Interval method for interval linear program

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Abstract. *In the problem of interval linear programming (i)*

$$\begin{array}{c}\max c^T x\\ a \le Ax \le b\end{array}\tag{i}$$

$x, c \in \mathbf{R}^n$; $a, b \in \mathbf{R}^m$; $A \in \mathbf{R}_m^{m\times n}$ (A is of full row rank) we introduce the new variable $t = c_1x_1 + \cdots + c_kx_k + \cdots + c_nx_n$ and eliminate the old variable ($c_k \neq 0$)

$$x_k = \frac{1}{c_k}t - \frac{c_1}{c_k}x_1 - \cdots - \frac{c_{k-1}}{c_k}x_{k-1} - \frac{c_{k+1}}{c_k}x_{k+1} - \cdots - \frac{c_n}{c_k}x_n.$$

So we come to the second form

$$\begin{array}{c}\max t\\ a + \mathrm{th} \le Bx' \le b + \mathrm{th}\end{array}\tag{ii}$$

$x' \in \mathbf{R}^{n-1}$; $a, b, h \in \mathbf{R}^m$; $B \in \mathbf{R}_{m-1}^{m\times(n-1)}$. It is known when $c \in \mathcal{R}(A^T)$ that problem (i) has an explicit solution. In this article we formulate the analogous theorem for the second form (ii), and then show the application of those results on the problem of sensitivity analyses.

Key words: *interval method, interval linear program, first form of interval linear program, second form of interval linear program, explicit solution, sensitivity analyses*

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1. First form of interval linear program (I. L. P.)

Let $a, b \in \mathbf{R}^m$; $c, x \in \mathbf{R}^n$; $A \in \mathbf{R}^{m\times n}$. For the problem of I. L. P.

$$\begin{array}{c}\max c^T x\\ a \le Ax \le b\end{array}\tag{1}$$

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we say that it is in the first form. Denote by K the set

$$K = \{x \in \mathbf{R}^n : a \leq Ax \leq b\}. \tag{2}$$

Two basic results on the exsistence of solution of (1) and on the representation of the solution when $A \in \mathbf{R}_m^{m \times n}$ are as follows (see [1]).

Theorem 1. *Let $K \neq \emptyset$. Then I. L. P. (1) has a bounded optimal solution iff $c \in \mathcal{R}(A^T)$, where by $\mathcal{R}(A)$ we denote the set $\{y \in \mathbf{R}^m : y = Ax$ for some $x \in \mathbf{R}^n\}$.*

Theorem 2. *Let $A \in \mathbf{R}_m^{m \times n}$ and $c \in \mathcal{R}(A^T)$. Accordingly, vector c may be represented in the form*

$$c^T = \sum_{i=1}^{m} \alpha_i A^i, \tag{3}$$

where A^i denotes the i-th row of A. Moreover, let the functions $\beta_i : \mathbf{R} \to \mathbf{R}$ $i = 1, 2, \ldots, m$ be defined by

$$\beta_i(\alpha) = \begin{cases} b_i & \text{if } \alpha \geq 0 \\ a_i & \text{if } \alpha < 0 \end{cases} \tag{4}$$

and let $I = \{i \in \{1, 2, \ldots, m\};\ \alpha_i \neq 0\}$. Then the optimal value of (1) is

$$\max_{x \in K} c^T x = \sum_{i=1}^{m} \alpha_i \beta_i(\alpha_i), \tag{5}$$

and the optimal solution is every vector x which satisfies the next system of equalities and inequalities

$$\begin{aligned} &A^i x = \beta_i(\alpha_i),\ \ i \in I \\ &a_i \leq A^i x \leq b_i,\ i \in \{1, 2, \ldots, m\} \setminus I. \end{aligned} \tag{6}$$

2. Second form of interval linear program

To get the second form from the first form we introduce the new variable

$$t = \sum_{i=1}^{n} c_i x_i. \tag{7}$$

Let for some k is $c_k \neq 0$ (otherwise problem (1) is trivial), then from (7) we express x_k

$$x_k = \frac{1}{c_k} t - \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{c_i}{c_k} x_i \tag{8}$$

and insert it in the system of inequalities (1). Thus, after arranging we come to the second form of I. L. P.

$$\begin{gathered} \max t \\ a + \mathrm{th} \leq Bx' \leq b + \mathrm{th}, \end{gathered} \tag{9}$$

where it is denoted by

$$x' = [x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n]^T$$
$$h = \left[-\frac{a_{1k}}{c_k}, -\frac{a_{2k}}{c_k}, \dots, -\frac{a_{mk}}{c_k}\right]^T \tag{10}$$
$$B = \begin{bmatrix} a_{11} - \frac{a_{1k}c_1}{c_k} & \dots & a_{1k-1} - \frac{a_{1k}c_{k-1}}{c_k} & a_{1k+1} - \frac{a_{1k}c_{k+1}}{c_k} & \dots & a_{1n} - \frac{a_{1k}c_n}{c_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} - \frac{a_{mk}c_1}{c_k} & \dots & a_{mk-1} - \frac{a_{mk}c_{k-1}}{c_k} & a_{mk+1} - \frac{a_{mk}c_{k+1}}{c_k} & \dots & a_{mn} - \frac{a_{mk}c_n}{c_k} \end{bmatrix}$$

It is important to note that relation (8) has a double role. First, it has a role of transition from the first to the second form. Second, when problem (9) is solved by means of (8) we calculate the value of x_k.

Now we want to formulate and prove theorem which is analogous to *Theorem 2*, but corresponding to the second form of I. L. P. With that goal in mind we firstly formulate and prove two auxiliary lemmas (about rank and other connections between matrices A and B).

Lemma 1.

$$r \begin{bmatrix} c^T \\ \vdots \\ A \end{bmatrix} = r(B) + 1. \tag{11}$$

Proof. Multiply the first row of the block-matrix from the left-hand side of (11) by $-a_{ik}(c_k)^{-1}$ and then add to the $(i+1)$th row for $i = 1, 2, \dots, m$. After that multiply the kth column by $-c_i(c_k)^{-1}$ and then add to the ith column for $i = 1, 2, \dots, k-1, k+1, \dots, n$. As a result of these calculations we get

$$D = \begin{bmatrix} 0 & \dots & 0 & c_k & 0 & \dots & 0 \\ a_{11} - \frac{a_{1k}c_1}{c_k} & \dots & a_{1k-1} - \frac{a_{1k}c_{k-1}}{c_k} & 0 & a_{1k+1} - \frac{a_{1k}c_{k+1}}{c_k} & \dots & a_{1n} - \frac{a_{1k}c_n}{c_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} - \frac{a_{mk}c_1}{c_k} & \dots & a_{mk-1} - \frac{a_{mk}c_{k-1}}{c_k} & 0 & a_{mk+1} - \frac{a_{mk}c_{k+1}}{c_k} & \dots & a_{mn} - \frac{a_{mk}c_n}{c_k} \end{bmatrix}. \tag{12}$$

From (12) it is obvious that

$$r \begin{bmatrix} c^T \\ \vdots \\ A \end{bmatrix} = r(D) = r(B) + 1.$$

□

Lemma 2. *Let $A \in \mathbf{R}_m^{m \times n}$ and $c \in \mathcal{R}(A^T)$. Whereupon*

$$c^T = \delta_1 A^1 + \delta_2 A^2 + \dots + \delta_m A^m, \tag{13}$$

where A^i denotes i-th row of A for $i = 1, 2, \dots, m$. Then

$$\delta_1 B^1 + \delta_2 B^2 + \dots + \delta_m B^m = 0, \tag{14}$$

where B^i denotes the i-th row of B for $i = 1, 2, \dots, m$.

Proof. Multiply the first row of B by δ_1, the second by δ_2 and so on; after that sum up all rows. Then we get

$$\sum_{j=1}^{m}\delta_j\left(a_{ji}-\frac{c_i}{c_k}a_{jk}\right)=\sum_{j=1}^{m}\delta_j a_{ji}-\frac{c_i}{c_k}\sum_{j=1}^{m}\delta_j a_{jk}=c_i-\frac{c_i}{c_k}c_k=0, \tag{15}$$

for $i=1,2,\dots,k-1,k+1,\dots,n$. □

The next corollary will show that the converse is also true.

Corollary 1. *Let conditions of* Lemma 2 *be fulfilled and let*

$$\varepsilon_1 B^1+\varepsilon_2 B^2+\cdots+\varepsilon_m B^m=0, \tag{16}$$

where at least one ε_i *is different from zero. Then* $\varepsilon_i=\lambda\delta_i$ $(\lambda\neq 0)$ *for* $i=1,2,\dots,m$.

Proof. From (16) it follows

$$\sum_{j=1}^{m}\varepsilon_j\left(a_{ji}-\frac{c_i}{c_k}a_{jk}\right)=\sum_{j=1}^{m}\varepsilon_j a_{ji}-\frac{c_i}{c_k}\sum_{j=1}^{m}\varepsilon_j a_{jk}=0, \tag{17}$$

for $i=1,2,\dots,k-1,k+1,\dots,n$. Now we suppose that

$$\lambda=\frac{1}{c_k}\sum_{j=1}^{m}\varepsilon_j a_{jk}=0. \tag{18}$$

Then from (17) it follows immediately

$$\sum_{j=1}^{m}\varepsilon_j a_{ji}=0,\quad i=1,2,\dots,n, \tag{19}$$

respectively

$$\sum_{j=1}^{m}\varepsilon_j A^j=0. \tag{20}$$

From the supposition of corollary $(r(A)=m)$ we conclude that $\varepsilon_1=\varepsilon_2=\ldots=\varepsilon_m=0$ which is the contradiction, hence $\lambda\neq 0$. From (17) we get

$$\sum_{j=1}^{m}\varepsilon_j a_{ji}=\lambda c_i\ (\lambda\neq 0)\ i=1,2,\dots,n, \tag{21}$$

respectively

$$\lambda c^T=\varepsilon_1 A^1+\varepsilon_2 A^2+\cdots+\varepsilon_m A^m. \tag{22}$$

From the facts $c\in\mathcal{R}(A^T)$ and (13) it follows $\varepsilon_i=\lambda\delta_i$ for $i=1,2,\dots,m$. □

Remark 1. *From conditions of* Corollary 1 *and from* Lemma 1 *it follows*

$$m=r(A)=r\begin{bmatrix}c^T\\ \vdots\\ A\end{bmatrix}=r(B)+1, \tag{23}$$

hence $r(B) = m - 1$.

Corollary 2. *Let the conditions of* Lemma 2 *be fulfilled and let*

$$\delta = [\delta_1, \delta_2, \dots, \delta_m]^T. \tag{24}$$

Moreover, let exactly r *components of* δ *be different from zero. Then, there exist exactly* r *submatrices from* B *of type* $(m-1)\times(n-1)$ *which are of full rank (their rank is* $m-1$*), and which we get from* B *by throwing out the row* B^{k_j} *for which* $\delta_{k_j} \neq 0$ $(j = 1, 2, \dots, r)$.

Proof. Immediately from *Corollary 1.* □

Before we formulate and prove the main theorem of this article let us denote by $\eta_i : \mathbf{R} \to \mathbf{R}$ for $i = 1, 2, \dots, m$ the functions

$$\eta_i(\alpha) = \begin{cases} a_i & \text{if } \alpha \geq 0 \\ b_i & \text{if } \alpha < 0. \end{cases} \tag{25}$$

Theorem 3. *Let vectors (the rows of* B*)* $B^2, B^3, \dots, B^m$ *be linearly independent and let*

$$B^1 = d_2B^2 + d_3B^3 + \dots + d_mB^m. \tag{26}$$

Then we distinguish two cases

(i)
$$-\sum_{i=2}^{m} d_ih_i + h_1 = 0 \quad \left(h_i = -\frac{a_{ik}}{c_k} \text{ for } i = 1, 2, \dots, m\right).$$

Then $c \notin \mathcal{R}(A^T)$ *and problem (1) has no bounded solution (*Theorem 1*).*

(ii)
$$-\sum_{i=2}^{m} d_ih_i + h_1 \neq 0.$$

Then $r(A) = m$, $c \in \mathcal{R}(A^T)$ *and problem (1) has a bounded solution*

$$t_0 = \max_{x\in K} c^T x = \begin{cases} \dfrac{\sum_{i=2}^{m} d_i\eta_i(d_i) - b_1}{-\sum_{i=2}^{m} d_ih_i + h_1} & \text{if } -\sum_{i=2}^{m} d_ih_i + h_1 < 0 \\[2ex] \dfrac{\sum_{i=2}^{m} d_i\beta_i(d_i) - a_1}{-\sum_{i=2}^{m} d_ih_i + h_1} & \text{if } -\sum_{i=2}^{m} d_ih_i + h_1 > 0, \end{cases} \tag{27}$$

where the optimal solution is every vector x *which satisfies the next system of equalities and inequalities*

$$B^ix' = \begin{cases} \eta_i(d_i) + t_0h_i & \text{if } -\sum_{i=2}^{m} d_ih_i + h_1 < 0 \\[1ex] \beta_i(d_i) + t_0h_i & \text{if } -\sum_{i=2}^{m} d_ih_i + h_1 > 0 \end{cases} ; \ i \in I \tag{28}$$

$$a_i + t_0h_i \leq B^ix' \leq b_i + t_0h_i; \quad i \in \{2, 3, \dots, m\} \setminus I \tag{29}$$

$$x_k = \frac{t_0}{c_k} - \frac{c_1}{c_k}x_1 - \dots - \frac{c_{k-1}}{c_k}x_{k-1} - \frac{c_{k+1}}{c_k}x_{k+1} - \dots - \frac{c_n}{c_k}x_n, \tag{30}$$

where I *denotes the set* $\{i \in \{2, 3, \dots, m\} : \ d_i \neq 0\}$.

Proof. Let vectors $B^2, B^3, \dots, B^m$ be linearly independent and let

$$B^1 = d_2B^2 + d_3B^3 + \cdots + d_mB^m. \tag{31}$$

Now we prove statement (i). Suppose that

$$0 = -\sum_{i=2}^{m} d_ih_i + h_1 = \frac{1}{c_k}\sum_{i=2}^{m} d_ia_{ik} - \frac{a_{1k}}{c_k} \tag{32}$$

and that $r(A) = m$. From relation (31) it follows $r(B) = m-1$ and from *Lemma 1* it follows $c \in \mathcal{R}(A^T)$. From *Lemma 2* and *Corollary 1* we conclude that $d_i = -\frac{\delta_i}{\delta_1}$ for $i = 2, 3, \dots, m$. We insert the expression for d_i in relation (32) and get

$$-\sum_{i=2}^{m} \frac{\delta_i a_{ik}}{c_k\delta_1} - \frac{a_{1k}}{c_k} = -\frac{1}{\delta_1}\sum_{i=1}^{m}\frac{\delta_i a_{ik}}{c_k} = -\frac{1}{\delta_1} = 0, \tag{33}$$

which is a contradiction. Hence $r(A) = m-1$, and from *Lemma 1* it follows

$$r\begin{bmatrix} c^T \\ \vdots \\ A \end{bmatrix} = r(B) + 1 = m, \tag{34}$$

whereupon we conclude that $c \notin \mathcal{R}(A^T)$. It follows from *Theorem 1* that problem (1) has no bounded solution.

Now we prove statement (ii). Let

$$-\sum_{i=2}^{m} d_ih_i + h_1 \neq 0. \tag{35}$$

It will be shown that problem (9) has a bounded solution (maximum), hence problem (1) has too, and from *Theorem 1* it follows that $c \in \mathcal{R}(A^T)$ respectively $r(A) = m$. At the end the equivalence between formulas (27) and (5) will be proved. With that goal in mind we write a system of inequalities in problem (9)

$$\begin{gathered} a_1 + th_1 \le B^1x' \le b_1 + th_1 \\ a_2 + th_2 \le B^2x' \le b_2 + th_2 \\ \cdots \\ a_m + th_m \le B^mx' \le b_m + th_m, \end{gathered} \tag{36}$$

where $h_i = -\frac{a_{ik}}{c_k}$ for $i = 1, 2, \dots, m$. From (31) and (36) we get

$$a_1 + th_1 \le B^1x' \le b_1 + th_1$$

$$\sum_{i=2}^{m} d_i\eta_i(d_i) + t\sum_{i=2}^{m} d_ih_i \le B^1x' \le \sum_{i=2}^{m} d_i\beta_i(d_i) + t\sum_{i=2}^{m} d_ih_i. \tag{37}$$

We introduce the following abbreviations

$$p_1 = \sum_{i=2}^{m} d_i\eta_i(d_i), \quad q_1 = \sum_{i=2}^{m} d_i\beta_i(d_i), \quad r_1 = \sum_{i=2}^{m} d_ih_i. \tag{38}$$

With the help of these abbreviations we form intervals[1]

$$\begin{aligned} A_t &= [a_1 + th_1,\ b_1 + th_1] \\ B_t &= [p_1 + tr_1,\ q_1 + tr_1]. \end{aligned} \tag{39}$$

By observing inequalities (37) we conclude that the necessary condition for the existence of the bounded solution of problem (9) is the existence of $t \in \mathbf{R}$ for which

$$A_t \cap B_t \neq \emptyset. \tag{40}$$

A sufficient condition for the existence of the bounded solution of problem (9) is the boundedness (from above) of the set

$$\{t \in \mathbf{R} : A_t \cap B_t \neq \emptyset\}. \tag{41}$$

It is easy to see that set (41) is bounded (from above) iff there exist the solutions of equations

$$\begin{aligned} a_1 + th_1 &= q_1 + tr_1 \\ b_1 + th_1 &= p_1 + tr_1. \end{aligned} \tag{42}$$

Equations (42) have a solution iff condition (35) is fulfilled. In that case the solution of equations (42) are

$$t_1 = \frac{q_1 - a_1}{-r_1 + h_1}; \quad t_2 = \frac{p_1 - b_1}{-r_1 + h_1}. \tag{43}$$

It remains to prove the equivalence between formulas (27) and (5). First we suppose that $-r_1 + h_1 < 0$, then by means of relations (33), (38) and (43) we get

$$\begin{aligned} t_2 &= \frac{p_1 - b_1}{-r_1 + h_1} = \delta_1 \left[b_1 - \sum_{i=2}^{m} d_i \eta_i(d_i) \right] = \delta_1 b_1 + \sum_{i=2}^{m} \delta_i \eta_i \left(\frac{-\delta_i}{\delta_1} \right) \\ &= \delta_1 b_1 + \sum_{i=2}^{m} \delta_i \beta_i(\delta_i) = \sum_{i=1}^{m} \delta_i \beta_i(\delta_i) = \max_{x \in K} c^T x. \end{aligned}$$

Fourth and fifth equalities are the consequence of the fact that in this case $\delta_1 > 0$.

Now we suppose that $-r_1 + h_1 > 0$ and we get

$$\begin{aligned} t_1 &= \frac{q_1 - a_1}{-r_1 + h_1} = \delta_1 \left[a_1 - \sum_{i=2}^{m} d_i \beta_i(d_i) \right] = \delta_1 a_1 + \sum_{i=2}^{m} \delta_i \beta_i \left(\frac{-\delta_i}{\delta_1} \right) \\ &= \delta_1 a_1 + \sum_{i=2}^{m} \delta_i \beta_i(\delta_i) = \sum_{i=1}^{m} \delta_i \beta_i(\delta_i) = \max_{x \in K} c^T x. \end{aligned}$$

Fourth and fifth equalities are the consequence of the fact that in this case $\delta_1 < 0$.

The equivalence between relations (28), (29) and (30) and relation (6) follows immediately from the linear independence of the vectors $B^2, B^3, \ldots, B^m$. □

[1] That is the main reason for the name of the method (interval method).

Remark 2. *From the proof of* Theorem 3 *and from* Corollary 2 *we conclude that on the left-hand side of relation (31) there can be any vector* B^j *if* $\delta_j \neq 0$.

Except this, it is easily seen that $\beta_i(-d_i)$ *can stay instead of* $\eta_i(d_i)$ *in formula (27).*

Example 1. *The next problem will be solved first by the application of* Theorem 2 *and then by the application of* Theorem 3

$$\begin{array}{c}\max(\ -y+4z)\\ -3\le 2x-y+3z\le 4\\ -2\le -x+2y-3z\le 5\\ -4\le 3x+y-z\le 2.\end{array}\tag{44}$$

Since

$$\begin{bmatrix}0\\-1\\4\end{bmatrix}=2\begin{bmatrix}2\\-1\\3\end{bmatrix}+\begin{bmatrix}-1\\2\\-3\end{bmatrix}-\begin{bmatrix}3\\1\\-1\end{bmatrix}\tag{45}$$

respectively $c^T = 2A^1 + A^2 - A^3$ *and from* Theorem 2 *we get*

$$\max = 2\cdot 4 + 1\cdot 5 + (-1)(-4) = 17.$$

If we want to use Theorem 3 *we need to transform (44) on the second form of I. L. P. with the help of the substitution* $t = -y + 4z$ *and elimination of variable* $y = 4z - t$*; in such a manner we get*

$$\begin{array}{c}\max t\\ -3-t\le 2x-z\le 4-t\\ -2+2t\le -x+5z\le 5+2t\\ -4+t\le 3x+3z\le 2+t.\end{array}\tag{46}$$

From (45), (46) and from Lemma 2 *it follows*

$$2\begin{bmatrix}2\\-1\end{bmatrix}+\begin{bmatrix}-1\\5\end{bmatrix}-\begin{bmatrix}3\\3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.\tag{47}$$

From (47) we get

$$B^2 = -2B^1 + B^3.\tag{48}$$

From (48) and (27), since it is $-(-2)(-1)-(1)(1)+2=-1<0$, *we get*

$$\max = \frac{(-2)(4)+1(-4)-5}{-(-2)(-1)-(1)(1)+2} = \frac{-17}{-1} = 17.\tag{49}$$

It will be shown that we can do the sensitivity analysis of the problem rather easily

$$\begin{array}{c}\max c^T x\\ a\le Ax\le b,\end{array}\tag{50}$$

with the help of formula (27) where A *is a regular matrix. As an example we take problem (44) but when* $a_{21} = -1 + s$. *It is easy to see that the second form of I. L. P. seems to be the same as (46) but only* $b_{21} = -1 + s$.

By means of (46) we can form the following tableau

$$\begin{array}{c|ccc|} & B^1 & B^2 & B^3 \\ \cline{2-4} E^1 & 2 & -1+s & 3 \\ E^2 & -1 & 5 & 3 \\ \cline{2-4} \end{array}\,, \tag{51}$$

with the purpose that by forming a new base one B^k is expressed as a linear combination of the remaining two, but with minimal problems about parameter s. That is best to do in the following way:

$$\begin{array}{c|ccc|cc|c|} & B^1 & B^2 & B^3 & B^2 & B^3 & B^2 \\ \cline{2-7} B^3\ \not{E}^1 & 2 & -1+s & 3 & 9+s & \boxed{9} & 1+\frac{s}{9} \\ B^1\ \not{E}^2 & \boxed{-1} & 5 & 3 & -5 & -3 & -2+\frac{s}{3} \\ \cline{2-7} \end{array}\,. \tag{52}$$

To be able to apply formula (27), we must first calculate the denominator

$$-(-1)\left(-2+\frac{s}{2}\right)-\left(1+\frac{s}{9}\right)+2=-1+\frac{2s}{9}.$$

With the help of this we get

$$\max(s)=\frac{\left(-2+\frac{s}{3}\right)\eta_1\left(-2+\frac{s}{3}\right)+\left(1+\frac{s}{9}\right)\eta_3\left(1+\frac{s}{9}\right)-5}{-1+\frac{2s}{9}},\quad \textit{if}\ \ s<\frac{9}{2}$$

$$\max(s)=\frac{\left(-2+\frac{s}{3}\right)\beta_1\left(-2+\frac{s}{3}\right)+\left(1+\frac{s}{9}\right)\beta_3\left(1+\frac{s}{9}\right)+2}{-1+\frac{2s}{9}},\quad \textit{if}\ \ s>\frac{9}{2}.$$

If we want to get the final form of the function $\max(s)$, *we need to examine a behavior (with respect to the sign) of expressions $-2+\frac{s}{3}$ and $1+\frac{s}{9}$. The zeros of these expressions are $s=6$ and $s=-9$, so we get*

$$\max(s)=\begin{cases} \dfrac{\left(-2+\frac{s}{3}\right)\cdot 4+\left(1+\frac{s}{9}\right)\cdot 2-5}{-1+\frac{2s}{9}}, & \textit{if}\ s\le -9 \\ \dfrac{\left(-2+\frac{s}{3}\right)\cdot 4+\left(1+\frac{s}{9}\right)\cdot(-4)-5}{-1+\frac{2s}{9}}, & \textit{if}\ -9<s<\frac{9}{2} \\ \dfrac{\left(-2+\frac{s}{3}\right)\cdot(-3)+\left(1+\frac{s}{9}\right)\cdot 2+2}{-1+\frac{2s}{9}}, & \textit{if}\ \frac{9}{2}<s\le 6 \\ \dfrac{\left(-2+\frac{s}{3}\right)\cdot 4+\left(1+\frac{s}{9}\right)\cdot 2+2}{-1+\frac{2s}{9}}, & \textit{if}\ 6<s \end{cases}$$

for $s=4.5$ the problem has no bounded solution. Finally, this function can be represented in the following way

$$\max(s)=\begin{cases} \dfrac{-11+\frac{14s}{9}}{-1+\frac{2s}{9}}, & \textit{if}\ s\le -9 \\ \dfrac{-17+\frac{8s}{9}}{-1+\frac{2s}{9}}, & \textit{if}\ -9<s<\frac{9}{2} \\ \dfrac{10-\frac{7s}{9}}{-1+\frac{2s}{9}}, & \textit{if}\ \frac{9}{2}<s\le 6 \\ \dfrac{-4+\frac{14s}{9}}{-1+\frac{2s}{9}}, & \textit{if}\ 6<s. \end{cases} \tag{53}$$

From (53) we conclude that the function $\max(s)$ *has a vertical asymptote for* $s = \frac{9}{2}$ *and a horizontal asymptote for* $\max = 7$*. Also we see that it consists of four linear fractional transformations of the form* $\frac{a+bs}{c+ds}$*. The graph of this function is presented in the next figure.*

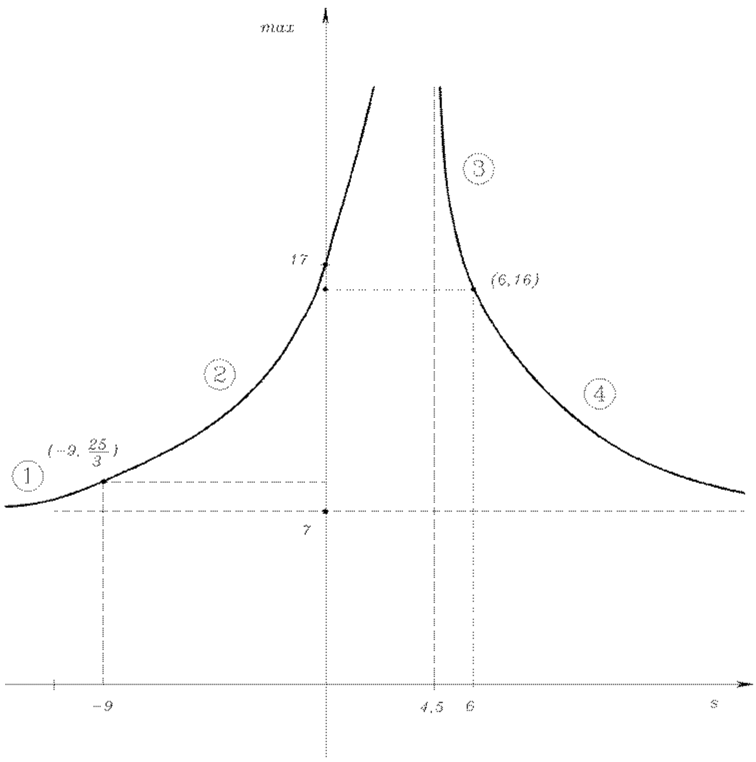

Figure 1.

General remark. The second form of I. L. P. and formula (27) can generalize the suboptimization method for interval linear programming (SUBOPT) see [4] and [5].

References

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