On a theorem of S. S. Bhatia and B. Ram

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Abstract. *In this paper some inequalities for Dirichlet's and Fejer's kernels proved in [6] are refined and extended. Then we have obtained the conditions for L^1-convergence of the r-th derivatives of complex trigonometric series. These results are extensions of corresponding Bhatia's and Ram's results for complex trigonometric series (case $r = 0$).*

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1. Introduction and preliminaries

Let $\{c_k \colon k = 0, \pm 1, \pm 2, \dots\}$ be a sequence of complex numbers and let the partial sums of the complex trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be denoted by

$$S_n(c,t) = \sum_{k=-n}^{n} c_k e^{ikt}\,, \quad t \in (0,\pi]\,. \tag{1}$$

If a trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$, for all n and $S_n(c,t) = S_n(f,t) = S_n(f)$.

S. S. Bhatia and B. Ram [6] introduced the following class $\Re^*$ of a complex sequence: a null sequence $\{c_n\}$ of complex numbers belongs to class $\Re^*$ if

$$\sum_{k=1}^{\infty} \left| \Delta \left(\frac{c_{-k} - c_k}{k} \right) \right| k \log k < \infty \quad \text{and}$$
$$\sum_{k=1}^{\infty} k^2 \left| \Delta^2 \left(\frac{c_k}{k} \right) \right| < \infty\,.$$

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Dirichlet's and respectively Feier's kernels are denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\overline{D}_n(t) = -\frac{1}{2}\operatorname{ctg}\frac{t}{2} + \tilde{D}_n(t) = -\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\tilde{K}_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} \tilde{D}_k(t) = \frac{1}{4\sin^2\frac{t}{2}}\left[\sin t - \frac{\sin(n+1)t}{n+1}\right]$$

Let $E_n(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{ikt}$ and $E_{-n}(t) = \frac{1}{2} + \sum_{k=1}^{n} e^{-ikt}$. Then the r-th derivatives $D_n^{(r)}(t)$ and $\tilde{D}_n^{(r)}(t)$ can be written as

$$\begin{aligned} 2\,D_n^{(r)}(t) &= E_n^{(r)}(t) + E_{-n}^{(r)}(t) \\ 2\,i\tilde{D}_n^{(r)}(t) &= E_n^{(r)}(t) - E_{-n}^{(r)}(t) \end{aligned} \tag{2}$$

S. S. Bhatia and B. Ram [6] introduced the following modified sums

$$g_n(c,t) = S_n(c,t) + \frac{i}{n+1}\left[c_{n+1}E_n'(t) - c_{-(n+1)}E_{-n}'(t)\right]$$

and proved the following result.

Theorem 1 [6]. *Let $\{c_n\} \in \Re^*$. Then there exists $f(t)$ such that*

(i) $\lim_{n\to\infty} g_n(c,t) = f(t)$ *for all* $0 < |t| \le \pi$.

(ii) $f(t) \in L^1(T)$ *and* $\|g_n(c,t) - f(t)\|_1 = o(1)$, $n \to \infty$.

(iii) $\|S_n(f,t) - f(t)\|_1 = o(1)$ *iff* $\hat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.

Now we define a new class $\Re^*(r)$, $r = 0,1,2,\dots$ of a complex sequence as follows: a null sequence $\{c_k\}$ of complex numbers belongs to class $\Re^*(r)$, $r = 0,1,2,\dots$ if

$$\sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right| k^{r+1}\log k < \infty$$

$$\sum_{k=1}^{\infty} k^{r+2}\left|\Delta^2\left(\frac{c_k}{k}\right)\right| < \infty\,.$$

If $r = 0$, class $\Re^*(r)$ reduces to $\Re^*$.

Č. V. Stanojević and V. B. Stanojević [7] introduced the following modified complex trigonometric sums:

$$U_n(c,t) = S_n(c,t) - \left(c_nE_n(t) + c_{-n}E_{-n}(t)\right).$$

The complex form of the r-th derivative of this sum obtained by Sheng [5] is

$$U_n^{(r)}(c,t) = S_n^{(r)}(c,t) - \big(c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)\big).$$

B. Ram and S. Kumari [4] introduced another set of modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\cos kx \quad \text{and}$$

$$h_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k\sin kx\,.$$

The complex form of the r-th derivative of these modified sums obtained by Bhatia and Ram [2] is

$$G_n^{(r)}(c,t) = S_n^{(r)}(c,t) + \frac{i}{n+1}\left[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)\right].$$

Using the modified complex sums $G_n^{(r)}$ we shall prove the following theorem:

Theorem 2. *Let* $\{c_n\} \in \Re^*(r)$, $r = 0,1,2,\dots$ *Then*

(i) $\lim_{n\to\infty} G_n^{(r)}(c,t) = f^{(r)}(t)$ *for all* $0 < |t| \le \pi$.

(ii) $f^{(r)} \in L^1(T)$ *and* $\|G_n^{(r)}(c,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$.

(iii) $\|S_n^{(r)}(f,t) - f^{(r)}(t)\|_1 = o(1)$, $n \to \infty$ *iff* $|n|^r \hat{f}(n)\log|n| = o(1)$, $|n| \to \infty$.

The case $r = 0$ of our Theorem yields *Theorem 1.*

For other criteria for L^1-convergence of the r-th derivative of a complex trigonometric series, see [8].

The real trigonometric series version of *Theorem 2* was established by Bhatia and Ram [3].

2. Lemmas

For the proof of our new theorem we need the following Lemmas.

Lemma 1 [5]. *For the r-th derivatives of the Dirichlet's kernels D_n and $\tilde{D}_n$ the following estimates hold*

(i) $\|D_n^{(r)}\|_1 = \frac{4}{\pi} n^r \log n + O(n^r)$, $r = 0,1,2,\dots$

(ii) $\|\tilde{D}_n^{(r)}\|_1 = O(n^r \log n)$, $r = 0,1,2,\dots$

Lemma 2 [5]. *For each non-negative integer n,* $\|c_n E_n^{(r)} + c_{-n} E_{-n}^{(r)}\|_1 = o(1)$, $|n| \to \infty$ *holds if and only if* $|n|^r c_n \log|n| = o(1)$, $|n| \to \infty$, *where* $\{c_n\}$ *is a complex sequence.* We note that this Lemma for $r = 0$, was obtained by Bray and Stanojević in [1].

Lemma 3 [10]. *Let r be a non-negative integer. Then for all* $0 < |t| \le \pi$ *and all* $n \ge 1$ *the following estimates hold* *(i)* $|E_{-n}^{(r)}(t)| \le \frac{4n^r\pi}{|t|}$.

(ii) $|\tilde{D}_n^{(r)}(t)| \le \frac{4n^r\pi}{|t|}$.

Proof. (i) The case $r = 0$ is trivial. Really, since $E_n(t) = D_n(t) + i\tilde{D}_n(t)$, we have

$$|E_n(t)| \le |D_n(t)| + |\tilde{D}_n(t)| \le \frac{\pi}{2|t|} + \frac{\pi}{|t|} = \frac{3\pi}{2|t|} < \frac{4\pi}{|t|}$$
$$|E_{-n}(t)| = |E_n(-t)| < \frac{4\pi}{|t|}.$$

Let $r \ge 1$. Applying Abels's transformation, we have:

$$E_n^{(r)}(t) = i^r \sum_{k=1}^{n} k^r e^{ikt} = i^r \left[\sum_{k=1}^{n-1} \Delta(k^r)\Big(E_k(t) - \frac{1}{2}\Big) + n^r \Big(E_n(t) - \frac{1}{2}\Big) \right]$$
$$|E_n^{(r)}(t)| \le \sum_{k=1}^{n-1} \big[(k+1)^r - k^r\big]\Big(\frac{1}{2} + |E_k(t)|\Big) + n^r\Big(|E_n(t)| + \frac{1}{2}\Big)$$
$$\le \Big(\frac{\pi}{2|t|} + \frac{3\pi}{2|t|}\Big)\left\{ \sum_{k=1}^{n-1} [(k+1)^r - k^r] + n^r \right\} = \frac{4\pi\, n^r}{|t|}.$$

Since $E_{-n}^{(r)}(t) = E_n^{(r)}(-t)$, we obtain $|E_{-n}^{(r)}(t)| \le \frac{4\pi n^r}{|t|}$.

(ii) Applying inequality (i) and equation 2, we obtain

$$|\tilde{D}_n^{(r)}(t)| = |i\tilde{D}_n^{(r)}(t)| \le \frac{1}{2}|E_n^{(r)}(t)| + \frac{1}{2}|E_{-n}^{(r)}(t)| \le \frac{4n^r\pi}{|t|}.$$

□

Lemma 4 [6]. *$\|\tilde{K}_n'(t)\|_1 = O(n)$.*

Lemma 5 [11]. *If $T_n(x)$ is a trigonometric polynomial of order n, then*

$$\|T_n^{(r)}\| \le n^r \|T_n\|.$$

This is Bernstein's inequality in the $L^1(0,\pi)$-metric (see [11], vol.2, p.11).

Lemma 6. *$\|\tilde{K}_n^{(r)}\|_1 = O(n^r)$, $r = 1, 2, \dots$*

Proof. Since $\tilde{K}_n(x) = \sum_{k=1}^{n} \frac{n+1-k}{n+1} \sin kx$, we have that

$$T_n(x) = \tilde{K}_n'(x) = \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1} \cos kx$$

is a cosine trigonometric polynomial of order n.

Applying first Bernstein's inequality, then *Lemma 4*, yields:

$$\|\tilde{K}_n^{(r)}\|_1 = \|T_n^{(r-1)}(x)\|_1 \le n^{r-1}\|T_n(x)\|_1 = O(n^r).$$

□

3. Proof of the main result

Applying Abel's transformation, we have:

$$\begin{aligned} G_n^{(r)}(c,t) &= S_n^{(r)}(c,t) + \frac{i}{n+1}\Big[c_{n+1}E_n^{(r+1)}(t) - c_{-(n+1)}E_{-n}^{(r+1)}(t)\Big] \\ &= 2\sum_{k=1}^{n}\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{n}\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)iE_{-k}^{(r+1)}(t)\,. \end{aligned}$$

By *Lemma 3*, we get:

$$\begin{aligned} \sum_{k=1}^{\infty}\Big|\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}\Big| &\le \frac{4\pi}{|t|}\sum_{k=1}^{\infty}k^{r+1}\Big|\Delta\Big(\frac{c_k}{k}\Big)\Big| \le \frac{4\pi}{|t|}\left\{\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}k^{r+1}\Big|\Delta^2\Big(\frac{c_j}{j}\Big)\Big|\right\} \\ &= \frac{4\pi}{|t|}\left\{\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j}k^{r+1}\right)\Big|\Delta^2\Big(\frac{c_j}{j}\Big)\Big|\right\} \\ &= O\left(\frac{1}{|t|}\sum_{j=1}^{\infty}j^{r+2}\Big|\Delta^2\Big(\frac{c_j}{j}\Big)\Big|\right) < \infty \end{aligned}$$

and

$$\begin{aligned} \sum_{k=3}^{\infty}\Big|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)E_{-k}^{(r+1)}(t)\Big| &\le \frac{4\pi}{|t|}\left\{\sum_{k=3}^{\infty}k^{r+1}\Big|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\Big|\right\} \\ &= O\left(\frac{1}{|t|}\sum_{k=3}^{\infty}k^{r+1}\log k\Big|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\Big|\right) < \infty\,. \end{aligned}$$

Consequently,

$$f^{(r)}(t) = 2\sum_{k=1}^{\infty}\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}(t) + \sum_{k=1}^{\infty}\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)iE_{-k}^{(r+1)}(t)$$

exists and thus (i) follows.

Now, for $t \neq 0$, we have:

$$\begin{aligned} f^{(r)}(t) - G_n^{(r)}(c,t) &= 2\sum_{k=n+1}^{\infty}\Delta\Big(\frac{c_k}{k}\Big)\tilde{D}_k^{(r+1)}(t) + i\sum_{k=n+1}^{\infty}\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)E_{-k}^{(r+1)}(t) \\ &= 2\sum_{k=n+1}^{\infty}(k+1)\Delta^2\Big(\frac{c_k}{k}\Big)\tilde{K}_k^{(r+1)}(t) - 2(n+1)\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\tilde{K}_{n+1}^{(r+1)}(t) \\ &\quad + i\sum_{k=n+1}^{\infty}\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)E_{-k}^{(r+1)}(t)\,. \end{aligned}$$

Then,

$$\begin{aligned}\|f^{(r)}(t)-G_n^{(r)}(c,t)\|_1 \le 2\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right|\int_{-\pi}^{\pi}|\tilde{K}_k^{(r+1)}(t)|dt\\ +2(n+1)\left|\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\right|\int_{-\pi}^{\pi}\left|\tilde{K}_{n+1}^{(r+1)}(t)\right|dt\\ +\sum_{k=n+1}^{\infty}\left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\right|\int_{-\pi}^{\pi}|E_{-k}^{(r+1)}(t)|dt\,.\end{aligned}$$

Applying *Lemma 6* and *Lemma 1*, we have:

$$\begin{aligned}\|f^{(r)}(t)-G_n^{(r)}(c,t)\|_1 = O\left(\sum_{k=n+1}^{\infty}(k+1)^{r+2}\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right|\right)\\ +O\left((n+1)^{r+2}\left|\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\right|\right)\\ +O\left(\sum_{k=n+1}^{\infty}\left|\Delta\Big(\frac{c_{-k}-c_k}{k}\Big)\right|k^{r+1}\log k\right).\end{aligned}$$

But

$$\begin{aligned}\left|\Delta\Big(\frac{c_{n+1}}{n+1}\Big)\right| = \left|\sum_{k=n+1}^{\infty}\Delta^2\Big(\frac{c_k}{k}\Big)\right| \le \sum_{k=n+1}^{\infty}\frac{k^{r+2}}{k^{r+2}}\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right|\\ \le \frac{1}{(n+1)^{r+2}}\sum_{k=n+1}^{\infty}k^{r+2}\left|\Delta^2\Big(\frac{c_k}{k}\Big)\right| = o\Big(\frac{1}{(n+1)^{r+2}}\Big),\quad n\to\infty\,.\end{aligned}$$

Hence, $\|f^{(r)}(t)-G_n^{(r)}(c,t)\|_1 = o(1)$, $n\to\infty$ by the hypothesis of the theorem.

Since $G_n^{(r)}(c,t)$ is a polynomial, it follows that $f^{(r)}\in L^1$.

The proof of (iii) follows from the estimate

$$\begin{aligned}\left|\,\|f^{(r)}-S_n^{(r)}(f)\|_1 - \left\|\frac{i}{n+1}\big(\hat{f}(n+1)E_n^{(r+1)}-\hat{f}\big(-(n+1)\big)E_{-n}^{(r+1)}\big)\right\|_1\right|\\ \le \|f^{(r)}-G_n^{(r)}(c,t)\|_1 = o(1)\,,\ \ n\to\infty\end{aligned}$$

and from *Lemma 2*.

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