *and* (4).

3. The equivalence for strongly accretive and accretive maps

The map $J : X \to 2^{X^*}$ given by

$$Jx := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}, \forall x \in X,$$

is called *the normalized duality mapping.* Let us denote the identity map by I.

Definition 2. *Let X be a Banach space. A map $T : X \to X$ is called strongly pseudocontractive if there exists $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that*

$$\operatorname{Re} \langle Tx - Ty, j(x - y) \rangle \le k \|x - y\|^2,$$

for all $x, y \in X$. This is equivalent with (6).

A map $S : X \to X$ is called strongly accretive if there exists $\gamma \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\operatorname{Re} \langle Sx - Sy, j(x - y) \rangle \ge \gamma \|x - y\|^2.$$

for all $x, y \in X$.

A map $S : X \to X$ is called accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\operatorname{Re} \langle Sx - Sy, j(x - y) \rangle \ge 0.$$

for all $x, y \in X$.

Let us denote the identity map by I.

Remark 1. *Map T is a strongly pseudocontractive map with $k \in (0, 1)$ if and only if $(I - T)$ is a strongly accretive map with $(1 - k)$.*

Let us consider the following operator equation

$$Sx = f,$$

where S is a strongly accretive map and f is given. Consider the map $Tx = f + (I - S)x, \forall x \in X$. A fixed point for T will be a solution for the equation $Sx = f$; such a solution exists, see [6]. We consider iterations (2) and (4) with $f + (I - S)x$ instead of Tx.

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n (f + (I - S)y_n), \\ y_n &= (1 - \beta_n)x_n + \beta_n (f + (I - S)x_n), \quad n = 1, 2, \dots, \end{aligned} \tag{15}$$

and

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n (f + (I - S)v_n) + p_n, \\ v_n &= (1 - \beta_n)u_n + \beta_n (f + (I - S)u_n) + q_n, \quad n = 1, 2, \dots . \end{aligned} \tag{16}$$

Sequences $(\alpha_n)_n, (\beta_n)_n \subset (0, 1)$, are convergent such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Errors $(p_n)_n, (q_n)_n$ satisfy (5).

Theorem 1 assures that the Ishikawa iteration and Ishikawa iteration with errors are equivalent models for a strongly pseudocontractive map. Using *Remark 1*, observe that if S is Lipschitzian and strongly accretive, then the map $Tx = f+(I-S)x$ is Lipschitzian strongly pseudocontractive. We obtain

Theorem 4. *Let X be a Banach space, let $S : X \to X$ be a Lipschitzian with $L \geq 1$, a strongly accretive map. If $u_1 = x_1 \in B$, then the following two assertions are equivalent:*

(i) Ishikawa iteration (15) *converges to $x^* \in F(T)$, which is the solution of $Sx = f$,*

(ii) Ishikawa iteration with errors (16) *converges to the same $x^* \in F(T)$, which is the solution of $Sx = f$.*

When we take $\beta_n = 0$, we get a similar result for Mann iteration and Mann iteration with errors. Thus Mann iteration with errors (3) is equivalent with Mann iteration (1), when we take $Tx = f + (I - S)x, \forall x \in X$.

According to observation from [8], *Theorem 3* holds if we take the above operator T, with S strongly accretive. *Theorem 4* and the equivalence between Mann iteration with errors (3) and Mann iteration (1) lead us to the following conclusion

Corollary 2. *In the same assumptions as in* Theorem 4 *we have the equivalence between the convergences of* (1), (2), (3) *and* (4) *for a Lipschitzian strongly accretive map S, and $Tx = f + (I - S)x, \forall x \in X$.*

Remark 2. *If S is an accretive map, then $T = f - S$ is a strongly pseudocontractive map.*

Proof. For all $x, y \in X$ and $j(x - y) \in J(x - y)$, we have

$$\begin{aligned}
\operatorname{Re}\langle Sx - Sy, j(x-y)\rangle &\geq 0 \Leftrightarrow \\
\operatorname{Re}\langle (f-T)x - (f-T)y, j(x-y)\rangle &\geq 0 \Leftrightarrow \\
-\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle &\geq 0 \Leftrightarrow \\
\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle &\leq 0 \leq k\,\|x-y\|^2, \forall k \in (0,1).
\end{aligned}$$

□

Let us consider the following operator equation

$$x + Sx = f,$$

where S is a strongly accretive map and f is given. The existence of the solution for $x + Sx = f$ follows from [5]. It is clear that $x + Sx$ is Lipschitzian if S is. Consider the map $Tx = f - Sx, \forall x \in X$. A fixed point for T will be a solution for the equation $Sx = f$. Using *Remark 2*, if S is an accretive map, then T is strongly pseudocontractive. Now let us consider iterations (2) with $Tx = f - Sx$

$$\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n (f - Sy_n), \\
y_n &= (1-\beta_n)x_n + \beta_n (f - Sx_n), \quad n = 1, 2, \ldots,
\end{aligned} \tag{17}$$

and the Ishikawa iteration with errors (4):

$$\begin{aligned}
u_{n+1} &= (1-\alpha_n)u_n + \alpha_n (f - Sv_n) + p_n, \\
v_n &= (1-\beta_n)u_n + \beta_n (f - Su_n) + q_n, \quad n = 1, 2, \ldots.
\end{aligned} \tag{18}$$

Sequences $(\alpha_n)_n, (\beta_n)_n \subset (0,1)$ are convergent such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The errors verify (5).

Theorem 1 assures that Ishikawa iteration and Ishikawa iteration with errors are equivalent models for a strongly pseudocontractive map. According to *Remark 2*, the map $Tx = f - Sx, \forall x \in X$, is (Lipschitzian) strongly pseudocontractive map when S is a (Lipschitzian) accretive. These arguments lead us to the following conclusion

Theorem 5. *Let X be a Banach space, let $S : X \to X$ be a Lipschitzian with $L \geq 1$, accretive map. If $u_1 = x_1 \in X$, then the following two assertions are equivalent:*

(i) Ishikawa iteration (17) *converges to $x^* \in F(T)$, which is the solution of $x + Sx = f$*

(ii) Ishikawa iteration with errors (18) *converges to the same $x^* \in F(T)$, which is the solution of $x + Sx = f$.*

When we take $\beta_n = 0$, we get a similar result for Mann iteration and Mann iteration with errors. Thus Mann iteration with errors (3) is equivalent with Mann iteration (1), when we take $Tx = f - Sx, \forall x \in X$.

According to second observation from [8], *Theorem 1* holds if we take $Tx = f - Sx, \forall x \in X$, with S accretive. Also *Theorem 5* and the remark concerning the equivalence between Mann iteration with errors and Mann iteration for $Tx = f - Sx, \forall x \in X$, lead us to

Corollary 3. *In the same assumptions as in* Theorem 5 *we have the equivalence between the convergences of* (1), (2), (3) *and* (4) *for a Lipschitzian accretive map S, and $Tx = f - Sx, \forall x \in X$.*

4. The multivalued case

In the multivalued case the following definitions hold

Definition 3. *Let X be a real Banach space. A map $T : X \to 2^X$ is called strongly pseudocontractive if there exists $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that*

$$\langle \xi - \theta, j(x-y) \rangle \leq k \left\| x - y \right\|^2 ,$$

for all $x, y \in X, \xi \in Tx, \theta \in Ty$.

Let $S : X \to 2^X$, the map S is called strongly accretive if there exists $\gamma \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle \xi - \theta, j(x-y) \rangle \geq \gamma \left\| x - y \right\|^2 .$$

for all $x, y \in X, \xi \in Tx, \theta \in Ty$, etc.

We remark that all results from this paper hold in the multivalued case, provided that these multivalued maps admit single valued selections.

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