The properties of spaces which admit a Whitney map

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Abstract. *Let X be a topological space and 2^X the hyperspace of closed subsets of X. It is known that there is a Whitney map on the hyperspace 2^X for separable metric spaces X. In this paper we study the properties of spaces which admit a Whitney map for some subspaces of 2^X. We shall show that, under some natural assumptions, such spaces are closely related to separable metric spaces.*

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1. Introduction

All spaces in this paper are Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$.

For a space X with a topology T, we put: a) $2^X = \{F : F$ is a nonempty closed subset of $X\}$ equipped with the Vietoris topology, i.e., the topology with base $\mathcal{B} = \{\langle U_1, ..., U_n\rangle : U_i \in T$ for each $i < \infty\}$, where $\langle U_1, ..., U_n\rangle = \{F \in 2^X : F \subset U_1 \cup ... \cup U_n$ and $F \cap U_i \neq \emptyset$ for each $i\}$ [3, p. 4], b) $CP(X) = \{K \in 2^X : K$ is compact$\}$ and c) $X(n) = \{K \in CP(X) : K$ contains at most n points$\}$. We consider $CP(X)$ and $X(n)$ as subspaces of 2^X. If F is a closed subset of X, then $\{F\}$ denotes the corresponding element of 2^X.

For mapping $f : X \rightarrow Y$ define $2^f : 2^X \rightarrow 2^Y$ by $2^f(\{F\}) = \{\mathrm{Cl} f(F)\}$ for $\{F\} \in 2^X$. If Y is normal, then 2^f is continuous. Furthermore, $2^f(CP(X)) \subset CP(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|CP(X)$ is denoted by $cp(f)$.

We say that a topological space X is a Lindelöf space if X is regular and every open cover of X has a countable subcover. Every regular second-countable space is a Lindelöf space and every Lindelöf space is normal [2, Theorem 3.8.2, p. 247].

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [6, p. 24, (0.50)] we will mean any mapping $g : \Lambda \rightarrow [0, +\infty)$ satisfying

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a) if $\{A\}, \{B\} \in \Lambda$ such that $A \subset B, A \neq B$, then $g(\{A\}) < g(\{B\})$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

H. Whitney first constructed special types of functions on spaces of sets for the purpose of studying families of curves ([8] and [9]). Now, we construct the Whitney map (as Whitney's function is now called) that was constructed in [9, pp. 245-246] for separable metric spaces. Let d denote a metric on X. Let $Z = \{z_1, z_2, ..., z_n, ...\}$ be a countable dense subset of X. For each $n = 1, 2, ...$, define $f_n : X \to [0, 1]$ as follows:

$$f_n(x) = \frac{1}{1 + d(x_n, x)}$$

for each $x \in X$. Next, for each $n = 1, 2, ...$, define $w_n : 2^X \to [0, 1]$ as follows:

$$w_n(A) = \text{ diameter } [f_n(A)]$$

for each $A \in 2^X$. Finally, define $w : 2^X \to [0, 1]$ as follows:

$$w(A) = \sum_{n=1}^{\infty} \frac{w_n(A)}{2^n}$$

for each $A \in 2^X$. For the proof that $w : 2^X \to [0, 1]$ is a Whitney map for 2^X see [3, Theorem 13.4, p. 107].

J. L. Kelley [4] was the first person who introduced Whitney maps in studying hyperspaces. Book [6] is the first book in which Whitney maps are used for a systematic study of hyperspaces. A modern approach to hyperspace theory is book [3].

If X is a compact metric space, then there exists a Whitney map for 2^X and $C(X)$ ([6, pp. 24-26], [3, pp. 106, 205-230]). On the other hand, if X is a non-metrizable compact space, then it admits no Whitney map for 2^X [1].

The present paper is inspired by the following theorem due to T. Watanabe [7, Theorem 1].

Theorem 1. *Let X be a metric space. Then the following conditions are equivalent to each other:*

(i) *X admits a Whitney map $\mu : (2^X, H_d) \to \mathbb{R}$ for some bounded metric d on X,*

(ii) *X admits a Whitney map $\mu : (2^X, VT) \to \mathbb{R}$, where VT is the Vietoris topology on 2^X,*

(iii) *X has the Lindelöf property,*

(iv) *X is separable.*

Theorem 1 shows limitations for the existence of Whitney maps for metric spaces.

Example 1. *[7, Example 7]. Let m be an uncountable ordinal. Let $J(m)$ be the hedgehog space of spines m defined in Example 4.1.5. of [2, p. 314]. The space $J(m)$ is a connected, locally connected, metric space which admits no Whitney map for $2^{J(m)}$ since it does not have the Lindelöf property.*

The following result is also known.

Lemma 1 [[7], Lemma 3]. *Any metric space X admits a Whitney map $\mu : (CP(X), VT) \to \mathbb{R}$.*

In the sequel we study the properties of spaces which admit a Whitney map for some subspaces of 2^X for non-metric space X.

2. Whitney map for $X(2)$

If a space X admits a Whitney map for 2^X or for $CP(X)$, then X admits a Whitney map for $X(2)$. Hence, it is natural to begin with the study of properties of spaces which admit a Whitney map for $X(2)$. Let Δ be the diagonal of the product $X \times X$, i.e., $\Delta = \{(x, x) : x \in X\}$. If X is a Hausdorff space, then Δ is a closed subset of $X \times X$ [2, Corollary 2.3.22, p. 114]. Define $j_2 : X \times X \to X(2)$ by $j_2((x, y)) = \{x, y\}$. The mapping j_2 is a closed surjection if X is a Hausdorff space [2, Problem 2.7.20.(c), p. 162].

We start with the following lemma which is a part of the proof of Theorem of [1] in the case of a compact space X.

Lemma 2. *If a Hausdorff space X admits a Whitney map for $X(2)$, then $X(1)$ is a G_δ-subset in $X(2)$ and the diagonal Δ is a G_δ-subset in $X \times X$.*

Proof. Let $\mu : X(2) \to \mathbb{R}$ be a Whitney map. It is known that $X(1) \subset X(2)$ is homeomorphic to X if X is a T_1-space [2, Problem 2.7.20.(b), p.163]. If X is a Hausdorff space, then every $X(n), n \in \mathbb{N}$, is closed in 2^X [2, Problem 2.7.20.(b), p.163]. By the definition of a Whitney map we have $\mu^{-1}(0) = X(1)$. We infer that $X(1)$ is a G_δ-subset in $X(2)$ since $0 \in \mathbb{R}$ is a G_δ-subset in $\mathbb{R}$. Since $j_2^{-1}(X(1)) = \Delta$ we conclude $\Delta = (\mu j_2)^{-1}(0)$. Since $0 \in \mathbb{R}$ is a G_δ-subset in $\mathbb{R}$, we conclude that Δ is a G_δ-subset in $X \times X$. □

Lemma 3. *The diagonal Δ in $X \times X$ is a G_δ-subset if and only if $X(1)$ is a G_δ-subset in $X(2)$.*

Proof. Consider the mapping $j_2 : X \times X \to X(2)$ which is a closed surjection. Let us observe that $j_2^{-1}(X(1)) = \Delta$. If $X(1)$ is a G_δ-set in $X(2)$, then there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of $X(2)$ such that $X(1) = \cap\{U_n : n \in \mathbb{N}\}$. From $j_2^{-1}(X(1)) = \Delta$ it follows that $\Delta = j_2^{-1}(X(1)) = j_2^{-1}(\cap\{U_n : n \in \mathbb{N}\}) = \cap\{j_2^{-1}(U_n) : n \in \mathbb{N}\}$. Hence, Δ is a G_δ-subset of $X \times X$. Conversely, let Δ be a G_δ-subset in $X \times X$. Now, j_2 is closed and $X(2) \setminus X(1)$ is F_σ. Hence, $j_2^{-1}(X(2) \setminus X(1)) = X \times X \setminus \Delta$ is F_σ. □

Lemma 4. *If $X(2)$ is a normal space, then X admits a Whitney map if and only if $X(1)$ is a G_δ-subset in $X(2)$.*

Proof. If $X(1)$ is a G_δ-subset in $X(2)$, then there exists a continuous function $f : X(2) \to I = [0, 1]$ such that $X(1) = f^{-1}(0)$ [2, Corollary 1.5.11, p. 64]. This means that for $\{x, y\} \in X(2) \setminus X(1)$ we have $f(\{x, y\}) > 0$. Hence, f is a Whitney map for $X(2)$. Conversely, if there exists a Whitney map $\mu : X(2) \to \mathbb{R}$, then $X(1)$ is a G_δ-subset in $X(2)$ (*Lemma 2*). □

Lemma 5. *If $X \times X$ is a normal space, then X admits a Whitney map for $X(2)$ if and only if the diagonal Δ is a G_δ-subset in $X \times X$.*

Proof. If $X \times X$ is normal, then $X(2)$ is a normal space since there exists a closed mapping $j_2 : X \times X \to X(2)$ [2, Theorem 1.5.20, p. 69]. If Δ is a G_δ-subset in

$X \times X$ then by *Lemma 3* $X(1)$ is a G_δ-subset in $X(2)$. Apply *Lemma 4*. Conversely, if X admits a Whitney for $X(2)$, then apply *Lemma 2*. □

A topological space X is *Čech-complete* if X is a Tychonoff space and the remainder $\beta X \setminus X$ is an F_σ-set in βX [2, pp. 251-251].

A Tychonoff space X is said to be *locally Čech-complete* if every point $x \in X$ has a Čech-complete neighbourhood [2, Problem 3.12.18.(c), p. 297]. Every locally Čech-complete paracompact space is Čech-complete [2, Problem 5.5.8.(c), p. 422].

Lemma 6. *If a space X is a Čech-complete Lindelöf space, then $X(2)$ is a Lindelöf space.*

Proof. If a space X is a Čech-complete Lindelöf space, then the Cartesian product $X \times X$ is a Lindelöf space [2, Exercise 3.9.F, p. 257]. Now, $X(2)$, as a continuous image of $X \times X$ [2, Problem 2.7.20.(b), p. 162] is a Lindelöf space. □

Theorem 2. *If X is a Čech-complete Lindelöf space, then X admits a Whitney map for $X(2)$ if and only if $X(1)$ is a G_δ-subset in $X(2)$.*

Proof. By virtue of *Lemma 6* $X(2)$ is a Lindelöf space and, consequently, normal. Apply *Lemma 4*. □

The following result is important in the sequel.

Lemma 7. *If a Hausdorff space X admits one-to-one continuous mapping $f : X \to Y$ onto a metric space Y, then X admits a Whitney map for $X(2)$ and for $CP(X)$.*

Proof. Compose $cp(f)$ with a Whitney map for $CP(Y)$. □

If X is a paracompact space, then we have the following important result.

Lemma 8. *[2, Problem 5.5.7, p. 421]. If X is a paracompact space and the diagonal Δ is G_δ-set in $X \times X$, then there exists a one-to-one continuous mapping of X onto a metrizable space.*

Theorem 3. *A paracompact space X admits a Whitney map for $X(2)$ if and only if there exists one-to-one continuous mapping $f : X \to Y$ onto a metric space Y.*

Proof. If there exists one-to-one continuous mapping $f : X \to Y$ onto a metric space Y, then apply *Lemma 7*. Conversely, if a paracompact space X admits a Whitney map for $X(2)$, then it has the diagonal Δ which is a G_δ-set in $X \times X$ because of *Lemma 2*. From *Lemma 8* it follows that there exists a one-to-one continuous mapping $f : X \to Y$ onto a metric space Y. □

Corollary 1. *A compact space X admits a Whitney map for $X(2)$ if and only if X is a metric space.*

Proof. If X is a compact metrizable space, then X admits a Whitney map for $X(2)$ since it admits a Whitney map for $CP(X)$ (*Lemma 1*). Conversely, if X admits a Whitney map for $X(2)$, then by *Theorem 3* there exists one-to-one continuous mapping $f : X \to Y$ onto a metric space Y. It is clear that f is a homeomorphism. Hence, X is a metrizable space. □

A topological space X is *countably compact* if X is a Hausdorff space and every countable open cover of X has a finite subcover [2, p. 258]. A Hausdorff space is countably compact if and only if every infinite set has an accumulation point [2, Theorem 3.10.3, p. 258]. Every countably compact paracompact space is compact [2, Theorem 5.1.20, p. 380].

Corollary 2. *A countably compact space X admits a Whitney map for $X(2)$ if and only if X is a metric space.*

Proof. If X is countably compact metrizable space, then X is a compact [2, Theorem 5.1.20, p. 380] metric space. By *Corollary 1* it follows that X admits a Whitney map for $X(2)$. Conversely, if X admits a Whitney map for $X(2)$, then the diagonal Δ is a G_δ-set in $X\times X$ (*Lemma 2*). By virtue of [2, Problem 3.12.22.(e), p. 303] we conclude that X is compact. Therefore X is metrizable by [2, Exercise 4.2.B, p. 330] or by *Corollary 1.* □

Lemma 9. *If a Hausdorff space X admits a Whitney map for $X(2)$, then every countably compact subset Y of X is metrizable.*

Proof. If X admits a Whitney map μ for $X(2)$, then $\mu|Y(2)$ is a Whitney map for $Y(2)$. Apply *Corollary 2.* □

We say that a Hausdorff space X is *locally countably compact* if for every $x \in X$ there exists a neighbourhood U of x such that $Cl(U)$ is a countably compact subspace of X.

Theorem 4. *If a locally countably compact paracompact space X admits a Whitney map for $X(2)$, then X is metrizable.*

Proof. Let $\mu : X(2) \to \mathbb{R}$ be a Whitney map. For every $x \in X$ there exists a neighbourhood U of x such that $Cl(U)$ is a countably compact subspace of X. From *Lemma 9* it follows that $Cl(U)$ is metrizable. Finally, using the Smirnov metrization theorem [2, 5.4.A, p. 415] we complete the proof. □

Theorem 5. *A linearly ordered space X admits a Whitney map for $X(2)$ if and only if it is metrizable.*

Proof. If X is metrizable, then X admits a Whitney map for $X(2)$. Conversely, if X admits a Whitney map for $X(2)$, then the diagonal Δ is a G_δ-set in $X \times X$ (*Lemma 2*). By [2, Problem 5.5.22.(k), p. 430] X is metrizable. □

3. Whitney map for $CP(X)$

If a space X admits a Whitney map for $CP(X)$, then it admits a Whitney map for $X(2)$. This means that X has the same properties as in the last section. We start with the following theorem.

Theorem 6. *If X is a paracompact space and the diagonal Δ is a G_δ-set in $X \times X$, then X admits a Whitney map for $CP(X)$.*

Proof. Apply *Lemma 8* and *Theorem 3.* □

Theorem 7. *A paracompact space X admits a Whitney map for $CP(X)$ if and only if there exists a one-to-one continuous mapping $f : X \to Y$ onto a metric space Y.*

Proof. The proof is the same as the proof of *Theorem 3.* □

Let us observe that for a compact space X we have $CP(X) = 2^X$ and if $f : X \to Y$ is a continuous one-to-one mapping, then f is a homeomorphism. Hence, we have the following corollary.

Corollary 3. *A compact space admits a Whitney map for 2^X if and only if it is metrizable.*

Remark 1. *For another proof of* Corollary 3 *see [1]. The proof by the inverse system method is given in [5].*

Theorem 8. *If a countably compact space admits a Whitney map for $CP(X)$, then X is metrizable.*

Proof. As in Corolary 2, X is compact and metrizable since X admits a Whitney map for $X(2)$. □

Corollary 4. *A countably compact space admits a Whitney map for $CP(X)$ if and only if it is a separable metric space.*

Theorem 9. *If X is a locally countably compact paracompact space which admits a Whitney map for $CP(X)$, then X is metrizable.*

Proof. From *Theorem 8* it follows that every $x \in X$ has a neighbourhood U such that $Cl(U)$ is metrizable. This means that every point of X has a metrizable neighbourhood. Using the Smirnov theorem [2, 5.4.A, p. 415] we conclude that X is metrizable since X is paracompact. □

Corollary 5. *A locally compact paracompact space X is metrizable if and only if X admits a Whitney map for $CP(X)$.*

Proof. If X is a locally compact paracompact space which admits a Whitney map, then X is metrizable because of *Theorem 9*. Conversely, if X is metrizable, then X admits a Whitney map (*Lemma 1*). □

4. Whitney map for 2^X

The assumption that a space X admits a Whitney map for 2^X is very restrictive. This is established by *Lemmas 10* and *11*.

Lemma 10. *If X admits a Whitney map for 2^X, then $hl(X) < \aleph_1$, i.e., X is hereditarily Lindelöf. Moreover, X is perfectly normal and hereditarily paracompact.*

Proof. Suppose that X contains a transfinite increasing sequence $U_0 \subset U_1 \subset ... \subset U_\xi \subset ..., \xi < \omega_1$ of open subsets of X. Then $X \setminus U_0 \supset X \setminus U_1 \supset ... \supset X \setminus U_\xi \subset ..., \xi < \omega_1$ is a decreasing sequence of closed subsets of X. Now, $\omega(X \setminus U_0) > \omega(X \setminus U_1) > ... > \omega(C_\xi) > ..., \xi < \omega_1$ is an decreasing transfinite sequence of real numbers $\omega(X \setminus U_0) > \omega(X \setminus U_1) > ... > \omega(X \setminus U_\xi) > ..., \xi < \omega_1$. This is impossible since $w(\mathbb{R}) = \aleph_0$. By virtue of [2, Problem 3.12.7(b), p. 284] it follows that $hl(X) < \aleph_1$, i.e., X is hereditarily Lindelöf. Furthermore, X is perfectly normal [2, Exercise 3.8.A(b), p. 249] and [2, Theorem 5.1.2, p. 373]. □

The following example shows that the assumption that X has the Lindelöf property (X is separable) cannot be omitted in *Corollary 9* (*Corollary 8*).

Example 2. *[7, Example 8]. Let L be the long segment defined in 3.12.18 of [2, p. 297]. Then L is a connected, locally connected and compact space which admits no Whitney map for 2^L since L is not separable and is not a hereditarily Lindelöf space.*

Lemma 11. *If X admits a Whitney map for 2^X, then $hd(X) < \aleph_1$.*

Proof. The proof is a straightforward modification of the proof of *Lemma 10* using [2, Problem 2.7.9(e), p.155]. □

Lemmas 10 and *11* show limitations for the existence of Whitney maps for 2^X. The space which admits a Whitney map for 2^X must be hereditarily Lindelöf and hereditarily separable. The following question is natural.

QUESTION. Does there exist a non-metric hereditarily Lindelöf and hereditarily separable space X which admits a Whitney map for 2^X?

Theorem 10. *If a space X admits a Whitney map for 2^X, then there exists a one-to-one mapping $f : X \to Y$ onto a separable metric space.*

Proof. By *Lemma 10* X is a paracompact space. Moreover, it has the diagonal Δ in $X \times X$ which is a G_δ-subset (*Lemma 2)*. From *Lemma 8* it follows that there exists a one-to-one continuous mapping $f : X \to Y$ onto a metrizable space Y. Now, since X is a Lindelöf space, Y is a Lindelöf space and, consequently, separable and second-countable. □

As an immediate consequence of the above Theorem we have the following result.

Theorem 11. *If a countably compact space X admits a Whitney map for 2^X, then it is compact and metrizable.*

Proof. By virtue of *Theorem 10* X is paracompact. Moreover, X is compact since each countably compact paracompact space is compact [2, Theorem 5.1.20, p. 380]. Finally, from *Theorem 10* it follows that X is metrizable. □

Corollary 6. *A countably compact space X is metrizable if and only if it admits a Whitney map for 2^X.*

Theorem 12. *If a locally countably compact space X admits a Whitney map for 2^X, then X is metrizable.*

Proof. Let $\mu : 2^X \to \mathbb{R}$ be a Whitney map. By virtue of *Theorem 10* X is paracompact. For every $x \in X$ there exists a neighbourhood U of x such that $Cl(U)$ is a countably compact subspace of X. It follows that $Cl(U)$ is compact [2, Theorem 5.1.20, p. 380]. From *Theorem 11* it follows that $Cl(U)$ is metrizable since the restriction $\mu|2^{Cl(U)}$ is a Whitney map for $2^{Cl(U)}$. Finally, using the Smirnov metrization theorem [2, 5.4.A, p. 415] we complete the proof. □

Corollary 7. *If X is a locally compact space which admits a Whitney map for 2^X, then X is metrizable.*

Proof. From *Theorem 11* it follows that every $x \in X$ has a neighbourhood U such that $Cl(U)$ is metrizable. This means that every point of X has a metrizable neighbourhood. Using the Smirnov theorem [2, 5.4.A, p. 415] we conclude that X is metrizable since X is paracompact (hereditarily) because of *Theorem 10.* □

Corollary 8. *A separable locally compact space X admits a Whitney map for 2^X if and only if it is metrizable.*

Proof. If X admits a Whitney map, then by *Theorem 7* X is metrizable. Conversely, if X is a separable metric space, then X admits a Whitney map because of *Theorem 1.* □

By the same method of proof we obtain the following result.

Corollary 9. *A locally compact space X with the Lindelöf property admits a Whitney map for 2^X if and only if it is metrizable.*

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