Weakly Picard pairs of some multivalued operators

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Abstract. *The purpose of this paper is to present a partial answer to the following problem:*

Let (X,d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. Determine the metric conditions which imply that (T_1, T_2) is a weakly Picard pair of multivalued operators and T_1, T_2 are weakly Picard multivalued operators.

Key words: *fixed point, common fixed point, weakly Picard multivalued operator, weakly Picard pair of multivalued operators*

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1. Introduction

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X, i. e. $P(X) := \{ Y \mid \emptyset \neq Y \subseteq X \}$.

Let $T_1, T_2 : X \to P(X)$ be two multivalued operators. We denote by G_{T_1} the graph of T_1, i. e. $G_{T_1} := \{ (x,y) \mid x \in X,\ y \in T_1(x) \}$, by F_{T_1} the fixed points set of T_1, i. e. $F_{T_1} := \{ x \in X \mid x \in T_1(x) \}$ and by $(CF)_{T_1,T_2}$ the common fixed points set of T_1 and T_2.

Let (X,d) be a metric space. Further on we shall need the following notation

$$P_{cl}(X) := \{ Y \mid Y \in P(X) \text{ and } Y \text{ is a closed set } \}$$

and the following functionals

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf \{ d(a,b) \mid a \in A,\ b \in B \},$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \max \left\{ \sup_{a\in A} D(a,B),\ \sup_{b\in B} D(b,A) \right\}.$$

Definition 1 [**[6]**, **[7]**]. *Let (X,d) be a metric space and $T : X \to P(X)$ a multivalued operator. We say that T is a* weakly Picard multivalued operator (*briefly* w. P. m. o.) *iff for each $x \in X$ and for every $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:*

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(*i*) $x_0 = x$, $x_1 = y$;

(*ii*) $x_{n+1} \in T(x_n)$, *for each* $n \in \mathbb{N}^*$;

(*iii*) *sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a fixed point of* T.

Remark 1. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *which satisfies conditions* (*i*) *and* (*ii*) *from* Definition 1 *is, by definition, a* sequence of successive approximations of T, starting from (x, y).

For examples of w. P. m. o. see for instance [6], [7].

Definition 2 [[7]]. *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *a w. P. m. o.. Then we define the multivalued operator* $T^\infty : G_T \to P(F_T)$ *by the formula* $T^\infty(x, y) = \{\ z \in F_T \mid$ *there exists a sequence of successive approximations of* T, *starting from* (x, y), *that converges to* $z\ \}$, *for each* $(x, y) \in G_T$.

Definition 3 [[7]]. *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *a w. P. m. o.. Then* T *is a* c-weakly Picard multivalued operator ($c \in [0, +\infty[$) (*briefly* c-w. P. m. o.) *iff there exists a selection* t^∞ *of* T^∞ *such that*

$$d(x, t^\infty(x, y)) \le c\ d(x, y),$$

for each $(x, y) \in G_T$.

Examples of c-w. P. m. o. are given in [7].

Definition 4 [[10]]. *Let* (X, d) *be a metric space and* $T_1, T_2 : X \to P(X)$ *two multivalued operators. By definition, we say that the pair of multivalued operators* (T_1, T_2) *is a* weakly Picard pair of multivalued operators (*briefly* w. P. p. m. o.) *iff for each* $x \in X$ *and for every* $y \in T_1(x) \cup T_2(x)$, *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *such that*:

(*i*) $x_0 = x$, $x_1 = y$;

(*ii*) $x_{2n-1} \in T_i(x_{2n-2})$ *and* $x_{2n} \in T_j(x_{2n-1})$, *for each* $n \in \mathbb{N}^*$, *where* $i, j \in \{1, 2\}$, $i \ne j$;

(*iii*) *sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a common fixed point of* T_1 *and* T_2.

Remark 2. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *which satisfies conditions* (*i*) *and* (*ii*) *from* Definition 4 *is a sequence of successive approximations for the pair* (T_1, T_2), *starting from* (x, y).

For examples of w. P. p. m. o. see [10].

Definition 5 [[10]]. *Let* (X, d) *be a metric space and* $T_1, T_2 : X \to P(X)$ *two multivalued operators which form a w. P. p. m. o.. Then we define the multivalued operator* $(T_1, T_2)^\infty : G_{T_1} \cup G_{T_2} \to P((CF)_{T_1,T_2})$ *by the formula* $(T_1, T_2)^\infty(x, y) = \{\ z \in (CF)_{T_1,T_2} \mid$ *there exists a sequence of successive approximations for the pair* (T_1, T_2), *starting from* (x, y), *that converges to* $z\ \}$, *for each* $(x, y) \in G_{T_1} \cup G_{T_2}$.

Definition 6 [[10]]. *Let* (X, d) *be a metric space and* $T_1, T_2 : X \to P(X)$ *two multivalued operators which form a w. P. p. m. o.. Then* (T_1, T_2) *is a* c-weakly Picard pair of multivalued operators ($c \in [0, +\infty[$) (*briefly* c-w. P. p. m. o.) *iff there exists a selection* $(t_1, t_2)^\infty$ *of* $(T_1, T_2)^\infty$ *such that*

$$d(x, (t_1, t_2)^\infty(x, y)) \le c\ d(x, y),$$

for each $(x,y) \in G_{T_1} \cup G_{T_2}$.

Examples of c-w. P. p. m. o. are given in [10].

The purpose of this paper is to study the following problem.

Problem 1. *Let* (X,d) *be a metric space and* $T_1, T_2 : X \to P(X)$ *two multivalued operators. Determine the metric conditions which imply that* (T_1,T_2) *is a weakly Picard pair of multivalued operators and* T_1, T_2 *are weakly Picard multivalued operators.*

2. Weakly Picard pairs of some multivalued operators

The following theorem was established by Sîntămărian in [10] and it is a partial answer to *Problem 1.*

Theorem 1 [[10]]. *Let* (X,d) *be a complete metric space and* $T_1, T_2 : X \to P_{cl}(X)$ *two multivalued operators for which there exists* $a \in [0,1/2[$ *such that*

$$H(T_1(x),T_2(y)) \leq a\ [D(x,T_1(x)) + D(y,T_2(y))],$$

for each $x, y \in X$.

Then $F_{T_1} = F_{T_2} \in P_{cl}(X)$, (T_1,T_2) *is c-w. P. p. m. o. and* T_1 *and* T_2 *are c-w. P. m. o., with* $c = (1-a)/(1-2a)$.

Another partial answer to *Problem 1* is the following result.

Theorem 2. *Let* (X,d) *be a complete metric space and* $T_1, T_2 : X \to P_{cl}(X)$ *two multivalued operators. We suppose that*:

(i) *there exists* $a_1 \in [0,1/2[$ *such that for each* $x \in X$, *any* $u_x \in T_1(x)$ *and for all* $y \in X$, *there exists* $u_y \in T_2(y)$ *so that*

$$d(u_x,u_y) \leq a_1\ [d(x,u_x) + d(y,u_y)];$$

(ii) *there exists* $a_2 \in [0,1/2[$ *such that for each* $x \in X$, *any* $u_x \in T_2(x)$ *and for all* $y \in X$, *there exists* $u_y \in T_1(y)$ *so that*

$$d(u_x,u_y) \leq a_2\ [d(x,u_x) + d(y,u_y)].$$

Then $F_{T_1} = F_{T_2} \in P_{cl}(X)$ *and* (T_1,T_2) *is c-w. P. p. m. o., with* $c = (1-a)/(1-2a)$, *where* $a = \max\{a_1,a_2\}$.

If in addition we have that $2\max\{a_1,a_2\} + \min\{a_1,a_2\} < 1$, *then* T_i *is* c_i*-w. P. m. o., with* $c_i = (1-a_1)(1-a_2)/(1-2a_i-a_j)$, $i,j \in \{1,2\}$, $i \neq j$.

Proof. First of all, we notice that from Theorem 4.2 given by Latif-Beg in [1] it follows that $(CF)_{T_1,T_2} \neq \emptyset$.

From Theorem 2.2 given by Sîntămărian in [8] we have that $F_{T_1} = F_{T_2} \in P_{cl}(X)$ and the fact that (T_1,T_2) is c-w. P. p. m. o. follows from Theorem 2.7 given by Sîntămărian in [10].

Furthermore, we suppose that $2\max\{a_1,a_2\} + \min\{a_1,a_2\} < 1$ and we shall prove that T_i is c_i-w. P. m. o., $i \in \{1,2\}$.

Let $i,j \in \{1,2\}$, $i \neq j$. Let $x_0 \in X$ and $x_1 \in T_i(x_0)$. It follows that there exists $y_1 \in T_j(x_1)$ such that

$$d(x_1,y_1) \leq a_i\ [d(x_0,x_1) + d(x_1,y_1)]$$

and there exists $x_2 \in T_i(x_1)$ such that

$$d(y_1, x_2) \leq a_j \; [d(x_1, y_1) + d(x_1, x_2)].$$

From these, using the triangle inequality, we obtain

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, y_1) + d(y_1, x_2) \\ &\leq d(x_1, y_1) + a_j \; [d(x_1, y_1) + d(x_1, x_2)] \\ &= (1 + a_j) \; d(x_1, y_1) + a_j \; d(x_1, x_2) \\ &\leq (1 + a_j) a_i/(1 - a_i) \; d(x_0, x_1) + a_j \; d(x_1, x_2). \end{aligned}$$

So

$$d(x_1, x_2) \leq a_i(1 + a_j)/[(1 - a_i)(1 - a_j)] \; d(x_0, x_1).$$

Now, there exists $y_2 \in T_j(x_2)$ such that

$$d(x_2, y_2) \leq a_i \; [d(x_1, x_2) + d(x_2, y_2)]$$

and there exists $x_3 \in T_i(x_2)$ such that

$$d(y_2, x_3) \leq a_j \; [d(x_2, y_2) + d(x_2, x_3)].$$

From these we have that

$$d(x_2, x_3) \leq a_i(1 + a_j)/[(1 - a_i)(1 - a_j)] \; d(x_1, x_2).$$

By induction, we obtain that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations of T_i, starting from (x_0, x_1), with the property that

$$d(x_n, x_{n+1}) \leq a_i(1 + a_j)/[(1 - a_i)(1 - a_j)] \; d(x_{n-1}, x_n),$$

for each $n \in \mathbb{N}^*$.

It follows that $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence, because (X, d) is a complete metric space and $a_i(1 + a_j)/[(1 - a_i)(1 - a_j)] < 1$. Let $x^* = \lim_{n \to \infty} x_n$.

From $x_n \in T_i(x_{n-1})$ we have that there exists $u_n \in T_j(x^*)$ such that

$$d(x_n, u_n) \leq a_i \; [d(x_{n-1}, x_n) + d(x^*, u_n)],$$

for all $n \in \mathbb{N}^*$.

Using the triangle inequality we obtain

$$d(x^*, u_n) \leq (1 - a_i)^{-1} \; [d(x^*, x_n) + a_i \; d(x_{n-1}, x_n)],$$

for all $n \in \mathbb{N}^*$.

This implies that $d(x^*, u_n) \to 0$, as $n \to \infty$. Since $u_n \in T_j(x^*)$, for all $n \in \mathbb{N}^*$ and $T_j(x^*)$ is a closed set, it follows that $x^* \in T_j(x^*)$. So $x^* \in F_{T_j} = F_{T_i}$.

It is not difficult to verify that

$$d(x_n, x^*) \leq [a_i(1+a_j)(1-a_i)^{-1}(1-a_j)^{-1}]^n (1-a_i)(1-a_j)/(1-2a_i-a_j) \; d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

For $n = 0$ we have

$$d(x_0, x^*) \leq (1 - a_i)(1 - a_j)/(1 - 2a_i - a_j) \; d(x_0, x_1),$$

which means that T_i is c_i-w. P. m. o., with $c_i = (1 - a_i)(1 - a_j)/(1 - 2a_i - a_j)$. □

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