Approximating negative and harmonic mean moments for the Poisson distribution

C. Matthew Jones*

Abstract. *Asymptotic expansions and the related approximations are constructed for the negative moments of the Poisson distribution and the moments of the harmonic mean of independent and identically distributed (i.i.d.) Poisson random variable (r.v.). The accuracy of the approximations are assessed and applications to multi-centre clinical trials are outlined.*

Key words: *Poisson distribution, negative moments, Poisson-Charlie polynomials, harmonic mean, multi-centre clinical trials*

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1. Introduction

Let *Poisson*(λ) denote the Poisson distribution with parameter λ, and let ξ be a random variable, $\xi \sim Poisson(\lambda)$. Let also ξ_+ be the so-called positive Poisson r.v. with parameter λ; that is,

$$\Pr(\xi_+ = k) = \frac{1}{1-e^{-\lambda}} \frac{\lambda^k}{k!} e^{-\lambda}, \quad \text{for} \quad k = 1, 2 \ldots$$

The negative moments of the Poisson distribution with parameter λ are defined as the negative moments of ξ_+:

$$\mu_{-\alpha} = E\left(\frac{1}{\xi_+^{\,\alpha}}\right) \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{k=1}^{\infty} \frac{\lambda^k}{k^\alpha\, k!}\,. \tag{1}$$

In *Section 2.* we derive asymptotic expansions and related approximations for the first four negative moments $\mu_{-1}, \ldots, \mu_{-4}$ assuming $\lambda \to \infty$.

The methods of deriving approximations for μ_{-1} and $\mu_{-\alpha}$ with $\alpha \geq 2$ are different. In *Section 2.1.* we use a simple recurrence formula (see Equation (8)) to derive approximations for μ_{-1}. The k-th order approximation we suggest is

$$\mu_{-1} \simeq \mu_{-1}^{(k)} = \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{2}{\lambda^3} + \ldots + \frac{(k-1)!}{\lambda^k}\,. \tag{2}$$

*School of Mathematics, Cardiff Research Consortium, Cardiff Medicentre, Heath Park, Cardiff, CF14 4UJ, United Kingdom, e-mail: `Matthew.Jones@Cardiff-Research-Consortium.co.uk`

As a benchmark for comparison we shall use the well-known Tiku's estimators (see Equation (11) in [16]). These estimators are defined as follows:

$$\mu_{-1} \simeq T_k \frac{1}{(\lambda-1)(1-e^{-\lambda})}\left(1+\sum_{r=3}^{k}\beta_r\right), \tag{3}$$

where $\beta_r = a^{(r)}/\left[\lambda(\lambda+1)(\lambda+2)\dots(\lambda+r-1)\right]$ for $r = 3, 4, \dots$ with $a^{(3)} = 1$, $a^{(4)} = 7$, $a^{(5)} = 43$, $a^{(6)} = 271$, etc. The suggested estimators (2) are more accurate than Tiku's estimators (if λ is not too small); this is illustrated in *Table 1* below.

Note that for small λ the moments can be easily computed using definition (1). In particular, the relative error of the following simple approximation

$$\mu_{-\alpha} = \frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{k=1}^{\infty}\frac{\lambda^k}{k^\alpha\, k!} \simeq \frac{e^{-\lambda}}{1-e^{-\lambda}}\sum_{k=1}^{3\lambda+10}\frac{\lambda^k}{k^\alpha\, k!}$$

is smaller than 10^{-10} in absolute value for all $\lambda > 0$ and $\alpha \geq 1$.

The relative error of the approximations is used as the criterion by which to determine accuracy, where

$$\text{Relative Error} = \frac{\text{Exact Value} - \text{Approximate Value}}{\text{Exact Value}}.$$

In *Sections 2.3.* and *2.4.* we use the estimators (2) and the Poisson-Charlie orthogonal polynomials (they are introduced in *Section 2.2.*) for deriving approximations for $\mu_{-\alpha}$ with $\alpha = 2, 3, 4$. *Tables 3*, *4* and *5* show that the accuracy of the derived approximations is quite good. The analytical formulae for some of these approximations are given in the Appendix.

In *Section 3.* we derive the asymptotic (as $\lambda \to \infty$) expansions and corresponding approximations for the first four moments of the harmonic mean $H = 2/(1/\xi + 1/\zeta)$ of two i.i.d.r.v. $\xi, \zeta \sim$ *Poisson*(λ); the results are extended to an arbitrary number of random variables.

The approximation of the negative moments of the Poisson distribution has independent interest and has attracted reasonable attention in literature, especially in the field of sampling. The first negative moment μ_{-1} is of particular importance. The main applications of μ_{-1} are related to the fact that if η_j are i.i.d.r.v. with variance σ^2 and the sample has random size $n \sim$ *Poisson*(λ), then the variance of the mean $(\eta_1 + \ldots + \eta_n)/n$ is $\sigma^2\mu_{-1}$. This is a standard problem, for example, in life testing, see e.g. Bartholomew [1], David and Johnson [3], Epstein et al. [5], Epstein and Sobel [6], Grab and Savage [8], Mendenhall [11] and Stephan [15]. The problem of approximating the first negative moment of the Poisson distribution was also considered in Chao and Strawderman [2] and in Stancu [14]. Equation (4) in Grab and Savage demonstrated that

$$E\left(\frac{1}{X}|X > 0\right) = \frac{e^{-\lambda}}{1-e^{-\lambda}}\left(Ei(\lambda) - \log\lambda - \gamma\right) ;$$

where γ is Euler's constant $(0.5772\dots)$.

Tiku's estimators of negative moments $\mu_{-\alpha}$ constructed in [16] (see (3) and (15)) have been cited in a number of reference books; see, for instance, [9], [10]. These estimators shall be used as a benchmark for comparison in subsequent calculations.

The authors have faced the problem of estimating negative moments and moments of the harmonic mean $H = 2/(1/\xi + 1/\zeta)$, for the Poisson distribution, in connection to work on randomized multi-center clinical studies, see [7]. Multi-centre trials are essential in the pharmaceutical industry, as enrollment is accelerated by the simultaneous recruitment of patients to numerous centres. An additional benefit is the ability to generalize the studies results to a wider variety of patients and treatment settings than would be possible in a single-centre study. A linear model is regarded as the conventional way to model the observed data; with terms for centres, treatments and treatment-by-centre interactions. These terms may be considered as being fixed or random effects, see e.g. [4] and [13] for details.

The treatment effect of interest is the (weighted) average of the true (but unknown) treatment difference over the centres. Three well-known estimators exist (as specified in [4]), their accuracy is typically measured in terms of their mean squared error (MSE). All estimators are the least squares estimators of the treatment effect, generated from three fixed effects models of increasing complexity.

The MSE's for the second and third estimators (a similar formula holds for the first estimator) are as follows:

$$MSE(\Delta_{III})\frac{\sigma^2}{N^2}\sum_{i=1}^{N}\left(\frac{1}{n_{i2}}+\frac{1}{n_{i1}}\right), \tag{4}$$

$$MSE(\Delta_{II}) = \sigma^2\sum_{i=1}^{N}W_i^2\left(\frac{1}{n_{i2}}+\frac{1}{n_{i1}}\right) + 4\sigma_\tau^2\sum_{i=1}^{N}\left(W_i-\frac{1}{N}\right)^2 \tag{5}$$

with the weights

$$W_i = \frac{n_{i2}n_{i1}/(n_{i2}+n_{i1})}{\sum_{k=1}^{N} n_{k2}n_{k1}/(n_{k2}+n_{k1})},$$

where N is the number of centres in the study, n_{ij} is the number of patients on the j^{th} $(j = 1,\ 2)$ treatment in the i^{th} centre, σ_τ^2, σ_μ^2 and σ^2 are the variances of the treatment-by-centre interactions, centre effects and measurement error, respectively.

In standard literature n_{ij} are assumed to be known. A more realistic model to describe the probability of arrival (see [7]) of a given number of patients for both treatment arms, at the i-th centre is the Poisson process with parameter λ_i, $i = 1, \dots, N$. If this occurs, we have randomised enrolment; the MSE's become random variables and the first two moments of the MSE's are now of interest.

The need for negative moments is easily seen from both (4) and (5). When the number of centres N is large, then W_i are asymptotically independent and proportional to $2/(1/\xi_i + 1/\zeta_i)$ with independent $\xi_i,\ \zeta_i \sim Poisson(\lambda_i)$. To estimate the first two moments of MSE(Δ_{II}) we will then require approximations for the first four moments of H for $\lambda = \lambda_i$ $(i = 1, \dots, N)$.

2. Approximations for the negative moments of the Poisson distribution

Let $\xi \sim \textit{Poisson}(\lambda)$ and the negative moments $\mu_{-\alpha}$ be defined as in (1).

2.1. First negative moment

Set

$$\Phi_m = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k+m-1)!k} \,. \tag{6}$$

In particular,

$$\mu_{-1} = \frac{\Phi_1}{1-e^{-\lambda}} \,.$$

For each $m > 0$ we have

$$\frac{1}{k} = \frac{1}{k+m} + \frac{m}{k(k+m)} \,. \tag{7}$$

By applying (7) to (6) we obtain the recurrence formula

$$\Phi_m \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k+m-1)!k} \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k+m)!} + m\Phi_{m+1}\Upsilon_m + m\Phi_{m+1} \,, \tag{8}$$

where

$$\Upsilon_m \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k+m)!} \frac{1}{\lambda^m} \sum_{k=m+1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = \frac{1}{\lambda^m}\left(1 - \sum_{k=0}^{m} \frac{e^{-\lambda}\lambda^k}{k!}\right). \tag{9}$$

This gives for each $k = 1, 2 \ldots$

$$\Phi_1 = \Upsilon_1 + \Phi_2 = \Upsilon_1 + \Upsilon_2 + 2\Phi_3 \ldots = \sum_{i=1}^{k} (i-1)!\,\Upsilon_i + k!\,\Phi_{k+1} \,. \tag{10}$$

Equations (8) and (9) imply

$$\Phi_k = \frac{1}{\lambda^k} + O\left(\frac{1}{\lambda^{k+1}}\right) \quad \text{as } \lambda \to \infty \,. \tag{11}$$

To construct k^{th} order approximations for μ_{-1}, we keep $\Upsilon_1 \ldots \Upsilon_k$ and ignore Φ_{k+1} in (10). In this way we obtain the estimator

$$\hat{\mu}_{-1}^{(k)} = \frac{1}{\lambda}\left(1 - \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right) + \frac{1}{\lambda^2}\left(1 - \frac{(\lambda + \frac{\lambda^2}{2})e^{-\lambda}}{1-e^{-\lambda}}\right) + \ldots + \frac{(k-1)!}{\lambda^k}\left(1 - \sum_{j=1}^{k} \frac{\frac{\lambda^j}{j!}e^{-\lambda}}{1-e^{-\lambda}}\right). \tag{12}$$

As $\lambda \to \infty$, $e^{-\lambda}$ tends to 0 exponentially fast. Replacing $e^{-\lambda}$ with 0 in (12), we obtain a simpler estimator (2), that is,

$$\mu_{-1}^{(k)} = \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{2}{\lambda^3} + \ldots + \frac{(k-1)!}{\lambda^k} \,.$$

The asymptotic relation (11) implies that

$$\left|\mu_{-1} - \hat{\mu}_{-1}^{(k)}\right| = O\left(\frac{1}{\lambda^{k+1}}\right) \quad \text{and} \quad \left|\mu_{-1} - \mu_{-1}^{(k)}\right| = O\left(\frac{1}{\lambda^{k+1}}\right) \quad \text{as } \lambda \to \infty\,.$$

	Exact Value	Relative Error				
λ	μ_{-1}	T_0	$\mu_{-1}^{(3)}$	$\mu_{-1}^{(4)}$	T_3	T_6
5	$2.578\ 10^{-1}$	$2.356\ 10^{-2}$	$6.865\ 10^{-3}$	$-3.038\ 10^{-2}$	$1.891\ 10^{-2}$	$1.032\ 10^{-2}$
8	$1.469\ 10^{-1}$	$2.712\ 10^{-2}$	$1.605\ 10^{-2}$	$6.079\ 10^{-3}$	$2.577\ 10^{-2}$	$2.426\ 10^{-2}$
10	$1.130\ 10^{-1}$	$1.686\ 10^{-2}$	$9.037\ 10^{-3}$	$3.729\ 10^{-3}$	$1.611\ 10^{-2}$	$1.546\ 10^{-2}$
15	$7.187\ 10^{-2}$	$6.117\ 10^{-3}$	$2.349\ 10^{-3}$	$7.003\ 10^{-4}$	$5.934\ 10^{-3}$	$5.799\ 10^{-3}$
25	$4.175\ 10^{-2}$	$1.912\ 10^{-3}$	$4.426\ 10^{-4}$	$7.467\ 10^{-5}$	$1.855\ 10^{-3}$	$1.837\ 10^{-3}$
50	$2.042\ 10^{-2}$	$4.350\ 10^{-4}$	$5.121\ 10^{-5}$	$4.190\ 10^{-6}$	$4.275\ 10^{-4}$	$4.264\ 10^{-4}$
100	$1.010\ 10^{-2}$	$1.042\ 10^{-4}$	$6.190\ 10^{-6}$	$2.504\ 10^{-7}$	$1.032\ 10^{-4}$	$1.031\ 10^{-4}$
200	$5.025\ 10^{-3}$	$2.551\ 10^{-5}$	$7.620\ 10^{-7}$	$1.572\ 10^{-8}$	$2.539\ 10^{-5}$	$2.538\ 10^{-5}$

Table 1. *Exact values for the first negative moment, along with comparison of the relative errors of the five estimators, against different values of* λ

The accuracy of these estimates may be seen in *Table 1* which gives the relative error against different values of λ. We compare our estimators to the Tiku's estimators T_k defined in (3); only the estimators $T_0 = 1/[(\lambda-1)(1-e^{-\lambda})]$, T_3 and T_6 are used.

For values of $\lambda > 8$ we find that the estimators $\mu^{(3)}$, and especially $\mu^{(4)}$, are very accurate, comparing favorably to the more complex estimators of Tiku. Additionally (this is not included in the table), it was found that for $\lambda \geq 8$ the simpler estimators $\mu_{-1}^{(k)}$ are marginally better than the respective estimators $\hat{\mu}_{-1}^{(k)}$.

2.2. Poisson-Charlie polynomials and the method of deriving approximations for higher order moments

The derivation of higher order moments uses the properties of the so-called Poisson-Charlie polynomials. In accordance with the theory of orthogonal polynomials, there exist so-called Poisson-Charlie polynomials, $p_0(k)$, $p_1(k)$, ... $(k = 0, 1, 2, \dots)$ such that for every function $f(k) \,\epsilon\, L_2(P_\lambda)$

$$f(k) = \sum_{m=0}^{\infty} C_m p_m(k)\,, \tag{13}$$

where

$$C_m = E\left(f(\xi)p_m(\xi)\right) = \sum_{k=0}^{\infty} f(k)p_m(k)\frac{\lambda^k}{k!}e^{-\lambda}\,.$$

Moment	Estimator		
$E\left(1/\xi_+^2\right)$	$\beta_1 = \sum_{i=0}^{2} \theta_i^2$ with $\theta_0 \simeq \mu_{-1}^{(3)}$	$\beta_2 = \sum_{i=0}^{2} \theta_i^2$ with $\theta_0 \simeq \mu_{-1}^{(4)}$	$\beta_3 = \sum_{i=0}^{3} \theta_i^2$ with $\theta_0 \simeq \mu_{-1}^{(4)}$
$E\left(1/\xi_+^3\right)$	$\varphi_1 = \sum_{i=0}^{2} \theta_i\theta_i'$ $\theta_0 \simeq \mu_{-1}^{(3)}$, $\theta_0' \simeq \beta_1$	$\varphi_2 = \sum_{i=0}^{2} \theta_i\theta_i'$ $\theta_0 \simeq \mu_{-1}^{(4)}$, $\theta_0' \simeq \beta_3$	$\varphi_3 = \sum_{i=0}^{3} \theta_i\theta_i'$ $\theta_0 \simeq \mu_{-1}^{(4)}$, $\theta_0' \simeq \beta_3$
$E\left(1/\xi_+^4\right)$	$\varrho_1 = \sum_{i=0}^{2} \theta_i'^2$ with $\theta_0' \simeq \beta_1$	$\varrho_2 = \sum_{i=0}^{2} \theta_i'^2$ with $\theta_0' \simeq \beta_2$	$\varrho_3 = \sum_{i=0}^{3} \theta_i'^2$ with $\theta_0' \simeq \beta_2$

Table 2. *Notation for estimators of the second, third and fourth negative moments; see Appendix for explicit formulae of the estimators*

By definition, $f(k) \in L_2(P_\lambda)$ if and only if

$$Ef^2(\xi) = \sum_{k=0}^{\infty} f^2(k) \frac{\lambda^k}{k!} e^{-\lambda} < \infty .$$

The Poisson-Charlie polynomials, $p_n(x)$, are defined as:

$$p_n(x) = \frac{\lambda^{n/2}}{\sqrt{n!}} P_n(x), \quad n = 0, 1, 2, \dots ,$$

where (see, for example, [12])

$$P_n(x) = \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \lambda^{-r} x(x-1) \dots (x-r+1) .$$

These polynomials are orthonormal in $L_2(P_\lambda)$; that is,

$$E\left(p_m(\xi) p_l(\xi)\right) = \sum_{k=0}^{\infty} p_m(k) p_l(k) \frac{\lambda^k}{k!} e^{-\lambda} = \begin{cases} 1 & \text{if } l = m, \\ 0 & \text{if } l \neq m . \end{cases} \tag{14}$$

The first four Poisson-Charlie polynomial are:

$$p_0(k) = 1, \;\; p_1(k) = \sqrt{\lambda}\left(\frac{k}{\lambda} - 1\right), \;\; p_2(k) = \frac{\lambda}{\sqrt{2}}\left(1 - \frac{2k}{\lambda} + \frac{k(k-1)}{\lambda^2}\right),$$

$$p_3(k) = \frac{\lambda^{3/2}}{\sqrt{6}}\left(-1 + 3\,\frac{k}{\lambda} - 3\,\frac{k\,(k-1)}{\lambda^2} + \frac{k\,(k-1)\,(k-2)}{\lambda^3}\right).$$

Let us consider the functions

$$f_\alpha(k) = \begin{cases} 0 & \text{if } k = 0, \\ k^{-\alpha} & \text{if } k \neq 0 . \end{cases}$$

Then

$$\mu_{-1} = E\left(1/\xi_+\right) = \frac{1}{(1 - e^{-\lambda})} E f_1(\xi)$$

and

$$\mu_{-\alpha} = E\left(1/\xi_+^{\alpha}\right) = \frac{1}{(1-e^{-\lambda})} E f_\alpha(\xi).$$

The approximation of $Ef_1(\xi)$ has already been considered. We now construct approximations for $Ef_\alpha(\xi)$, $\alpha = 2, 3, 4$. We are interested in $\lambda \geq 10$. As $e^{-10} \simeq 0.0000454$, for this range of λ the multiplier $1 - e^{-\lambda}$ will be neglected in subsequent derivations. The approximations for the higher order moments shall be compared against the corresponding Tiku estimators $T^{(\alpha)}$, which for $\alpha \geq 2$ are defined as (see (14) in [16])

$$T^{(\alpha)} = \frac{1}{(\lambda-1)(\lambda-2)\dots(\lambda-\alpha)}\,. \tag{15}$$

2.3. Second negative moment

Consider the expansion (13) for $f(k) = f_1(k)$:

$$f_1(k) = \sum_{m=0}^{\infty} \theta_m p_m(k)\,,$$

where

$$\theta_m = E\left(f_1(\xi) p_m(\xi)\right) = \sum_{k=1}^{\infty} p_m(k) \frac{\lambda^k}{k\, k!} e^{-\lambda}\,.$$

Then

$$f_2(k) = f_1^2(k) = \sum_{m=0}^{\infty} \theta_m p_m(k) \sum_{l=0}^{\infty} \theta_l p_l(k) \sum_{m,l=0}^{\infty} \theta_m \theta_l p_m(k) p_l(k)\,. \tag{16}$$

In particular,

$$\theta_0 = E(p_0(\xi) f_1(\xi)) = E f_1(\xi) = (1 - e^{-\lambda})\, \mu_{-1}\,. \tag{17}$$

Applying the orthonormality property (14) we obtain

$$E f_2(\xi) = E \sum_{m,l=0}^{\infty} \theta_m \theta_l p_m(k) p_l(k) = \sum_{m=0}^{\infty} \theta_m^2\,. \tag{18}$$

To approximate $Ef_2(\xi)$, we shall use either the first three or the first four terms in (18); μ_{-1} in (17) will be approximated by either $\mu_{-1}^{(3)}$ or $\mu_{-1}^{(4)}$ (see *Section 2.1.*). Note that when using $\theta_0 \simeq \mu_{-1}^{(3)}$ the approximations for μ_{-2} (as well as for μ_{-3} and μ_{-4}) with three and four terms coincide (as $\theta_3 = 0$ and $\theta_3' = 0$). In this way, we construct three estimators of μ_{-2} as stated in *Table 2* (similarly, we shall obtain three estimators of μ_{-3} and μ_{-4}).

Using the four terms in (18), the following approximation is obtained:

$$(1 - e^{-\lambda})\, E(1/\xi_+^2) = E f_2(\xi) \cong \theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2\,, \tag{19}$$

where

$$\theta_1 = E(p_1(\xi) f_1(\xi)) = \sqrt{\lambda} E\left(\frac{\xi}{\lambda} - 1\right) f_1(\xi) \frac{\sqrt{\lambda}}{\lambda} E(\xi f_1(\xi)) - \sqrt{\lambda} E f_1(\xi)\,.$$

Through using the fact that

$$E(\xi f_1(\xi)) = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 1 - e^{-\lambda} \simeq 1 \tag{20}$$

and $Ef_1(\xi) = \theta_0$ we obtain

$$\theta_1 \simeq \frac{1}{\sqrt{\lambda}} - \sqrt{\lambda}\theta_0\,.$$

Using (20) and the fact that $E(\xi^2 f_1(\xi)) = E\xi = \lambda$ we have

$$\theta_2 = E(p_2(\xi) f_1(\xi)) = \frac{\sqrt{2}}{2\lambda} E((\lambda - \xi)^2 - \xi) f_1(\xi) \simeq \frac{\lambda}{\sqrt{2}}\theta_0 - \frac{1}{\sqrt{2}\lambda}(2\lambda + 1) + \frac{1}{\sqrt{2}}\,.$$

Analogously, by using $E(\xi^3 f_1(\xi)) = E\xi^2 = \lambda^2 + \lambda$ we get

$$\begin{aligned}\theta_3 = E(p_3(\xi) f_1(\xi)) &= \frac{\lambda^{3/2}}{\sqrt{6}} E\left(\left(-1 + 3\frac{\xi}{\lambda} - 3\frac{\xi(\xi-1)}{\lambda^2} + \frac{\xi(\xi-1)(\xi-2)}{\lambda^3}\right) f_1(\xi)\right) \\ &\simeq \frac{\lambda^{3/2}}{\sqrt{6}} \left(-\theta_0 + \frac{3}{\lambda^2} + \frac{\lambda^2 - 2\lambda + 2}{\lambda^3}\right).\end{aligned}$$

Hence, by using (19) we obtain:

$$\begin{aligned}Ef_2(\xi) &\cong \theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_3^2 \cong \theta_0^2 + \left(\frac{1}{\lambda} - \sqrt{\lambda}\theta_0\right)^2 \\ &+ \left(\frac{\lambda}{\sqrt{2}}\theta_0 - \frac{1}{\sqrt{2}\lambda}(2\lambda+1) + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\lambda^{3/2}}{\sqrt{6}}\left(-\theta_0 + \frac{3}{\lambda^2} + \left(\lambda^2 - 2\lambda + 2\right)\right)\right)^2.\end{aligned}$$

For the approximation involving three terms we omit θ_3 and therefore the last term in this formula disappears. Relative errors for the estimators of the second negative moment can be seen in *Table 3.*

	Exact Value	Relative Error			
λ	μ_2	β_1	β_2	β_3	$T^{(2)}$
10	$1.532\ 10^{-2}$	$7.429\ 10^{-2}$	$4.685\ 10^{-2}$	$4.294\ 10^{-2}$	$9.355\ 10^{-2}$
15	$5.664\ 10^{-3}$	$1.810\ 10^{-2}$	$8.827\ 10^{-3}$	$7.432\ 10^{-3}$	$2.993\ 10^{-1}$
25	$1.827\ 10^{-3}$	$3.191\ 10^{-3}$	$1.047\ 10^{-3}$	$7.104\ 10^{-4}$	$8.338\ 10^{-3}$
50	$4.259\ 10^{-4}$	$3.624\ 10^{-4}$	$8.360\ 10^{-5}$	$3.852\ 10^{-5}$	$1.808\ 10^{-3}$
100	$1.031\ 10^{-4}$	$4.353\ 10^{-5}$	$8.093\ 10^{-6}$	$2.274\ 10^{-6}$	$4.244\ 10^{-4}$
200	$2.538\ 10^{-5}$	$5.343\ 10^{-6}$	$8.774\ 10^{-7}$	$1.387\ 10^{-7}$	$1.030\ 10^{-4}$

Table 3. *Exact values for the second negative moment, along with comparison of the relative errors of the four estimators, against different values of* λ

2.4. Third and fourth negative moments

Consider the expansion of the function $f_2(k)$:

$$f_2(k) = \sum_{m=0}^{\infty} \theta'_m p_m(k)$$

with $\theta'_m = E(f_2(\xi)p_m(\xi))$.

λ	Exact Value	Relative Error			
	μ_3	φ_1	φ_2	φ_3	$T^{(3)}$
10	$2.900\ 10^{-3}$	$3.059\ 10^{-1}$	$2.371\ 10^{-1}$	$2.275\ 10^{-1}$	$3.159\ 10^{-1}$
15	$5.071\ 10^{-4}$	$8.051\ 10^{-2}$	$4.961\ 10^{-2}$	$4.501\ 10^{-1}$	$9.704\ 10^{-2}$
25	$8.427\ 10^{-5}$	$1.288\ 10^{-2}$	$4.928\ 10^{-3}$	$3.688\ 10^{-3}$	$2.287\ 10^{-2}$
50	$9.089\ 10^{-6}$	$1.449\ 10^{-3}$	$3.688\ 10^{-4}$	$1.946\ 10^{-4}$	$4.700\ 10^{-3}$
100	$1.064\ 10^{-6}$	$1.740\ 10^{-4}$	$3.432\ 10^{-5}$	$1.142\ 10^{-5}$	$1.081\ 10^{-3}$
200	$1.289\ 10^{-7}$	$2.136\ 10^{-5}$	$3.626\ 10^{-6}$	$6.938\ 10^{-7}$	$2.597\ 10^{-4}$

Table 4. *Exact values for the third negative moment, along with comparison of the relative errors of the four estimators, against different values of* λ

Analogously to (16), we have

$$f_3(k) = f_1(k)f_2(k) = \sum_{m,l=0}^{\infty} \theta_m \theta'_l p_m(k) p_l(k)$$

and

$$f_4(k) = f_2(k)f_2(k) = \sum_{m,l=0}^{\infty} \theta'_m \theta'_l p_m(k) p_l(k).$$

Thus, we have

$$Ef_3(\xi) = \sum_{m=0}^{\infty} \theta_m \theta'_m \simeq \theta_0\theta'_0 + \theta_1\theta'_1 + \theta_2\theta'_2 + \theta_3\theta'_3\,,$$

$$Ef_4(\xi) = \sum_{m=0}^{\infty} (\theta'_m)^2 \simeq (\theta'_0)^2 + (\theta'_1)^2 + (\theta'_2)^2 + (\theta'_3)^2\,.$$

Deriving expressions for θ'_0, θ'_1, θ'_2 and θ'_3 yields the following:

$$\theta'_0 = Ef_2(\xi),$$

$$\theta'_1 = E(p_1(\xi)f_2(\xi)) = \sqrt{\lambda}E\left(\frac{\xi}{\lambda} - 1\right) f_2(\xi) \simeq \frac{\sqrt{\lambda}}{\lambda}E(\xi f_2(\xi)) - \sqrt{\lambda}Ef_2(\xi) \simeq \frac{\theta_0}{\sqrt{\lambda}} - \sqrt{\lambda}\theta'_0,$$

$$\theta'_2 = E(p_2(\xi)f_2(\xi)) = \frac{1}{\sqrt{2}\lambda}E(\lambda^2 + \xi^2 - (2\lambda+1)\xi)f_2(\xi) \simeq \frac{1}{\sqrt{2}}\lambda\theta'_0 + \frac{1}{\sqrt{2}\lambda} - \frac{(2\lambda+1)}{\sqrt{2}\lambda}\theta_0$$

$$\theta'_3 = E(p_3(\xi)f_2(\xi))\frac{\lambda^{3/2}}{\sqrt{6}}E\left(\left(-1 + \frac{3\xi}{\lambda} - \frac{3\xi(\xi-1)}{\lambda^2} + \frac{\xi(\xi-1)(\xi-2)}{\lambda^3}\right) f_2(\xi)\right)$$

$$\simeq \frac{\lambda^{3/2}}{\sqrt{6}}\left(-\theta'_0 + \frac{3}{\lambda}\theta_0 - \frac{3}{\lambda^2}(1-\theta_0) + \frac{1}{\lambda^3}(\lambda - 3 + 2\theta_0)\right)$$

λ	Exact Value	Relative Error			
	μ_4	ϱ_1	ϱ_2	ϱ_3	$T^{(4)}$
10	$9.865\ 10^{-4}$	$6.835\ 10^{-1}$	$6.120\ 10^{-1}$	$5.986\ 10^{-1}$	$6.648\ 10^{-1}$
15	$5.782\ 10^{-5}$	$2.755\ 10^{-1}$	$2.163\ 10^{-1}$	$2.043\ 10^{-1}$	$2.801\ 10^{-1}$
25	$4.130\ 10^{-6}$	$3.820\ 10^{-2}$	$1.816\ 10^{-2}$	$1.385\ 10^{-2}$	$5.064\ 10^{-2}$
50	$1.986\ 10^{-7}$	$4.294\ 10^{-3}$	$1.370\ 10^{-3}$	$7.125\ 10^{-4}$	$9.780\ 10^{-3}$
100	$1.109\ 10^{-8}$	$5.187\ 10^{-4}$	$1.310\ 10^{-4}$	$4.180\ 10^{-5}$	$2.202\ 10^{-3}$
200	$6.576\ 10^{-10}$	$6.388\ 10^{-5}$	$1.412\ 10^{-5}$	$2.541\ 10^{-6}$	$5.241\ 10^{-4}$

Table 5. *Exact values for the fourth negative moment, along with comparison of the relative errors of the four estimators, against different values of λ*

3. Approximating the moments of the harmonic mean

Let ξ and ζ be i.i.d.r.v., $\xi, \zeta \sim \mathit{Poisson}(\lambda)$. Set

$$H = \frac{2\xi\zeta}{\xi+\zeta} = \frac{2}{1/\xi + 1/\zeta}$$

to be the harmonic mean of ξ and ζ. In the following theorem we derive an asymptotic expansion (as $\lambda \to \infty$) for $E(H)$. *Table 6* demonstrates that the corresponding approximation is extremely accurate.

3.1. First moment of the harmonic mean

Theorem 1. *Let ξ and ζ be i.i.d.r.v. with $\xi, \zeta \sim \mathit{Poisson}(\lambda)$. Then*

$$E\left(\frac{2\xi\zeta}{\xi+\zeta}\right) \lambda - \frac{1}{2} + o\left(\frac{1}{\lambda^{40}}\right) \quad \text{as} \quad \lambda \to \infty\,. \tag{21}$$

Proof. We have

$$H = H(\xi,\zeta) = \frac{2\xi\zeta}{\xi+\zeta-2\lambda+2\lambda} = \frac{1}{\lambda}\frac{\xi\zeta}{1+z}$$

with

$$z = z(\xi,\zeta) = \begin{cases} \frac{\xi+\zeta}{2\lambda} - 1 & \text{if} \ \ \xi+\zeta > 0, \\ 0 & \text{if} \ \ \xi+\zeta = 0\,. \end{cases} \tag{22}$$

Note that $z > -1$. The moments of the r.v. $\xi \sim \mathit{Poisson}(\lambda)$ can be expressed explicitly as (see e.g. [9], p.6)

$$m_\alpha = m_\alpha(\lambda) = E\,\xi^\alpha = \sum_{i=1}^{\alpha} \lambda^i S_{i:\alpha}, \tag{23}$$

where α is a positive integer and $S_{i:\alpha}$ are Stirling numbers of the second kind:

$$S_{i:\alpha} = \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j} j^\alpha\,.$$

Denote

$$A_n = A_n(\lambda) = (-1)^n \frac{1}{\lambda} E\, \xi\zeta z^n$$

with z defined in (22).

Using the binomial formula we obtain

$$\begin{aligned} A_n &= (-1)^n \frac{1}{\lambda}\, E\, \xi\zeta \left(\frac{\xi+\zeta}{2\lambda} - 1\right)^n = (-1)^n \frac{1}{\lambda} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{2^k \lambda^k} E\xi\zeta(\xi+\zeta)^k \\ &= \frac{\xi\zeta}{\lambda} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{2^k\lambda^k} \sum_{l=0}^{k} \binom{k}{l} m_{l+1} m_{k-l+1} \end{aligned}$$

with the moments m_α defined in (23).

Analytical manipulations using the computer package Maple 7 show that for all positive integers $\kappa \le N+2$ with $N = 40$ we have

$$A_{2\kappa} = O\left(\frac{1}{\lambda^{\kappa-1}}\right) \text{ and } A_{2\kappa+1} = O\left(\frac{1}{\lambda^{\kappa}}\right) \text{ as } \lambda \to \infty\,; \tag{24}$$

additionally,

$$\sum_{n=0}^{2\kappa} A_n = \frac{1}{\lambda} E\, \xi\zeta(1-z+z^2-\ldots+z^{2\kappa}) = \lambda - \frac{1}{2} + O\left(\frac{1}{\lambda^\kappa}\right),\ \lambda \to \infty\,. \tag{25}$$

For all $z > -1$ and any integer $\kappa > 0$ we have

$$\frac{1}{1+z} \ge 1 - z + z^2 - \ldots + z^{2\kappa} - z^{2\kappa+1}\,.$$

This yields for all positive integers κ

$$EH = \frac{1}{\lambda} E\, \frac{\xi\zeta}{1+z} \ge \frac{1}{\lambda} E\, \xi\zeta(1-z+\ldots+z^{2\kappa}-z^{2\kappa+1}) \sum_{n=0}^{2\kappa+1} A_n\,. \tag{26}$$

The asymptotic relations (25) and (24) imply

$$\sum_{n=0}^{2\kappa+1} A_n \lambda - \frac{1}{2} + O\left(\frac{1}{\lambda^\kappa}\right),\ \lambda \to \infty\,,$$

with $\kappa = 42$. Combining this with (26) we obtain

$$EH \ge \lambda - \frac{1}{2} + O\left(\frac{1}{\lambda^{42}}\right),\ \ \lambda \to \infty\,. \tag{27}$$

For any $c \ge 1$, any positive integer κ and $z \ge -1 + \frac{1}{c}$ we have

$$\frac{1}{1+z} \le 1 - z + z^2 - \ldots + z^{2\kappa-2} - z^{2\kappa-1} + cz^{2\kappa}\,. \tag{28}$$

Since $\xi,\ \zeta$ are non-negative, we have

$$\frac{\xi\zeta}{\lambda(1+z)} = \frac{2\xi\zeta}{\xi+\zeta} \le 2\xi\,. \tag{29}$$

Through combining (28) and (29) we obtain

$$\frac{1}{\lambda}\frac{\xi\zeta}{1+z} \le \frac{\xi\zeta}{\lambda}(1-z+z^2-\ldots+z^{2\kappa-2}-z^{2\kappa-1}+cz^{2\kappa})+f(\xi,\zeta)\,, \tag{30}$$

where

$$f(\xi,\zeta) = \begin{cases} 0 & \text{if } z \ge -1+\frac{1}{c}, \\ 2\xi & \text{if } -1 < z < -1+\frac{1}{c}\,. \end{cases}$$

Using (30) and (25) we have for all $\kappa \le 42$

$$\begin{aligned} E\ H &\le \frac{1}{\lambda}E\ \xi\zeta(1-z+z^2-\ldots+cz^{2\kappa})+E\ f_\lambda(\xi,\zeta) \\ &= \lambda-\frac{1}{2}+O\left(\frac{1}{\lambda^\kappa}\right)+(c-1)\frac{1}{\lambda}E\ \xi\zeta z^{2\kappa}+E\ f_\lambda(\xi,\zeta)\,. \end{aligned} \tag{31}$$

Consider the term $E\ f_\lambda(\xi,\zeta)$ on the right-hand side of (31). The inequality $z > -1+\frac{1}{c}$ is equivalent to $\xi+\zeta < \frac{2\lambda}{c}$ and therefore (using the fact that ζ is non-negative)

$$f_\lambda(\xi,\zeta) \le \tilde{f}_\lambda(\xi,\zeta) = \begin{cases} 0 & \text{if } \xi > \frac{2\lambda}{c}, \\ 2\xi & \text{if } \xi \le \frac{2\lambda}{c}\,. \end{cases}$$

This implies

$$\begin{aligned} E\,f_\lambda(\xi,\zeta) \le E\,\tilde{f}_\lambda(\xi,\zeta) &= \sum_{j=0}^{\lfloor\frac{2\lambda}{c}\rfloor} 2je^{-\lambda}\frac{\lambda^j}{j!} \\ &= 2e^{-\lambda}\sum_{j=1}^{\lfloor\frac{2\lambda}{c}\rfloor}\frac{\lambda^j}{(j-1)!} \le 2e^{-\lambda}\sum_{j=1}^{\lfloor\frac{2\lambda}{c}\rfloor}\lambda^j \end{aligned} \tag{32}$$

for any $c \ge 1$.

By estimating each term on the right-hand side of (32) by the largest term (the last term), we get

$$E\ f_\lambda(\xi,\zeta) \le E\ \tilde{f}_\lambda(\xi,\zeta) \le 2e^{-\lambda}\left(\frac{2\lambda}{c}\right)\lambda^{2\lambda/c}\,. \tag{33}$$

Substituting $c=\sqrt{\lambda}$ into the right-hand side (33) we obtain

$$g_\lambda = 2e^{-\lambda}\left(\frac{2\lambda}{c}\right)\lambda^{2\lambda/c}4e^{-\lambda}\sqrt{\lambda}\,\lambda^{2\sqrt{\lambda}}\,.$$

Since $\log g_\lambda = -\lambda + 2\sqrt{\lambda}\log\lambda + \frac{1}{2}\log\lambda + 2\log 2$, we have

$$\frac{1}{\lambda}\log g_\lambda = -1 + o\left(\lambda^{-\frac{1}{2}+\epsilon}\right) \text{ as } \lambda \to \infty, \text{ for any } \epsilon > 0\,. \tag{34}$$

Hence, $E\, f_\lambda(\xi,\zeta)$ tends to zero as $\lambda \to \infty$ exponentially fast implying

$$E\, f_\lambda(\xi,\zeta) = o\left(\frac{1}{\lambda^\kappa}\right) \text{ for any } \kappa > 0 \text{ as } \lambda \to \infty\,. \tag{35}$$

Consider now

$$h_\lambda = (c-1)\frac{1}{\lambda}E\,\xi\zeta\, z^{2\kappa} = (c-1)A_{2\kappa}.$$

on the right-hand side of (31).

Using (24) and recalling that $c = \sqrt{\lambda}$, we obtain

$$h_\lambda = (\sqrt{\lambda}-1)A_{2\kappa} = O\left(\frac{1}{\lambda^{\kappa-\frac{3}{2}}}\right)\,, \text{ as } \lambda \to \infty\,. \tag{36}$$

Substituting (35) and (36) into (31) we obtain

$$E\,H \le \lambda - \frac{1}{2} + O\left(\frac{1}{\lambda^{\kappa-\frac{3}{2}}}\right) \tag{37}$$

for $\kappa = N + 2 = 42$.

Combining (27) and (37) gives (21). This completes the proof. □

The constant 40 in (21) is attributable to the capabilities of our computers (Pentium III) and the version of the computer package (Maple 7) used. For example, formula (25) for $\kappa = 40$ can be specified as

$$\sum_{n=0}^{80} A_n = \lambda - \frac{1}{2} + C\lambda^{-40} + O\left(\frac{1}{\lambda^{41}}\right),\ \lambda \to \infty\ ,\quad C \simeq 0.421196957388\, 10^{50}\,.$$

Without the aid of a computer we can attain accuracy only up to $o(\lambda^{-2})$ rather than $o(\lambda^{-40})$ in (21).

3.2. Higher order moments of the harmonic mean

Theorem 2. *Let ξ and ζ be i.i.d.r.v., $\xi,\zeta \sim$ Poisson(λ). Then, as $\lambda \to \infty$, we have*

$$E\left(\frac{2\xi\zeta}{\xi+\zeta}\right)^2 = \lambda^2 - \frac{\lambda}{2} + \frac{3}{4} - \frac{1}{4\lambda} - \frac{1}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\,,$$

$$E\left(\frac{2\xi\zeta}{\xi+\zeta}\right)^3 = \lambda^3 + \frac{7\lambda}{4} - \frac{21}{8} + \frac{15}{8\lambda} + \frac{7}{16\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\,,$$

$$E\left(\frac{2\xi\zeta}{\xi+\zeta}\right)^4 = \lambda^4 + \lambda^3 + \frac{13\lambda^2}{4} - \frac{57\lambda}{8} + \frac{225}{16} - \frac{121}{8\lambda} + \frac{13}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\,.$$

Proof. Firstly, consider the second moment

$$\left(\frac{2\xi\zeta}{\xi+\zeta}\right)^2 = \frac{4\xi^2\zeta^2}{(\xi+\zeta-2\lambda+2\lambda)^2} = \frac{1}{\lambda^2}\frac{\xi^2\zeta^2}{(1+z)^2}$$

with z defined in (22). Similarly to the proof of *Theorem 1* we have the asymptotic expansion (as $\lambda \to \infty$):

$$\frac{1}{\lambda^2}E\,\frac{\xi^2\zeta^2}{(1+z)^2} = \frac{1}{\lambda^2}E\,\xi^2\zeta^2(1-z+z^2-z^3+z^4-z^5+z^6-z^7+z^8)^2 + O\left(\frac{1}{\lambda^3}\right)$$
$$= \lambda^2 - \frac{\lambda}{2} + \frac{3}{4} - \frac{1}{4\lambda} - \frac{1}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right).$$

The same methodology can be applied to obtain asymptotic expressions for higher order moments of H. This completes the proof. □

λ	h_1	h_2	h_3	h_4
10	2.5031 10^{-7}	-2.7617 10^{-4}	1.8860 10^{-4}	1.3331 10^{-4}
15	2.7586 10^{-10}	-7.9107 10^{-5}	3.7366 10^{-5}	-1.8299 10^{-5}
25	5.8163 10^{-16}	-1.6647 10^{-5}	4.8328 10^{-6}	-1.4754 10^{-6}
50	3.5267 10^{-30}	-2.0402 10^{-6}	3.0121 10^{-7}	-4.7286 10^{-8}
100	1.8177 10^{-58}	-2.5251 10^{-7}	1.8791 10^{-8}	-1.4954 10^{-9}
200	6.7957 10^{-115}	-3.1406 10^{-8}	1.1173 10^{-9}	-4.7001 10^{-11}

Table 6. *Relative errors for the approximations (38) of the first four moments of H against different values of* λ

Theorems 1 and *2* suggest the following approximations for the first four moments of the harmonic mean $H = 2\xi\zeta/(\xi+\zeta)$:

$$\begin{cases} E(H) \simeq h_1 = \lambda - \frac{1}{2}, \\ E\left(H^2\right) \simeq h_2 = \lambda^2 - \frac{\lambda}{2} + \frac{3}{4}, \\ E\left(H^3\right) \simeq h_3 = \lambda^3 + \frac{7}{4}\lambda - \frac{21}{8}, \\ E\left(H^4\right) \simeq h_4\lambda^4 + \lambda^3 + \frac{13}{4}\lambda^2 - \frac{57}{8}\lambda + \frac{225}{16}. \end{cases} \tag{38}$$

The accuracy of the suggested four approximations is demonstrated in *Table 6.*

3.3. Harmonic mean of several variables

Let $n \geq 1$ be fixed and $\xi_1, \dots, \xi_n$ be i.i.d. *Poisson*(λ) r.v. Set

$$H_n = \frac{n}{\sum_{k=1}^n 1/\xi_k}$$

to be the harmonic mean of random variables $\xi_1, \dots, \xi_n$. Then, similarly to *Theorems 1* and *2*, we have, as $\lambda \to \infty$,

$$E(H_n) = \lambda - \frac{n-1}{n} + O\left(\frac{1}{\lambda}\right)\ ;\ \operatorname{Var}(H_n)\frac{\lambda}{n} - \frac{2(n-1)(n-3)}{n^2} + O\left(\frac{1}{\lambda}\right).$$

Appendix

The estimators in *Table 2* may be expressed as follows:

$$\beta_1 = \frac{1}{\lambda^2}+\frac{3}{\lambda^3}+\frac{11}{\lambda^4}+\frac{8}{\lambda^5}+\frac{4}{\lambda^6};$$

$$\beta_2 = \frac{1}{\lambda^2}+\frac{3}{\lambda^3}+\frac{11}{\lambda^4}+\frac{44}{\lambda^5}+\frac{58}{\lambda^6}+\frac{60}{\lambda^7}+\frac{36}{\lambda^8};$$

$$\beta_3 = \frac{1}{\lambda^2}+\frac{3}{\lambda^3}+\frac{11}{\lambda^4}+\frac{50}{\lambda^5}+\frac{58}{\lambda^6}+\frac{60}{\lambda^7}+\frac{36}{\lambda^8};$$

$$\varphi_1 = \frac{1}{\lambda^3}+\frac{6}{\lambda^4}+\frac{35}{\lambda^5}+\frac{57}{\lambda^6}+\frac{58}{\lambda^7}+\frac{28}{\lambda^8}+\frac{8}{\lambda^9};$$

$$\varphi_2 = \frac{1}{\lambda^3}+\frac{6}{\lambda^4}+\frac{35}{\lambda^5}+\frac{183}{\lambda^6}+\frac{472}{\lambda^7}+\frac{892}{\lambda^8}+\frac{1\,196}{\lambda^9}+\frac{1\,044}{\lambda^{10}}+\frac{648}{\lambda^{11}}+\frac{216}{\lambda^{12}};$$

$$\varphi_3 = \frac{1}{\lambda^3}+\frac{6}{\lambda^4}+\frac{35}{\lambda^5}+\frac{201}{\lambda^6}+\frac{508}{\lambda^7}+\frac{940}{\lambda^8}+\frac{1\,232}{\lambda^9}+\frac{1\,044}{\lambda^{10}}+\frac{648}{\lambda^{11}}+\frac{216}{\lambda^{12}};$$

$$\varrho_1 = \frac{1}{\lambda^4}+\frac{10}{\lambda^5}+\frac{85}{\lambda^6}+\frac{231}{\lambda^7}+\frac{379}{\lambda^8}+\frac{360}{\lambda^9}+\frac{224}{\lambda^{10}}+\frac{80}{\lambda^{11}}+\frac{16}{\lambda^{12}};$$

$$\varrho_2 = \frac{1}{\lambda^4}+\frac{10}{\lambda^5}+\frac{85}{\lambda^6}+\frac{567}{\lambda^7}+\frac{2\,179}{\lambda^8}+\frac{6\,084}{\lambda^9}+\frac{12\,644}{\lambda^{10}}+\frac{19\,412}{\lambda^{11}}+\frac{22\,804}{\lambda^{12}}+\frac{20\,064}{\lambda^{13}} +\frac{12\,744}{\lambda^{14}}+\frac{5\,616}{\lambda^{15}}+\frac{1\,296}{\lambda^{16}};$$

$$\varrho_3 = \frac{1}{\lambda^4}+\frac{10}{\lambda^5}+\frac{85}{\lambda^6}+\frac{621}{\lambda^7}+\frac{2\,395}{\lambda^8}+\frac{6\,588}{\lambda^9}+\frac{13\,436}{\lambda^{10}}+\frac{20\,228}{\lambda^{11}}+\frac{23\,380}{\lambda^{12}}+\frac{20\,280}{\lambda^{13}} +\frac{12\,744}{\lambda^{14}}+\frac{5\,616}{\lambda^{15}}+\frac{1\,296}{\lambda^{16}}.$$

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