CODEN STJSAO ISSN 0562-1887
ZX470/1578 UDK 624.073.8:519.61/.63

Dynamic Analogies for Proving the Convergence of Boundary Value Plate Problems

Željana NIKOLIĆ, Ante MIHANOVIĆ* and *Pavao MAROVIĆ

Građevinsko arhitektonski fakultet,
Sveučilište u Splitu (Faculty of Civil Engineering and Architecture, University of Split),
Matice hrvatske 15, HR-21000 Split,
Republic of Croatia

zeljana.nikolic@gradst.hr

Original scientific paper

This paper describes a procedure for proving the convergence and for estimating the numerical solution error of boundary value problems. The procedure is based on the transformation of a discrete boundary value problem into an equivalent discrete dynamic eigenproblem. Discrete dynamic eigenproblem has physical meaning in convergence analysis because a mass represents the measure of domain discretization. The convergence and accuracy of numerical solution of boundary value problem depend on the convergence of discrete dynamic eigenproblem spectrum. The developed procedure is relatively simple, easy to perform; in this paper it is used to evaluate the convergence and accuracy of numerical solution of the thin plate bending problem. The plate is discretized with four-node finite elements. One translational and two rotational degrees of freedom, which are independent of each other, are associated to each node of the plate. The shape functions satisfy a homogenous differential equation of plate bending. The developed procedure gives the greatest global error which can appear for a chosen discretization. The performance of the proposed method is illustrated by the solution procedure of two examples: a simply supported square thin plate and a cantilever square thin plate.

Keywords
Accuracy
Computational mechanics
Convergence
Dynamic analogies
Finite element modelling
Plate elements

Ključne riječi
Dinamička analogija
Konvergenicaja
Modeliranje konačnim elementima
Pločasti elementi
Računalna mehanika
Točnost

Received (primljeno): 2007-12-04
Accepted (prihvaćeno): 2008-02-18

Dinamička analogija za dokazivanje konvergencije rubnih zadaća ploča

Izvornoznanstveni članak

U ovome radu se opisuje postupak dokazivanja konvergencije i ocjene točnosti numeričkog rješenja rubnih zadaća. Postupak se temelji na transformaciji diskretne rubne zadaće u ekvivalentnu diskretnu dinamičku vlastitu zadaću. Diskretna dinamička vlastita zadaća ima fizikalno značenje jer masa predstavlja mjeru diskretizacije područja. Konvergencija i točnost numeričkog rješenja rubne zadaće ovisi o konvergenciji diskretne dinamičke vlastite zadaće. Razvijeni postupak je relativno jednostavan i lako izvediv, a u ovome radu je upotrijebljen za analizu konvergencije i točnosti numeričkog rješenja tankih ploča. Ploča je diskretizirana s četveročvornim konačnim elementima. Svakom čvoru ploče su pridruženi jedan translacijski i dva rotacijska stupnja slobode koji su međusobno neovisni. Bazne funkcije zadovoljavaju homogenu diferencijalnu jednadžbu savijanja ploča. Razvijenim postupkom dobivamo najveću moguću pogrešku koja se može pojaviti kod određene diskretizacije. Učinkovitost predloženog postupka je prikazana u postupku rješavanja dva primjera: slobodno oslonjene kvadratne tanke ploče i uklij eštene kvadratne tanke ploče.

1. Introduction

The application of spectra of matrices and operators has a great importance in many branches of engineering science. The purporse of this work is to use the spectrum of generalized eigenproblem for proving the existance and convergence of the solution of boundary value problem and, also, for estimating the global discretization error.

Yosifian, Oleinik and Shamaev [11] found a suitable set of conditions that ensure convergence of the spectrum of selfadjoint operators A_n $(n \in N)$, as $n \to \infty$ towards the spectrum of corresponding limit-operator. Under the same conditions convergence of the eigenvectors is also proved.

A sequence of generalized eigenproblems in different Hilbert spaces was considered by Jurak [2] and conditions ensuring convergence of the eigenvalues and eigenvectors were given. An application to the Laplace type boundary value problem was presented.

Mihanović and Radelja [6] proved that, if the applied numerical method is based on the weak formulation,

Symbols/Oznake

$\mathbf{a}$	- load vector - vektor opterećenja
A	- differential operator - diferencijalni operator
a_i	- component of load vector - komponenta vektora opterećenja
B	- differential operator - diferencijalni operator
E	- Young's modulus - Youngov modul elastičnosti
$f(\Omega)$	- action on the domain Ω - djelovanje na području Ω
$\mathbf{f}_n$	- vector of the nodal forces - vektor čvornih sila
$g(\Gamma)$	- tractions at the boundary Γ of the domain Ω - djelovanje na rubu Γ područja Ω
H	- Hilbert space - Hilbertov prostor
h_{11}	- shape function of the translation - bazna funkcija translacije
h_{12}	- shape function of the rotation around ξ axis - bazna funkcija zaokreta oko osi ξ
h_{13}	- shape function of the rotation around η axis - bazna funkcija zaokreta oko osi η
$\mathbf{I}$	- unit matrix - jedinična matrica
$\mathbf{K}_n$	- stiffness matrix - matrica krutosti
$m_i(\Omega)$	- mass distributed over the domain Ω - masa raspodijeljena po području Ω
$\mathbf{M}_n$	- mass matrix - matrica masa
Q	- volume of the domain Ω - volumen područja Ω
$R^l_{u_i}$	- error of the i-th component of the load vector - pogreška *i*-te komponente vektora opterećenja
s_i	- component of load vector - komponenta vektora opterećenja
$u(\Gamma)$	- displacements at the boundary Γ of the domain Ω - pomaci na rubu Γ područja Ω
$u(\Omega)$	- displacements over the domain Ω - pomaci na području Ω
u_j	- component of displacement - komponenta pomaka
$\mathbf{u}_n$	- displacement vector - vektor pomaka
γ_i	- influence of the *i*-th eigenvector to the total solution - učešće *i*-tog vlastitog vektora u ukupnom rješenju
δ_i	- error of the *i*-th eigenvector - pogreška *i*-tog vlastitog vektora
$\hat{\delta}_i$	- relative error of the *i*-th eigenvector - relativna pogreška *i*-tog vlastitog vektora
δ^l_i	- relative error of the *i*-th component of the load vector - relativna pogreška *i*-te komponente vektora opterećenja
δ_{kl}	- Kronecker delta - Kronecker delta
$\hat{\delta}_{uk}$	- total displacement error - ukupna pogreška pomaka
Φ_{ij}	- component of eigenvector matrix - komponenta vlastite matrice
$\mathbf{\Phi}_n$	- eigenvector matrix - vlastita matrica
λ, λ^i_j	- eigenvalue - vlastita vrijednost
φ^i_j	- eigenvector - vlastiti vektor
$\lambda^n_j,\ \varphi^n_j$	- pairs of eigenvalues and eigenvectors in Hilbert spaces H_n - parovi vlastitih vrijednosti i vlastih vektora u Hilbertovim prostorima H_n
$\mathbf{\Lambda}$	- spectral matrix - spektralna matrica
ν	- Poisson's ratio - Poissonov koeficijent
Ω	- domain - područje

the generalized eigenproblem can be associated to the boundary value problem. In that case the numerical solution of the boundary value problem exist and converge to the exact solution. The described procedure can be used for proving the existence of solution of observed method and convergence of the numerical solution to the exact one. In this paper the method is applied for proving the convergence and calculating the discretization error of four-node plate finite element with one translational and two rotational degrees of freedom and the shape functions satisfying the homogenous differential equation of the thin plate bending [8].

The bending of thin plate is associated with fourth-order differential equation. The mathematical proof of the conditions ensuring convergence of the solution of thin plate is very complex and it is not the aim of this paper. The convergence analysis of the numerical solution for thin plate is performed indirectly. Namely, the numerical model of the boundary value problem has an appertained discrete standard eigenproblem and discrete dynamic eigenproblem. Dynamic eigenproblem has physical meaning because a mass represents the measure of the domain discretization. If eigenvalues of the associated discrete dynamic eigenproblem converge to the exact eigenvalues of generalized eigenproblem, it can be stated that, the numerical solution of the boundary value problem exists and converges to the exact solution with the increasing of a number of degrees of freedom [6, 7]. Therefore, convergence of the numerical solution depends on the convergence of discrete dynamic eigenproblem spectrum. This conclusion can be applied in convergence analysis of different boundary value problems.

The global discretization error of the finite element approximation can be estimated by using the eigenvalues and eigenvectors of discrete dynamic eigenproblem. It is different procedure from the estimation of global error based on energy norm in papers of Zienkiewicz and Taylor [12], Kelly, Gago, Zienkiewicz and Babuška [3], Zienkiewicz and Zhu [13], Li and Wiberg [4].

The organization of this paper is as follows. In Section 2, we present theoretical considerations of the proposed procedure and the transformation of the discrete boundary value problem into an equivalent discrete dynamic problem. Some considerations concerning convergence criteria of the numerical solution and numerical solution accuracy are shown in Sections 3 and 4, while Section 5 introduces the necessary mass and stiffness matrices used in the thin plate bending problem. The application of the proposed procedure is illustrated in Section 6 by means of two examples, i.e. a simply supported square thin plate and a cantilever square thin plate, together with appropriate comments. The general conclusion is presented in Section 7.

2. Dynamic eigenproblem of the equilibrium boundary value problem

In this paper we are dealing with the equilibrium boundary value problem over the domain, consisting of several parts which are deformable solids. The governing equation is:

$$A\left[u(\Omega)\right] = f(\Omega), \tag{1}$$

where A is differential operator, $u(\Omega)$ are unknown displacements over the domain Ω and $f(\Omega)$ are the known action on the domain Ω, with the appropriate boundary conditions:

$$B\left[u(\Gamma)\right] = g(\Gamma), \tag{2}$$

where B is a differential operator, $u(\Gamma)$ are displacements at the boundary Γ of the domain Ω and $g(\Gamma)$ are tractions at the same boundary. We assume that the boundary conditions (2) are such that the whole domain is supported as a kinematics rigid body.

The standard eigenproblem:

$$A\left[u(\Omega)\right] = \lambda\, u(\Omega), \tag{3}$$

where λ represents eigenvalues, can be associated with the boundary value problem. The solutions of the standard eigenproblem are pairs of eigenvalues and orthonormalized eigenfunctions $\lambda^i{}_j, \varphi^i{}_j(\Omega),\ j = 1,2,3,\ldots$ spanning the separable Hilbert space H. If that is the case, then for $f(\Omega)$ limited, the solution of the boundary value problem (1) exists.

Let us introduce the generalized eigenproblems from the standard eigenproblem (3) as follows:

$$\begin{aligned} &A\left[u(\Omega)\right] = \lambda\, m_i(\Omega)\, u(\Omega), \quad i = 1,2,3,\ldots \\ &0 < m_i(\Omega) < +\infty, \qquad \int_\Omega m_i(\Omega)\ d\Omega = \int_\Omega d\Omega = Q, \quad \forall i \end{aligned} \tag{4}$$

and name them equivalent dynamic eigenproblems. The functions $m_i(\Omega)$ can be treated as a mass distributed over the whole domain, and Q is a constant equal to a volume over domain Ω. Let eigenproblems (4) have solutions $\lambda^i{}_j, \varphi^i{}_j(\Omega),\ j = 1,2,3,\ldots$ such that:

$$\int_\Omega \varphi_k\, m_i(\Omega)\, \varphi_l\, d\Omega = \delta_{kl}, \tag{5}$$

where δ_{kl} is Kronecker delta function.

The numerical model of the boundary value problem (1) and (2) is:

$$\mathbf{K}_n\, \mathbf{u}_n = \mathbf{f}_n \tag{6}$$

and it has an appertained standard eigenproblem:

$$\mathbf{K}_n\, \mathbf{\Phi}_n = \mathbf{\Lambda I \Phi}_n. \tag{7}$$

The appropriate dynamic eigenproblem can be written in the following form:

$$\mathbf{K}_n\, \mathbf{\Phi}_n = \mathbf{\Lambda M}_n \mathbf{\Phi}_n, \tag{8}$$

where solutions are eigenvalues and eigenvectors λ''_j, φ''_j spanning a separable Hilbert spaces H_n. If a mass matrix is diagonal and especially if $\mathbf{M}_n = \frac{1}{n}\mathbf{I}$, then matrix $\mathbf{M}_n$ represents a discretization measure of the domain . In that case the eigenproblem (8) represents an equivalent dynamic eigenproblem for a standard eigenproblem, where eigenvectors are identical, but the quotient of eigenvalues is n.

When n, the discrete mass is transformed into a continuous function derivable over parts of the domain:

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{I} : \Omega \to m_n(\Omega). \tag{9}$$

If eigenvalues of the dynamic eigenproblem (8) converge when n, the numerical solution of boundary value problem (1) and (2) exists and converges towards the exact solution. In the special case when:

$$\lim_{k\to\infty}\frac{1}{k}\mathbf{I}:\Omega\to m_k(\Omega)\equiv\text{const},\tag{10}$$

eigenpairs from the Eq. (8) are identical to eigenpairs obtained from Eq. (3).

3. Convergence criteria of numerical solution

The mathematical proof of the conditions ensuring convergence of the solutions in different boundary value problems is complex and is not the aim of this paper. The convergence analysis can be performed indirectly.

Namely, it was proved in in the works [2, 11] that the convergence in the Laplace type boundary value problem is ensured if the numerical method is based on the weak formulation or any other formulation which produces the same matrix operator **K**. In that case the numerical solution of the boundary value problem exists and converges to the exact solution when the number of degrees of freedom increases, while the spectrum of the operators $\mathbf{K}_n$ of discrete dynamics problem converges towards the spectrum of the corresponding limit-operator. This proof can be inverted. If the convergence of eigenvalues of the associated discrete dynamic problem is achieved:

$$\lim_{n\to\infty}\lambda_k^n=\lambda_k,\tag{11}$$

it can be stated that, when *n*, the numerical solutions of the boundary value problem converge towards the exact solutions [6, 7].

4. Numerical solution accuracy

Once we have proved that a numerical solution is convergent to the given class of boundary value problems, the next step is to find out how the discretization affects the accuracy of the results.

The load type influences the discretization error in addition to other factors. It is necessary to represent the load over the domain as a linear combination of eigenvectors if the equivalent discrete dynamic problem can be used to estimate the numerical solution error. The influence of each eigenvector on the numerical solutions can be determined by computing the coefficients in the load representation by a linear combination of eigenvectors.

The procedure for the estimation of the error consists of two phases. The relative error in any component of each eigenvector has to be computed firstly. It is independent of the applied load and, for the observed discretization, it can be evaluated in relation to the analytical solution or to the solution obtained by another discretization. Afterwards, the error of each eigenvector, based on the influence of each eigenvector and its relative error upon the numerical solutions, is computed. The total error is obtained by summing the errors of all eigenvectors [7].

The vector of the nodal forces $\mathbf{f}_n$ from Eq. (6) can be represented as a linear combination:

$$\mathbf{f}_n=\mathbf{\Phi}_n\,\mathbf{a},\tag{12}$$

where $\mathbf{f}_n$ is the nodal forces vector and $\mathbf{\Phi}_n$ is the eigenvector matrix of dynamic eigenproblem (8). The components of vector $\boldsymbol{a}$ follow from:

$$\mathbf{a}=\mathbf{\Phi}_n^{\mathrm{T}}\,\mathbf{f}_n.\tag{13}$$

Because of **K** and **I** orthogonality of $\mathbf{\Phi}_n$, the solution **u** can be represented as:

$$\mathbf{u}=\sum_{i=1}^{k}u_i=\sum_{i=1}^{k}\frac{a_i}{\lambda_i}\varphi_i,\tag{14}$$

where *k* is a number of degrees of freedom for a given discretization and it corresponds to the maximal possible number of the eigenvalues of the dynamic eigenproblem.

If we define the norm:

$$|\mathbf{u}|=\sum_{i=1}^{k}|u_i|,\qquad |u_i|=\sqrt{\sum_{i=1}^{k}u_{i,j}^2}=\frac{|a_i|}{|\lambda_i|}\sqrt{\varphi_i^T\varphi_i}=\frac{|a_i|}{|\lambda_i|},\tag{15}$$

we get:

$$|\mathbf{u}|=\sum_{i=1}^{k}\frac{|a_i|}{|\lambda_i|}.\tag{16}$$

Let us consider two special cases of a load vector **f**. The first case is a concentrated unit force in the *m* direction. Therefore, Eq. (13) becomes:

$$a_i=\varphi_{i,m}\tag{17}$$

and the components of displacements are:

$$u_j=\sum_{i=1}^{k}\frac{a_i}{\lambda_i}\varphi_{i,j}=\sum_{i=1}^{k}\frac{\varphi_{i,m}\varphi_{i,j}}{\lambda_i}.\tag{18}$$

The second case is a uniformly distributed load of intensity *k*. The components of the load vector $\boldsymbol{a}$ given by Eq. (13) are:

$$\mathbf{a}=\Phi^{\mathrm{T}}\mathbf{1}=[a_1,a_2,\dots a_i,\dots,a_k]\tag{19}$$

where:

$$a_i=s_i=\sum_{j=1}^{k}\varphi_{i,j}.\tag{20}$$

Because of the property of orthonormalized eigenvectors:

$$s_{i-1} > s_i > s_{i+1}, \tag{21}$$

the components of displacement u_j (j = 1, ..., k) can be evaluated as:

$$u_j = \sum_{i=1}^{k} \frac{s_i}{\lambda_i} \varphi_{i,j}. \tag{22}$$

Let us suppose that we have two discretizations E_l and E_k, where $k \rightarrow \infty$, under condition that $k>l$ and that k mesh contains all degrees of freedom of the l mesh. Let us suppose that the load has as many components as the l mesh.

We can formally reduce the k mesh to the k_r mesh by omitting those degrees of freedom which are not contained in mesh l to obtain:

$$\varphi_i^k \rightarrow \varphi_i^{k_r}, \qquad a_i^{k_r} = a_i^k, \qquad \lambda_i^{k_r} = \lambda_i^k. \tag{23}$$

According to Eqs. (13) and (14), the error of the i-th component of the load vector in the l mesh caused by eigenvector i can be presented as:

$$R_{u_i}^l = \frac{a_i^l}{\lambda_i^l}\varphi_i^l - \frac{a_i^{k_r}}{\lambda_i^{k_r}}\varphi_i^{k_r}. \tag{24}$$

If l is large enough, then for a few first eigenvectors where for $i<l$, we obtain:

$$\varphi_i^l \approx \varphi_i^{k_r}, \qquad a_i^l \approx a_i^{k_r}. \tag{25}$$

Then, the error can be rewritten as:

$$R_{u_i}^l = \left(\frac{1}{\lambda_i^l} - \frac{1}{\lambda_i^{k_r}}\right) a_i^l \varphi_i^l. \tag{26}$$

In this case the relative error in any component of the i-th vector is:

$$\delta_i^l = \frac{\left(\dfrac{1}{\lambda_i^l} - \dfrac{1}{\lambda_i^{k_r}}\right)}{\dfrac{1}{\lambda_i^{k_r}}} = \frac{\lambda_i^{k_r} - \lambda_i^l}{\lambda_i^l}. \tag{27}$$

The total error is obtained by summing up the errors of all eigenvectors (Nikolić 1999). The evaluation of the total error is represented by a few examples in the Section 6.

5. Mass and stiffness matrices in the thin plate bending problem

The convergence and the accuracy of the numerical solution are analyzed for a four-node plate element with three independent degrees of freedom per node, one translation perpendicular on the plate mid-surface and two rotations. This is in accordance with a unified approach to structural system modelling, described by Ibrahimbegović [1], which eliminates a problem of joining individual elements to the compatible whole. The shape functions for the first node of the element are given by the following equation:

$$\begin{aligned} h_{11}(\xi,\eta) &= \frac{1}{8}\left(2 - 3\xi - 3\eta + 4\xi\eta + \xi^3 + \eta^3 - \xi^3\eta - \xi\eta^3\right), \\ h_{12}(\xi,\eta) &= \frac{1}{8}\left(1 - \xi - \eta + \xi\eta - \eta^2 + \xi\eta^2 + \eta^3 - \xi\eta^3\right), \\ h_{13}(\xi,\eta) &= \frac{1}{8}\left(1 - \xi - \eta + \xi\eta - \xi^2 + \xi^2\eta + \xi^3 - \xi^3\eta\right), \end{aligned} \tag{28}$$

where h_{11} is the function of the translation, h_{12} is the function of the rotation around ξ axis, and h_{13} is the function of the rotation around η axis of the local coordinate system ξ-η.

The solution of a dynamic eigenproblem requires the formation of the mass and stiffness matrices. The computation can employ either the consistent mass matrix, whose elements are computed with shape functions approximating the displacement field, or the diagonal mass matrix when the inertial forces are distributed uniformly over nodes. In solving the standard eigenproblem the mass matrix is diagonal with units on the diagonal. Due to the equivalence of the dynamic and standard eigenproblem, matrix $\mathbf{M}_n$ must be diagonal when analyzing the convergence and proving the accuracy of the equilibrium problem solution. The translational masses on the diagonal are computed according to Eq. (9), while the rotational masses are neglected since their influence is lost due to mesh refinement.

6. Numerical examples

The appropriate numerical algorithm is developed on the basis of the theoretical considerations. The governing equilibrium equation is of a biharmonic form [7]. The finite element method is applied. The structure is discretized by four-node plate elements with a total of 12 degrees of freedom. One translational and two rotational displacements, which are independent of each other, are associated to each element node. The shape functions satisfy a homogenous differential equation of plate bending. In order to demonstrate the proposed method for proving the convergence and to estimate the numerical solution error two examples are solved: (a) a simply supported square thin plate, and (b) a cantilever square thin plate.

6.1. Example 1: Simply supported square thin plate

A simply supported square thin plate subjected to flexure by a uniformly distributed load is analyzed. The plate length is L = 1 m and the thickness of the plate is h = 0.001 m. Young's modulus is $E = 1.2 \times 10^{10}$ kN/m^2 and the Poisson ratio is v = 0. The unit mass is uniformly distributed over the plate area.

The equivalent dynamic eigenproblem is analyzed for several discretizations, namely 4×4, 8×8, 12×12, 16×16 and 24×24 element mesh. Figure 1 shows the first four eigenvectors while Table 1 shows the first eight eigenvalues. It is obvious that the first eight eigenvalues obtained numerically, converge to the analytical eigenvalues [5, 9] with the increasing number of degrees of freedom. These results show that the numerical solution converges to the analytical one.

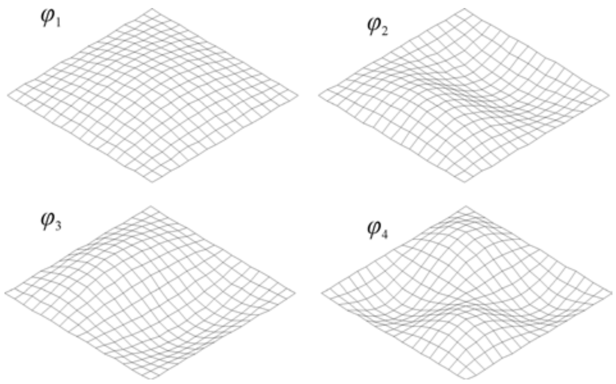

Figure 1. Eigenvectors of a simply supported square thin plate

Slika 1. Vlastiti vektori slobodno oslonjene kvadratne tanke ploče

The plate loads are developed into series by eigenvectors for proving the numerical solution accuracy, the coefficients $\hat{a}_{ij}$ are determined [10] and the influence of each of the eigenvectors upon the total solution is computed as shown in Table 2.

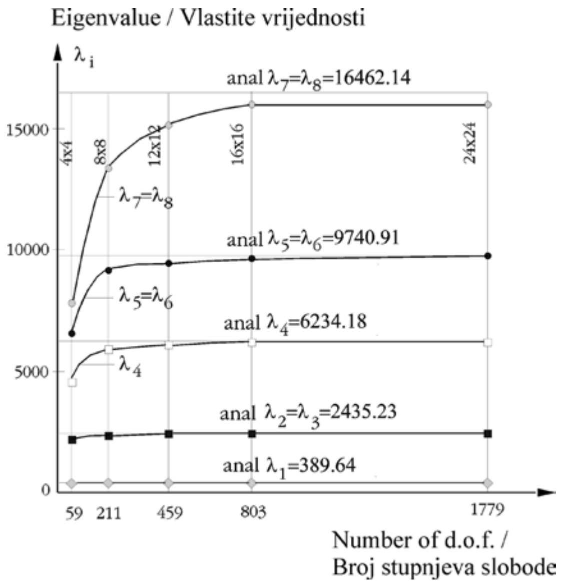

Figure 2. Eigenvalues of a simply supported square thin plate

Slika 2. Vlastite vrijednosti slobodno oslonjene kvadratne tanke ploče

The relative error δ_i and total error $\hat{\delta}_i$ for each of the eigenvectors are computed according to the influence of each vector upon the total solution. Table 3, Figure 2 and Figure 3 show the relative and total eigenvector errors. The relative error for the given discretization is the lowest

Table 1. Eigenvalues of a simply supported square thin plate

Tablica 1. Vlastite vrijednosti slobodno oslonjene kvadratne tanke ploče

Eigenvalue / Vlastite vrijednosti	2×2 k=19	4×4 k=59	8×8 k=211	12×12 k=459	16×16 k=803	24×24 k=1779	Analytical / Analitički k=∞
$\lambda_1 = \hat{\lambda}_{11}$	109.10	368.30	384.60	387.58	388.32	388.88	389.64
$\lambda_2 = \hat{\lambda}_{12}$	-	2177.46	2344.55	2418.14	2428.24	2435.60	2435.23
$\lambda_3 = \hat{\lambda}_{21}$	-	2177.46	2344.55	2418.14	2428.24	2435.60	2435.23
$\lambda_4 = \hat{\lambda}_{22}$	-	4551.94	5899.34	6091.89	6155.93	6201.26	6234.18
$\lambda_5 = \hat{\lambda}_{13}$	-	6559.39	9142.61	9428.13	9625.76	9693.13	9740.91
$\lambda_6 = \hat{\lambda}_{31}$	-	6559.39	9142.61	9428.13	9625.76	9693.13	9740.91
$\lambda_7 = \hat{\lambda}_{23}$	-	7797.62	13352.20	15129.80	15972.10	15981.00	16462.14
$\lambda_8 = \hat{\lambda}_{32}$	-	7797.62	13352.20	15129.80	15972.10	15981.00	16462.14

for the first eigenvector and it increases for the higher-order eigenvectors. It is distinct because, for the same discretization, the first eigenvector, which is relatively simple, cannot be obtained with the same accuracy as the higher-order vectors with a more complex form. The relative error of the eigenvector components is reduced when the number of degrees of freedom increases.

Table 2. Computation of the influence of eigenvectors upon the total solution for a simply supported square thin plate

Tablica 2. Proračun učešća vlastitih vektora u ukupnom rješenju za slobodno oslonjenu kvadratnu tanku ploču

$\hat{\lambda}_{ij}$	$\hat{a}_{ij} = \dfrac{16q_0}{\pi^2 ij}$	$\dfrac{\hat{a}_{ij}}{\hat{\lambda}_{ij}}$	$\gamma_{ij} = \dfrac{\hat{a}_{ij}/\hat{\lambda}_{ij}}{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\hat{a}_{ij}/\hat{\lambda}_{ij}}$
$\hat{\lambda}_{11}$= 389.64	1.62113894	0.00416065	0.96908432
$\hat{\lambda}_{12}$= 2435.23	0	0	0
$\hat{\lambda}_{21}$= 2435.23	0	0	0
$\hat{\lambda}_{22}$= 6234.18	0	0	0
$\hat{\lambda}_{13}$= 9740.91	0.54037965	0.00005548	0.01292112
$\hat{\lambda}_{31}$= 9740.91	0.54037965	0.00005548	0.01292112
$\hat{\lambda}_{23}$= 16462.14	0	0	0
$\hat{\lambda}_{32}$= 16462.14	0	0	0
$\hat{\lambda}_{41}$= 28151.23	0	0	0
$\hat{\lambda}_{14}$= 28151.23	0	0	0
$\hat{\lambda}_{33}$= 31560.55	0.18012655	0.00000571	0.00132933
S	S	S	S
	$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\dfrac{\hat{a}_{ij}}{\hat{\lambda}_{ij}}$	0.00429338	

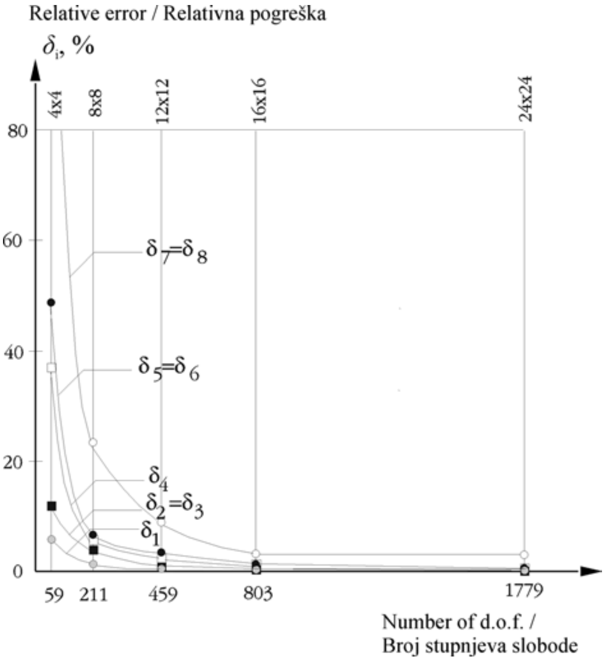

Figure 3. Relative eigenvector error (%) of a simply supported square thin plate

Slika 3. Relativna pogreška vlastitih vektora (%) za slobodno oslonjenu kvadratnu tanku ploču

The total error $\hat{\delta}_i$ of the *i*-th eigenvector is the greatest for the first eigenvector while it decreases for the higher-order eigenvectors because the influence of these vectors upon the total solution becomes smaller. The total solution error can be computed by summing up the errors of all eigenvectors. The fifth and sixth eigenvector have a total error less than 1 per cent (%) for all discretizations. The total error for the higher-order eigenvectors becomes smaller so the solution accuracy can be estimated quite precisely with several first eigenvectors. The total error of the numerical solution, in relation to the exact solution, is estimated upon the basis of the first eight eigenvectors and for discretizations 4x4, 8x8, 12x12, 16x16 and 24x24 is successively equal to 6.8207 %, 1.4284 %,

Table 3. Relative and total eigenvector error (%) for a simply supported square thin plate

Tablica 3. Relativna i ukupna pogreška vlastitih vektora (%) za slobodno oslonjenu kvadratnu tanku ploču

φ_i	4×4		8×8		12×12		16×16		24×24	
	δ_i	$\hat{\delta}_i$	δ_i	$\hat{\delta}_i$	δ_i	$\hat{\delta}_i$	δ_i	$\hat{\delta}_i$	δ_i	$\hat{\delta}_i$
φ_1	5.7942	5.5673	1.3105	1.2592	0.5315	0.5107	0.3399	0.3266	0.1954	0.1877
φ_2	11.8381	0.0000	3.8677	0.0000	0.7067	0.0000	0.2879	0.0000	0.0152	0.0000
φ_3	11.8381	0.0000	3.8677	0.0000	0.7067	0.0000	0.2879	0.0000	0.0152	0.0000
φ_4	36.9566	0.0000	5.6759	0.0000	2.3357	0.0000	1.2711	0.0000	0.5309	0.0000
φ_5	48.5033	0.6267	6.5441	0.0846	3.3175	0.0429	1.1963	0.0155	0.4929	0.0064
φ_6	48.5033	0.6267	6.5441	0.0846	3.3175	0.0429	1.1963	0.0155	0.4929	0.0064
φ_7	111.1175	0.0000	23.2916	0.0000	8.8061	0.0000	3.0681	0.0000	3.0107	0.0000
φ_8	111.1175	0.0000	23.2916	0.0000	8.8061	0.0000	3.0681	0.0000	3.0107	0.0000
$\sum_{i=1}^{8}\hat{\delta}_i$		6.8207	-	1.4284	-	0.5965	-	0.3576	-	0.2005

0.596 %, 0.3576 % and 0.2005 % (see Table 3). The error of the numerical solution cannot be as high as those values at any plate node. The developed procedure for the estimation of the numerical solution shown in this paper guarantees the upper bounds of the error at the finite element nodes for a given discretization.

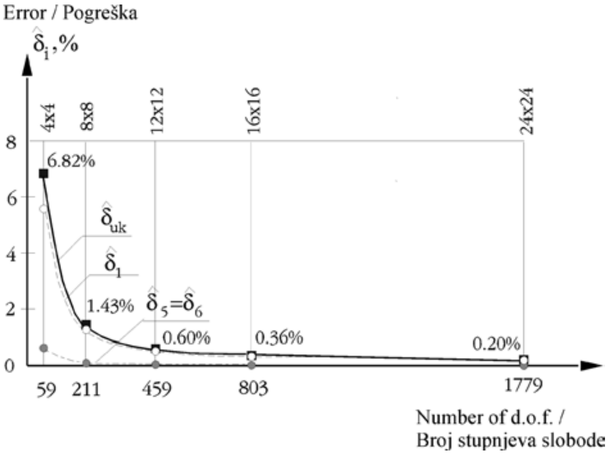

Figure 4. Displacement error of a simply supported square thin plate subjected to a uniform load

Slika 4. Pogreška pomaka slobodno oslonjene kvadratne tanke ploče izložene jednoliko raspodijeljenom opterećenju

6.2. Example 2: Cantilever square thin plate

A cantilever square thin plate subjected to flexure by a uniformly distributed load is analyzed. The plate length is L = 1 m, Young's modulus is $E = 1.2 \times 10^{10}$ kN/m^2, the thickness of the plate is h = 0.001 m.

The equivalent dynamic eigenproblem is analyzed for several discretizations, namely 4×4, 8×8, 12×12 and 16×16 element mesh. Table 4 and Figure 5 show the first six numerical eigenvalues and eigenvectors.

Table 4. Eigenvalues of a cantilever square thin plate

Tablica 4. Vlastite vrijednosti ukliještene kvadratne tanke ploče

Eigenvalue / Vlastita vrijednost	4×4 k=60	8×8 k=216	12×12 k=468	16×16 k=816
λ_1	11.68	12.22	12.30	12.29
λ_2	82.64	90.83	92.42	92.95
λ_3	467.80	465.33	477.93	482.41
λ_4	884.85	806.131	819.02	827.44
λ_5	999.61	1187.21	1221.67	1233.53
λ_6	4381.51	3946.26	3859.84	3924.34

The first two eigenvalues converge fast to the exact values. The convergence is slower for the higher-order eigenvalues. The reason is that for the same discretization, the first eigenvector, which is relatively simple, cannot be obtained with the same accuracy as the higher-order vectors with a more complex form. Because of that the third, fourth and fifth eigenvalue start to converge with 8×8 mesh while the sixth eigenvalue starts to converge with 12×12 mesh.

Consequently, it is obvious from Table 4 that the first six eigenvalues obtained numerically converge with the increase of the number of degrees of freedom. According to the Eq. (11) it follows that the numerical solution, obtained by the proposed method, converges to the exact one.

The accuracy of the solution in this example is analyzed with regard to the solution obtained by 16×16 element mesh. The given load is presented as a linear combination of the eigenvectors. The components of vector **a** are calculated for the first several eigenvectors and different discretizations. According to the computed components of vector **a** and its eigenvalues the influence of each eigenvector for uniformly distributed loads according to Eq. (20) is computed and the data are given in Table 5. The influence of each eigenvector upon the total solution slightly differs for different discretizations and the results converge with the mesh refinement.

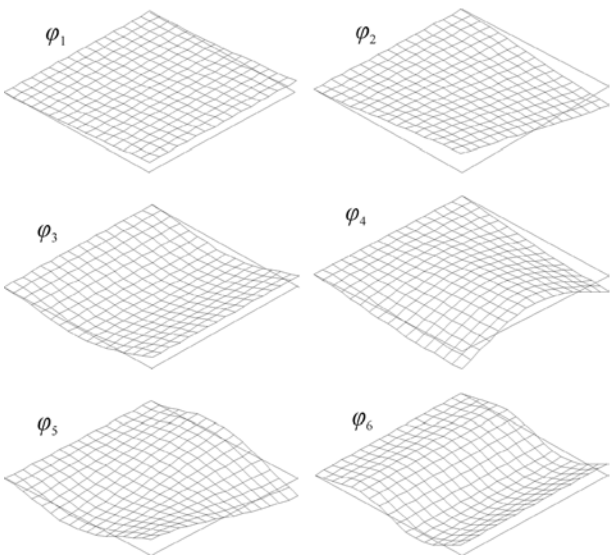

Figure 5. Eigenvectors of a cantilever square thin plate

Slika 5. Vlastiti vektori ukliještene kvadratne tanke ploče

The relative error δ_i of each eigenvector and the total error $\hat{\delta}_i$ calculated on the basis of the first six eigenvectors related to the values obtained by the 16×16 element mesh is shown in Table 6.

Table 5. Computation of the influence of eigenvectors upon the total solution for a cantilever square thin plate

Tablica 5. Proračun učešća vlastitih vektora u ukupnom rješenju za uklije̦štenu kvadratnu ploču

φ_i	Components of the vector **a** / Komponente vektora **a**, $a_i = \sum_j \varphi_{i,j}$				Influence of the eigenvectors to the total solution / Utjecaj vlastitih vektora u ukupnom rješenju, γ_i			
	4×4	8×8	12×12	16×16	4×4	8×8	12×12	16×16
φ_1	2.8810	9.2371	19.0750	32.4969	0.98764	0.98631	0.98655	0.98581
φ_2	0.0000	0.0000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000
φ_3	0.7295	3.1959	7.7870	14.3718	0.00624	0.00892	0.01036	0.01111
φ_4	1.0978	2.0932	3.1025	4.1859	0.00497	0.00339	0.00241	0.00189
φ_5	0.0000	0.0000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000
φ_6	1.2565	4.0520	4.0729	12.5776	0.00115	0.00134	0.00067	0.00119

Table 6. Relative and total eigenvector error (%) for a cantilever square thin plate

Tablica 6. Relativna i ukupna pogreška vlastitih vektora (%) za uklije̦štenu kvadratnu ploču

φ_i	4x4		8x8		12x12	
	δ_i	$\lvert\hat{\delta}_i\rvert$	δ_i	$\lvert\hat{\delta}_i\rvert$	δ_i	$\lvert\hat{\delta}_i\rvert$
φ_1	5.2226	5.1485	0.5728	0.5646	0.0813	0.0801
φ_2	12.4758	0.0000	2.3340	0.0000	0.5735	0.0000
φ_3	3.1231	0.0347	3.6705	0.0407	0.9374	0.0104
φ_4	-6.4987	0.0123	2.6435	0.0050	1.0281	0.0019
φ_5	23.4011	0.0000	3.9016	0.0000	0.9708	0.0000
φ_6	-10.4954	0.0125	-0.5555	0.0007	1.6711	0.0020
$\sum\lvert\hat{\delta}_i\rvert$		5.2080	-	0.6110	-	0.0944

The relative error is the least for the first eigenvector. Generally, the relative error of eigenvectors decreases with the increase of the number of degrees of freedom. The total error for the uniformly distributed load decreases with the increase of the number of degrees of freedom. The displacement field error for a given discretization in the analyzed example at any place will not exceed 5.2080 % for 4×4, 0.6110 % for 8×8 and 0.0944 % for 12×12 element mesh discretization. Although the error is computed according to the first six eigenvectors, the influence of other vectors upon the total solution is significantly smaller so that it will not affect the obtained results.

For the same discretization, the total error for the cantilever plate is smaller than for the simply supported square plate if both are exposed to uniformly distributed loads, which is expected since the eigenvectors accuracy is greater for the cantilever plate.

7. Conclusion

The procedure for proving the convergence and for estimating the numerical solution error of boundary value plate problems is shown in this paper. The procedure is based on the transformation of a discrete boundary value problem into an equivalent discrete dynamic eigenproblem. Discrete dynamic eigenproblem has physical meaning in convergence analysis because a mass represents the measure of domain discretization. Convergence of the numerical solution depends on the convergence of discrete dynamic eigenproblem spectrum.

The numerical experiment is proposed for the estimation of the convergence of numerical method instead a mathematical proof of convergence. The convergence of eigenvalues of discrete dynamic eigenproblem shows that numerical solution exists and converges to exact solution for used numerical method. If the numerical solution is convergent, the global discretization error of the finite element approximation can be estimated by using the eigenvalues and eigenvectors of discrete dynamic eigenproblems. Furthermore, the procedure can be applied in convergence analysis of different boundary value problems in linear and non-linear analysis.

The developed procedure is relatively simple, easy to perform and irrespective of the applied load. It gives the greatest global error which can appear for a chosen discretization. The only condition is to apply the algorithm for determining the number of eigenvalues and eigenvectors which is sufficient to estimate the error reliably and accurately.

Acknowledgement

The partial financial support, provided by the Ministry of Science, Education and Sports of the Republic of Croatia under the projects Non-linear Dynamic Analysis of Three-dimensional Reinforced Concrete Structures, Grant No. 083-0831541-1532, Modelling of Structural Stability under Large Displacements, Grant No. 083-0831541-1545, and Numerical and Experimental Investigations of Engineering Structures Behaviour, Grant No. 083-0831541-1547 is gratefully acknowledged.

REFERENCES

[1] IBRAHIMBEGOVIĆ, A.: *A unified formulation for modelling a structural system: The finite elements with rotational degrees of freedom*. *Int. J. Engineering Modelling*, 3 (1-2): 21-27, 1990.

[2] JURAK, M.: *On the limit behavior of the solutions of a sequence of generalized eigenproblems in different Hilbert spaces*. *Glasnik matematički*, 29: 79-95, 1994.

[3] KELLY, D.W.; Gago, S.R.; Zienkiewicz, O.C.; Babuška, I.: *A posteriori error analysis and adaptive procces in the finite element method: Part 1 – Error analysis*. *Int. J. Num. Meth. Eng.*, 19: 1593-1619, 1983.

[4] lI , X.D; WIBERG, N.E.: *A posteriori error estimate by element pach postprocessing, adaptive analysis in energy and L_2 norm*. *Computers and Structures*, 53: 907-919, 1994.

[5] MIHANOVIĆ, A.: *Dynamics of Structures*, Faculty of Civil Engineering, University of Split, Split, 1995. (in Croatian)

[6] MIHANOVIĆ, A.; RADELJA. T.: *Convergence and accuracy of numerical methods in structural boundary problems*. Int. J. for Engineering Modelling, **7** (3-4): 9-16, 1994.

[7] NIKOLIĆ, Ž.: *Contributions to the improvement of solutions on finite elements in frame structures and thin plates*. Ph.D. Thesis, Faculty of Civil Engineering, University of Split, Split, 1999. (in Croatian)

[8] NIKOLIĆ, Ž.; MIHANOVIĆ, A.; MAROVIĆ, P.: *Finite element solution improved by full clamping element functions*, Engineering Computations, 18 (5/6): 786-801, 2001.

[9] NOVACKI, V.: *Dynamics of Elastic Systems*. Chapman & Hall, London, 1963.

[10] TIMOSHENKO, S.P.; WOINOWSKY-KRIEGER, S.: *Theory of Plates and Shells*. 2nd edition, McGraw-Hill, New York, 1959.

[11] YOSIFIAN, G.A.; Ol EINIK, O.A.; SHAMAEV, A.S.: *On the limit behaviour of the spectrum of a sequence of operators, given in different Hilbert spaces*. Uspehi Mat. Nauk, 44: 157-158, 1989.

[12] ZIENKIEWICZ, O.C.; TAYLOR, R.L.: *The Finite Element Method, Vol 1*. 5th edition, Butterworth & Heinemann, Oxford, 2000.

[13] ZIENKIEWICZ, O.C.; ZHU, J.Z.: *The superconvergent pach recovery (SPR) and a posteriori error estimates: Part 2 – Error estimates and adaptivity*. Int. J. Num. Meth. Eng., 33: 1365-1382, 1992.