On the Ishikawa iterative approximation with mixed errors for solutions to variational inclusions with accretive type mappings in Banach spaces*

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Abstract. *Using the new analysis techniques, the existence and iterative approximation problem of a solution for a class of nonlinear variational inclusions with accretive type mappings are discussed in arbitrary Banach spaces. The results extend and improve some recent results.*

Key words: *variational inclusion; accretive mapping; Ishikawa iterative sequence with mixed errors; Mann iterative sequence with mixed errors.*

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1. Introduction

Throughout this paper we suppose that X is a real Banach space, X^* is its dual space, $\langle\cdot,\cdot\rangle$ is the pairing of X and X^*. $D(T)$ and $R(T)$ denote the domain and the range of T, respectively.

Let $T,\ A:\ X\to X,\ g:\ X\to X^*$ be three mappings and $\varphi : X^*\to R\cup\{+\infty\}$ be a proper convex lower semicontinuous function.

In 1999, Chang [1] introduced and studied the existence and approximation problem of solutions for a class of nonlinear variational inclusions with accretive mappings in uniformly smooth Banach space as follows:

For any given $f\in X$, to find an $u\in X$ such that

$$\begin{cases} g(u)\in D(\partial\varphi), \\ \langle Tu-Au-f, v-g(u)\rangle \geq \varphi(g(u))-\varphi(v),\ \forall v\in X^*, \end{cases} \tag{1}$$

where $\partial\varphi$ denotes the subdifferential of φ.

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The purpose of this paper is to study further the existence and uniqueness of solutions and the convergence problem of Ishikawa and Mann iterative processes with mixed errors for a class of accretive type variational inclusion in arbitrary Banach spaces. The results presented in this paper not only extend and improve the main results in Chang [1], but also extend and improve the corresponding results in Chang [2,3], Chang, Cho, Lee et al [4], Ding [5,6], Hassouni and Moudafi [7], Huang [8-10], Kazmi [12], Noor [15,16], Siddiqi and Ansari [17], Siddiqi, Ansari and Kazmi [18] and Zeng [19].

2. Preliminaries

A mapping $J : X \to 2^{X^*}$ is said to be a *normalized duality mapping*, if it is defined by

$$J(x) = \left\{ f \in X^* : \langle x, f \rangle = \|f\|^2 = \|x\|^2 \right\}, \quad \forall x \in X.$$

Definition 1. *A mapping $T : D(T) \subset X \to X$ is said to be accretive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y) \rangle \geq 0.$$

If T is accretive and $R(I + rT) = X$ for all $r > 0$, then T is called m-accretive.

In the sequel we shall use the following Proposition and Lemmas.

Proposition 1 [14]. *Let X be a real Banach space, $T : D(T) \subset X \to X$ is accretive and continuous, and $D(T) = X$. Then T is m-accretive.*

Lemma 1 [13]. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the inequality.:*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 [20]. *Let X be a real Banach space, $T : D(T) \subset X \to X$ an m-accretive mapping. Then the equation $x + Tx = f$ has a unique solution in $D(T)$ for any $f \in X$.*

Lemma 3 [2]. *Let X be an arbitrary real Banach space, $\partial\varphi \circ g : X \to 2^X$ a mapping, then the following conclusions are equivalent to each other:*

(i) $x^* \in X$ is a solution of variational inclusion problem (1);

(ii) $x^* \in X$ is a fixed point of the mapping $S: X \to 2^X$:

$$S(x) = f - (Tx - Ax + \partial\varphi(g(x))) + x;$$

(iii) $x^* \in X$ is a solution of the equation $f \in Tx - Ax + \partial\varphi(g(x))$.

3. Main results

Theorem 1. *Let X be an arbitrary real Banach space, $T, A : X \to X$, $g : X \to X^*$ three mappings, and $\varphi : X^* \to R \cup \{+\infty\}$ a function with a continuous Gâteaux differential $\partial\varphi$. For any given $f \in X$, define a mapping $S : X \to X$ by*

$$Sx = f - (Tx - Ax + \partial\varphi(g(x))) + x.$$

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Ishikawa iterative sequence with mixed errors defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n S x_n + v_n, \ \forall n \geq 0, \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0,1]$, and $\{u_n\}$, $\{v_n\}$ are two sequences in X such that $u_n = u'_n + u''_n$ for any sequences $\{u'_n\}$ and $\{u''_n\}$ in X satisfying the following conditions

(i) $T - A + \partial\varphi \circ g - I : X \to X$ *is accretive,*

(ii) $T - A + \partial\varphi \circ g : X \to X$ *is a Lipschitz operator with constant L,*

(iii) $K_n = (1+L_*)(1+L_*^2)\alpha_n + L_*(1+L_*)\beta_n \leq 1 - r$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \|u'_n\| < \infty$, $\|u''_n\| = \gamma_n \alpha_n$ *and* $\|v_n\| \to 0$ $(n \to \infty)$,

where $L_* = 1 + L$, $r \in (0,1)$ is a constant and $\gamma_n \to 0$ $(n \to \infty)$. Then the following conclusions hold:

(1) The nonlinear variational inclusion problem (1) has a unique solution $x^* \in X$,

(2) The Ishikawa iterative sequence $\{x_n\}$ with mixed errors converges strongly to the unique solution $x^* \in X$ of the variational inclusion problem (1).

Proof. (**1**) First we prove that the variational inclusion problem (1) has a unique solution $x^* \in X$.

From conditions (i) and (ii), the mapping $T - A + \partial\varphi \circ g - I : X \to X$ is continuous and accretive. By *Proposition 1* we know that $T - A + \partial\varphi \circ g - I$ is m-accretive. Therefore, by *Lemma 2*, for any given $f \in X$, the equation

$$f = x + (T - A + \partial\varphi \circ g - I)(x)$$

has a unique solution $x^* \in X$. Hence, by *Lemma 3*, we know that x^* is a unique solution of the variational inclusion problem (1), and it is also a fixed point of S, i.e., $Sx^* = x^*$.

(**2**) Next we prove that the Ishikawa iterative sequence $\{x_n\}$ with mixed errors converges strongly to x^*.

By condition (i), for any $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle = -\langle (T-A+\partial\varphi \circ g - I)x - (T-A+\partial\varphi \circ g - I)y, j(x-y)\rangle \leq 0.$$

It follows from Lemma 1.1 of Kato [11] that

$$\|x-y\| \le \|x-y-t(Sx-Sy)\| \tag{3}$$

for all $x, y \in X$ and $t > 0$. Using (2), we easily conclude that for all $n \ge 0$,

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n S y_n - u_n \\ &= (1+\alpha_n)x_{n+1} - \alpha_n S x_{n+1} + {\alpha_n}^2 (x_n - S y_n) \\ &\quad + \alpha_n (S x_{n+1} - S y_n) - (1+\alpha_n) u_n. \end{aligned} \tag{4}$$

Note that

$$x^* = (1+\alpha_n)x^* - \alpha_n S x^* \tag{5}$$

for all $n \ge 0$. It follows from (3), (4) and (5) that

$$\begin{aligned} \|x_n - x^*\| &\ge (1+\alpha_n)\left\| x_{n+1} - x^* - \frac{\alpha_n}{1+\alpha_n}(S x_{n+1} - S x^*) \right\| \\ &\quad - \alpha_n^2 \|x_n - S y_n\| - \alpha_n \|S x_{n+1} - S y_n\| - (1+\alpha_n)\|u_n\| \\ &\ge (1+\alpha_n)\|x_{n+1} - x^*\| - {\alpha_n}^2 \|x_n - S y_n\| \\ &\quad - \alpha_n \|S x_{n+1} - S y_n\| - (1+\alpha_n)\|u_n\|, \end{aligned}$$

which implies that

$$\|x_{n+1} - x^*\| \le \frac{1}{1+\alpha_n}\|x_n - x^*\| + {\alpha_n}^2 \|x_n - S y_n\| + \alpha_n \|S x_{n+1} - S y_n\| + \|u_n\|. \tag{6}$$

Since $T - A + \partial\varphi \circ g$ is a Lipschitz mapping with the constant L, it is easy to verify that S is also Lipschitz with the constant $L_* = 1 + L$. Furthermore, we have the following estimates:

$$\begin{aligned} \|x_n - S y_n\| &\le \|x_n - x^*\| + \|S y_n - S x^*\| \\ &\le \|x_n - x^*\| + L_* \|y_n - x^*\| \\ &\le \|x_n - x^*\| + L_*[(1-\beta_n)\|x_n - x^*\| + \beta_n L_* \|x_n - x^*\| + \|v_n\|] \\ &\le (1 + L_*^2)\|x_n - x^*\| + L_* \|v_n\| \end{aligned} \tag{7}$$

and

$$\begin{aligned} \|S x_{n+1} - S y_n\| &\le L_* \|x_{n+1} - y_n\| = L_* \|\alpha_n (S y_n - x_n) + \beta_n (x_n - S x_n) + u_n - v_n\| \\ &\le L_* \alpha_n \|x_n - S y_n\| + L_* \beta_n (\|x_n - x^*\| + \|S x_n - x^*\|) + L_* \|u_n\| + L_* \|v_n\| \\ &\le [L_*(1+L_*^2)\alpha_n + L_*(1+L_*)\beta_n]\|x_n - x^*\| + L_*(1+L_*\alpha_n)\|v_n\| + L_* \|u_n\|. \end{aligned} \tag{8}$$

Substituting (7) and (8) into (6), and by conditions (iii) and (iv), we infer that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \frac{1}{1+\alpha_n}\left\{1 + [(1+L_*)(1+L_*^2)\alpha_n + L_*(1+L_*)\beta_n]\alpha_n\right\}\|x_n - x^*\| \\ &\quad + \frac{1}{1+\alpha_n} L_*[1+(1+L_*)\alpha_n]\alpha_n \|v_n\| + \frac{1}{1+\alpha_n}[1+(1+L_*)\alpha_n]\|u_n\| \\ &\le \left(1 - \frac{1-K_n}{1+\alpha_n}\alpha_n\right)\|x_n - x^*\| + L_*(2+L_*)\alpha_n \|v_n\| + (2+L_*)(\|u_n'\| + \|u_n''\|) \\ &\le \left(1 - \frac{r}{2}\alpha_n\right)\|x_n - x^*\| + (L_* \|v_n\| + \gamma_n)(2+L_*)\alpha_n + (2+L_*)\|u_n'\|. \end{aligned} \tag{9}$$

Set

$$a_n = \|x_n - x^*\|,\ \ t_n = \frac{r}{2}\alpha_n,\ \ b_n = \frac{2}{r}(L_*\|v_n\| + \gamma_n)(2+L_*),\ \text{ and }\ c_n = (2+L_*)\|u'_n\|$$

Then (9) is equivalent to the following inequality:

$$a_{n+1} \le (1-t_n)a_n + b_n t_n + c_n,\ \ \forall n \ge 0.$$

In view of *Lemma 1*, conditions (iii) and (iv), we know that $a_n \to 0$ $(n \to \infty)$, that is, $x_n \to x^*$ $(n \to \infty)$. This completes the proof. □

Remark 1. Theorem 1 *improves and extends the corresponding results of [1] in its four aspects:*

(1) It abolishes the condition that X is uniformly smooth,

(2) The Ishikawa iterative process is replaced by the more general Ishikawa iterative process with mixed errors,

(3) It abolishes the condition that the range $R(S)$ of S is bounded,

(4) Sequences $\{\alpha_n\}$ and $\{\beta_n\}$ need not converge to zero.

Remark 2. Theorem 1 *extends and improves the main results of [2] in the following ways:*

(1) Sequences $\{\alpha_n\}$ and $\{\beta_n\}$ need not converge to zero,

(2) It abolishes the condition that the $\{Sx_n\}$ and $\{Sy_n\}$ are bounded,

(3) The Ishikawa and Mann iterative process with errors is replaced by the more general Ishikawa iterative process with mixed errors.

Remark 3. Theorem 1 *also extends and improves the corresponding results of Chang [3], Chang, Cho and Lee et al [4], Ding [5,6], Hassouni and Moudafi [7], Huang [8-10], Kazmi [12], Noor [15,16], Siddiqi and Ansari [17], Siddiqi, Ansari and Kazmi [18] and Zeng [19].*

In *Theorem 1*, if $\beta_n \equiv 0,\ v_n \equiv 0,\ \forall n \ge 0$, then $y_n = x_n$, hence we have the following result.

Theorem 2. *Let X be an arbitrary real Banach space, $T, A : X \to X$, $g : X \to X^*$ be three mappings, and $\varphi : X^* \to R \cup \{+\infty\}$ a function with a continuous Gâteaux differential $\partial\varphi$. For any given $f \in X$, define a mapping $S : X \to X$ by*

$$Sx = f - (Tx - Ax + \partial\varphi(g(x))) + x.$$

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Mann iterative sequence with mixed errors defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sx_n + u_n,\ \forall n \ge 0, \tag{10}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$, and $\{u_n\}$ is a sequence in X such that $u_n = u'_n + u''_n$ for any sequences $\{u'_n\}$ and $\{u''_n\}$ in X satisfying the following conditions:

(i) $T - A + \partial\varphi \circ g - I : X \to X$ *is accretive,*

(ii) $T - A + \partial\varphi \circ g : X \to X$ *is a Lipschitz operator with the constant* L,

(iii) $\alpha_n \le \frac{1-r}{(1+L_*)(1+L_*^2)}$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \|u'_n\| < \infty$ *and* $\|u''_n\| = \gamma_n \alpha_n$, *where* $L_* = 1 + L$, $r \in (0, 1)$ *is a constant and* $\gamma_n \to 0$ $(n \to \infty)$.

Then the following conclusions hold:

(1) *The nonlinear variational inclusion problem (1) has a unique solution* $x^* \in X$,

(2) *The Mann iterative sequence* $\{x_n\}$ *with mixed errors converges strongly to the unique solution* $x^* \in X$ *of the variational inclusion problem (1).*

If $\varphi \equiv 0$ in *Theorem 1*, we have the following result.

Theorem 3. *Let* X *be an arbitrary real Banach space and let* $T, A : X \to X$, $g : X \to X^*$ *be three mappings. For any given* $f \in X$, *define a mapping* $S : X \to X$ *by*

$$Sx = f - (Tx - Ax) + x.$$

Let $x_0 \in X$ *be any given point and* $\{x_n\}$ *the Ishikawa iterative sequence with mixed errors defined by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n S x_n + v_n, \ \forall n \ge 0, \end{cases} \tag{11}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two real sequences in* $[0, 1]$, *and* $\{u_n\}$, $\{v_n\}$ *are two sequences in* X *such that* $u_n = u'_n + u''_n$ *for any sequences* $\{u'_n\}$ *and* $\{u''_n\}$ *in* X *satisfying the following conditions:*

(i) $T - A - I : X \to X$ *is accretive,*

(ii) $T - A : X \to X$ *is a Lipschitz operator with the constant* L,

(iii) $K_n = (1 + L_*)(1 + L_*^2)\alpha_n + L_*(1 + L_*)\beta_n \le 1 - r$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \|u'_n\| < \infty$, $\|u''_n\| = \gamma_n \alpha_n$ *and* $\|v_n\| \to 0$ $(n \to \infty)$, *where* $L_* = 1 + L$, $r \in (0, 1)$ *is a constant and* $\gamma_n \to 0$ $(n \to \infty)$.

Then the following conclusions hold:

(1) *The variational inequality*

$$\langle Tx - Ax - f, v - g(x) \rangle \ge 0, \ \forall v \in X^* \tag{12}$$

has a unique solution $x^* \in X$,

(2) *The Ishikawa iterative sequence* $\{x_n\}$ *with mixed errors converges strongly to the unique solution* $x^* \in X$ *of the variational inequality (12).*

In *Theorem 3*, if $\beta_n \equiv 0$, $v_n \equiv 0$, $\forall n \geq 0$, then $y_n = x_n$, hence we have the following result.

Theorem 4. *Let X be an arbitrary real Banach space, and let $T, A : X \to X$, $g : X \to X^*$ be three mappings. For any given $f \in X$, define a mapping $S : X \to X$ by*

$$Sx = f - (Tx - Ax) + x.$$

Let $x_0 \in X$ be any given point and $\{x_n\}$ the Mann iterative sequence with mixed errors defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + u_n, \ \forall n \geq 0, \tag{13}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, and $\{u_n\}$ is a sequence in X such that $u_n = u'_n + u''_n$ for any sequences $\{u'_n\}$ and $\{u''_n\}$ in X satisfying the following conditions:

(i) $T - A - I : X \to X$ is accretive,

(ii) $T - A : X \to X$ is a Lipschitz operator with the constant L,

(iii) $\alpha_n \leq \frac{1-r}{(1+L_*)(1+L_*^2)}$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \|u'_n\| < \infty$ and $\|u''_n\| = \gamma_n \alpha_n$, where $L_* = 1 + L$, $r \in (0, 1)$ is a constant and $\gamma_n \to 0$ $(n \to \infty)$.

Then the following conclusions hold:

(1) The variational inequality (12) has a unique solution $x^* \in X$,

(2) The Mann iterative sequence $\{x_n\}$ with mixed errors converges strongly to the unique solution $x^* \in X$ of the variational inequality (12).

Remark 4. *The following example reveals that* Theorem 1 *extends properly Theorem 3.1 of Chang [1] and Theorem 2.1 of Chang [2].*

Example 1. *Let $X, T, A, g, f, S, \varphi$ be as in* Theorem 1 *and*

$$\alpha_n = \frac{1-r}{2(1+L_*)(1+L_*^2)}, \quad \beta_n = \frac{1-r}{2L_*(1+L_*)},$$

$$\|u'_n\| = \frac{1}{(n+1)^2}, \quad \|u''_n\| = \frac{1}{(n+1)} \frac{1-r}{2(1+L_*)(1+L_*^2)}, \quad \|v_n\| = \frac{1}{n+1}$$

for all $n \geq 0$. Then the conditions of Theorem 1 *are satisfied. But Theorem 3.1 in [1] and Theorem 2.1 in [2] are not applicable since $\{\alpha_n\}$ and $\{\beta_n\}$ do not converge to* 0.

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