Metrical relations in barycentric coordinates

VLADIMIR VOLENEC*

Abstract. *Let Δ be the area of the fundamental triangle ABC of barycentric coordinates and let $\alpha = \cot A$, $\beta = \cot B$, $\gamma = \cot C$. The vectors $\boldsymbol{v}_i = [x_i, y_i, z_i]$ $(i = 1, 2)$ have the scalar product $2\Delta(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)$. This fact implies all important formulas about metrical relations of points and lines. The main and probably new results are Theorems 1 and 8.*

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Let ABC be a given triangle with the sidelengths $a = |BC|,\ b = |CA|,\ c = |AB|$, the measures A, B, C of the opposite angles and the area Δ. For any point P let $\boldsymbol{P}$ be the radiusvector of this point with respect to any origin. Then we have $\overrightarrow{PQ} = \boldsymbol{Q} - \boldsymbol{P}$. There are uniquely determined numbers $y, z \in \mathbb{R}$ so that $\overrightarrow{AP} = y \cdot \overrightarrow{AB} + z \cdot \overrightarrow{AC}$, i.e. $\boldsymbol{P} - \boldsymbol{A} = y(\boldsymbol{B} - \boldsymbol{A}) + z(\boldsymbol{C} - \boldsymbol{A})$. If we put $x = 1 - y - z$, i.e.

$$x + y + z = 1, \tag{1}$$

then we have

$$\boldsymbol{P} = x\boldsymbol{A} + y\boldsymbol{B} + z\boldsymbol{C}. \tag{2}$$

Numbers x, y, z, such that (1) and (2) are valid, are uniquely determined by point P and triangle ABC, i.e. these numbers do not depend on the choice of the origin. We say that x, y, z, are the absolute barycentric coordinates of point P with respect to triangle ABC and write $P = (x, y, z)$. Obviously $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. Actually, point P is the barycenter of the mass point system of points A, B, C with masses x, y, z, respectively. The centroid of triangle ABC is point $G = (1/3, 1/3, 1/3)$.

Any three numbers x', y', z' proportional to the coordinates x, y, z are said to be relative barycentric coordinates of point P with respect to triangle ABC and we write $P = (x' : y' : z')$. Here we have $x' + y' + z' \neq 0$. Point P is uniquely

*Department of Mathematics, University of Zagreb, Bijenička 30, HR-10 000 Zagreb, Croatia, e-mail: `volenec@math.hr`

determined by its relative barycentric coordinates $x^{'}$, $y^{'}$, $z^{'}$ because its absolute barycentric coordinates are

$$x = \frac{x^{'}}{x^{'} + y^{'} + z^{'}}, \;\; y = \frac{y^{'}}{x^{'} + y^{'} + z^{'}}, \;\; z = \frac{z^{'}}{x^{'} + y^{'} + z^{'}}.$$

If point P divides two different points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the ratio $(P_1P_2P) = \lambda$, i.e. if $\overrightarrow{P_1P} = \lambda \cdot \overrightarrow{P_2P}$, then from $\boldsymbol{P} - \boldsymbol{P}_1 = \lambda(\boldsymbol{P} - \boldsymbol{P}_2)$ with $\boldsymbol{P}_i = x_i\boldsymbol{A} + y_i\boldsymbol{B} + z_i\boldsymbol{C}$ $(i = 1, 2)$ we obtain

$$(1 - \lambda)\boldsymbol{P} = (x_1 - \lambda x_2)\boldsymbol{A} + (y_1 - \lambda y_2)\boldsymbol{B} + (z_1 - \lambda z_2)\boldsymbol{C}.$$

Because of $x_i + y_i + z_i = 1$ $(i = 1, 2)$ we have

$$\frac{1}{1-\lambda}(x_1 - \lambda x_2 + y_1 - \lambda y_2 + z_1 - \lambda z_2) = 1$$

and therefore

$$P = \left(\frac{x_1 - \lambda x_2}{1-\lambda}, \frac{y_1 - \lambda y_2}{1-\lambda}, \frac{z_1 - \lambda z_2}{1-\lambda}\right), \tag{3}$$

$$P = ((x_1 - \lambda x_2) : (y_1 - \lambda y_2) : (z_1 - \lambda z_2)). \tag{4}$$

Specially, with $\lambda = -1$, point P is the midpoint of the points P_1 and P_2 with

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right),$$

i.e. $P = \frac{1}{2}P_1 + \frac{1}{2}P_2$. Sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ have the midpoints

$$\left(0, \frac{1}{2}, \frac{1}{2}\right) = (0:1:1), \;\; \left(\frac{1}{2}, 0, \frac{1}{2}\right) = (1:0:1), \;\; \left(\frac{1}{2}, \frac{1}{2}, 0\right) = (1:1:0).$$

If $\lambda = 1$, then equality (3) has no sense, but equality (4) obtains the form

$$P = ((x_1 - x_2) : (y_1 - y_2) : (z_1 - z_2)) \tag{5}$$

and represents the point at infinity of the straight line P_1P_2. For this point P we have equality $(P_1P_2P) = 1$ and the relative coordinates in (5) have the zero sum. Therefore, this point does not have the absolute coordinates. Because of $P_1 \neq P_2$ point P at infinity cannot be of the form $(0:0:0)$. Specially, straight lines BC, CA, AB have the points at infinity $(0:1:-1)$, $(-1:0:1)$, $(1:-1:0)$, respectively.

For any vector $\boldsymbol{v}$ numbers y and z are uniquely determined such that $\boldsymbol{v} = y \cdot \overrightarrow{AB} + z \cdot \overrightarrow{AC}$, i.e. $\boldsymbol{v} = y(\boldsymbol{B} - \boldsymbol{A}) + z(\boldsymbol{C} - \boldsymbol{A})$. If we put $x = -(y + z)$, then we have

$$\boldsymbol{v} = x\boldsymbol{A} + y\boldsymbol{B} + z\boldsymbol{C}, \;\; x + y + z = 0. \tag{6}$$

Numbers x, y, z are uniquely determined and are said to be the barycentric coordinates of vector $\boldsymbol{v}$ with respect to triangle ABC. We write $\boldsymbol{v} = [x, y, z]$. For two points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ we have $\boldsymbol{P}_i = x_i\boldsymbol{A} + y_i\boldsymbol{B} + z_i\boldsymbol{C}$ and therefore

$$\overrightarrow{P_1P_2} = \boldsymbol{P}_2 - \boldsymbol{P}_1 = (x_2 - x_1)\boldsymbol{A} + (y_2 - y_1)\boldsymbol{B} + (z_2 - z_1)\boldsymbol{C} = [x_2 - x_1, y_2 - y_1, z_2 - z_1].$$

Specially, $\overrightarrow{BC} = [0, -1, 1]$, $\overrightarrow{CA} = [1, 0, -1]$, $\overrightarrow{AB} = [-1, 1, 0]$. We conclude that the (relative) barycentric coordinates of the point at infinity of a straight line are proportional to the barycentric coordinates of any vector parallel to this line. Therefore, parallel lines have the same point at infinity.

The formulas for metrical relations can be written in a more compact form if we use numbers

$$\alpha = \cot A, \quad \beta = \cot B, \quad \gamma = \cot C \tag{7}$$

We have e.g.

$$b^2 + c^2 - a^2 = 2bc \cos A = 2bc \sin A \cot A = 4\Delta\alpha$$

and therefore

$$b^2 + c^2 - a^2 = 4\Delta\alpha, \quad c^2 + a^2 - b^2 = 4\Delta\beta, \quad a^2 + b^2 - c^2 = 4\Delta\gamma.$$

Adding, we obtain

$$a^2 = 2\Delta(\beta + \gamma), \quad b^2 = 2\Delta(\gamma + \alpha), \quad c^2 = 2\Delta(\alpha + \beta). \tag{8}$$

Now, let us prove the most important theorem about metrical relations in barycentric coordinates.

Theorem 1. *The scalar product of two vectors* $\boldsymbol{v}_i = [x_i, y_i, z_i]$ $(i = 1, 2)$ *is given by*

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2\Delta(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2),$$

where Δ *is the area of the fundamental triangle* ABC *and numbers* α, β, γ *are given by (7).*

Proof. Squaring the equality $\overrightarrow{AB} = \boldsymbol{B} - \boldsymbol{A}$ we obtain $c^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 - 2\boldsymbol{A} \cdot \boldsymbol{B}$, i.e. $2\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A}^2 + \boldsymbol{B}^2 - c^2$ and analogously $2\boldsymbol{A} \cdot \boldsymbol{C} = \boldsymbol{A}^2 + \boldsymbol{C}^2 - b^2$ and $2\boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{B}^2 + \boldsymbol{C}^2 - a^2$. Owing to the equalities $x_i + y_i + z_i = 0$ $(i = 1, 2)$ and (8) we obtain successively

$$\begin{aligned}
2\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 &= 2(x_1\boldsymbol{A} + y_1\boldsymbol{B} + z_1\boldsymbol{C})(x_2\boldsymbol{A} + y_2\boldsymbol{B} + z_2\boldsymbol{C}) \\
&= 2x_1x_2\boldsymbol{A}^2 + 2y_1y_2\boldsymbol{B}^2 + 2z_1z_2\boldsymbol{C}^2 + (x_1y_2 + y_1x_2)(\boldsymbol{A}^2 + \boldsymbol{B}^2 - c^2) \\
&\quad + (x_1z_2 + z_1x_2)(\boldsymbol{A}^2 + \boldsymbol{C}^2 - b^2) + (y_1z_2 + z_1y_2)(\boldsymbol{B}^2 + \boldsymbol{C}^2 - a^2) \\
&= (x_1 + y_1 + z_1)(x_2\boldsymbol{A}^2 + y_2\boldsymbol{B}^2 + z_2\boldsymbol{C}^2) \\
&\quad + (x_2 + y_2 + z_2)(x_1\boldsymbol{A}^2 + y_1\boldsymbol{B}^2 + z_1\boldsymbol{C}^2) \\
&\quad - a^2(y_1z_2 + z_1y_2) - b^2(z_1x_2 + x_1z_2) - c^2(x_1y_2 + y_1x_2) \\
&= -2\Delta[(\beta + \gamma)(y_1z_2 + z_1y_2) + (\gamma + \alpha)(z_1x_2 + x_1z_2) \\
&\quad + (\alpha + \beta)(x_1y_2 + y_1x_2)] \\
&= -2\Delta\{\alpha\,[x_1(y_2 + z_2) + (y_1 + z_1)x_2] + \beta\,[y_1(z_2 + x_2) + (z_1 + x_1)y_2] \\
&\quad + \gamma[z_1(x_2 + y_2) + (x_1 + y_1)z_2]\} \\
&= -2\Delta[\alpha(-2x_1x_2) + \beta(-2y_1y_2) + \gamma(-2z_1z_2)] \\
&= 4\Delta(\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2).
\end{aligned}$$

□

Corollary 1. *The length of the vector* $\boldsymbol{v} = [x, y, z]$ *is given by*

$$|\boldsymbol{v}|^2 = 2\Delta(\alpha x^2 + \beta y^2 + \gamma z^2).$$

Corollary 2. *The angle between two vectors* $\boldsymbol{v}_i = [x_i, y_i, z_i]$ $(i = 1, 2)$ *is given by*

$$\cos\angle(\boldsymbol{v}_1, \boldsymbol{v}_2) = \frac{1}{\Omega_1\Omega_2}(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2),$$

where

$$\Omega_i^2 = \alpha x_i^2 + \beta y_i^2 + \gamma z_i^2 \ (i = 1, 2).$$

Corollary 3. *Two points* $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ *have the distance* $|P_1P_2|$ *given by*

$$|P_1P_2|^2 = 2\Delta[\alpha(x_1 - x_2)^2 + \beta(y_1 - y_2)^2 + \gamma(z_1 - z_2)^2].$$

Specially, with $P_1 = P = (x, y, z)$ and $P_2 = A = (1, 0, 0)$ or $P_2 = B = (0, 1, 0)$ or $P_2 = C = (0, 0, 1)$ we obtain further:

Corollary 4. *For any point* $P = (x, y, z)$ *we have equalities*

$$\begin{aligned}
|AP| &= 2\Delta[\alpha(1 - x)^2 + \beta y^2 + \gamma z^2],\\
|BP| &= 2\Delta[\alpha x^2 + \beta(1 - y)^2 + \gamma z^2],\\
|CP| &= 2\Delta[\alpha x^2 + \beta y^2 + \gamma(1 - z)^2].
\end{aligned}$$

Theorem 2. *For the point* $P = (x, y, z)$ *and any point* S *we have*

$$|SP|^2 = x \cdot |SA|^2 + y \cdot |SB|^2 + z \cdot |SC|^2 - a^2yz - b^2zx - c^2xy.$$

Proof. Let S be the origin. Squaring the equality $\boldsymbol{P} = x\boldsymbol{A} + y\boldsymbol{B} + z\boldsymbol{C}$ and using the equalities from the proof of *Theorem 1* we get

$$\begin{aligned}
|SP|^2 = \boldsymbol{P}^2 &= x^2\boldsymbol{A}^2 + y^2\boldsymbol{B}^2 + z^2\boldsymbol{C}^2 + yz(\boldsymbol{B}^2 + \boldsymbol{C}^2 - a^2)\\
&\quad + zx(\boldsymbol{C}^2 + \boldsymbol{A}^2 - b^2) + xy(\boldsymbol{A}^2 + \boldsymbol{B}^2 - c^2)\\
&= (x + y + z)(x\boldsymbol{A}^2 + y\boldsymbol{B}^2 + z\boldsymbol{C}^2) - a^2yz - b^2zy - c^2xy
\end{aligned}$$

and because of $x + y + z = 1$ the statement of *Theorem 2* follows. □

With $P = G = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ we obtain:

Corollary 5 [Leibniz]. *For centroid* G *of triangle* ABC *and for any point* P *we have*

$$3 \cdot |SG|^2 = |SA|^2 + |SB|^2 + |SC|^2 - \frac{1}{3}(a^2 + b^2 + c^2).$$

If S is the circumcenter O of triangle ABC, then $|OA| = |OB| = |OC| = R$ and by (8) it follows:

Corollary 6. *For any point* $P = (x, y, z)$ *and the circumscribed circle* (O, R) *of triangle* ABC *the equality*

$$|OP|^2 = R^2 - a^2yz - b^2zx - c^2xy$$

holds, i.e. $|OP|^2 = R^2 - 2\Delta\Pi$, *where*

$$\Pi = (\beta+\gamma)yz + (\gamma+\alpha)zx + (\alpha+\beta)xy = \frac{1}{2\Delta}(a^2yz + b^2zx + c^2xy). \tag{9}$$

With $S = P$ *Theorem 2* implies:

Corollary 7. *For the point* $P = (x, y, z)$ *the equality*

$$x \cdot |AP|^2 + y \cdot |BP|^2 + z^2|CP|^2 = 2\Delta\Pi$$

holds, where the number Π *is given by (9).*

The equalities from *Corollary 4* can be written in another form because of (9), (8) and the equality $x + y + z = 1$. We obtain e.g.

$$\begin{aligned}
\frac{1}{2\Delta}|AP|^2 &= \alpha(1-x)^2 + \beta y^2 + \gamma z^2 \\
&= \alpha - 2\alpha x + \alpha x(1-y-z) + \beta y(1-z-x) + \gamma z(1-x-y) \\
&= \alpha - \alpha x + \beta y + \gamma z - (\beta+\gamma)yz - (\gamma+\alpha)zx - (\alpha+\beta)xy \\
&= \alpha(y+z) + \beta y + \gamma z - \Pi = \frac{1}{x}[(\gamma+\alpha)zx + (\alpha+\beta)xy] - \Pi \\
&= \frac{1}{x}[\Pi - (\beta+\gamma)yz] - \Pi = \frac{1}{x}[\Pi(1-x) - (\beta+\gamma)yz] \\
&= \frac{1}{x}[\Pi(y+z) - \frac{a^2}{2\Delta}yz].
\end{aligned}$$

Therefore:

Corollary 8. *For any point* $P = (x, y, z)$ *the equalities*

$$\begin{aligned}
x \cdot |AP|^2 &= 2\Delta\Pi(y+z) - a^2yz, \\
y \cdot |BP|^2 &= 2\Delta\Pi(z+x) - b^2zx, \\
z \cdot |CP|^2 &= 2\Delta\Pi(x+y) - c^2xy
\end{aligned}$$

hold, where number Π *is given by (9).*

For any point $P = (x, y, z)$ on the circumcircle (O, R) of triangle ABC we have $|OP| = R$ and because of *Corollary 6* it follows $\Pi = 0$. Therefore, the equalities of *Corollary 8* have now the form

$$|AP|^2 = -a^2\frac{yz}{x}, \quad |BP|^2 = -b^2\frac{zx}{y}, \quad |CP|^2 = -c^2\frac{xy}{z}. \tag{10}$$

We have:

Corollary 9. *The circumcircle of triangle* ABC *has the equation*

$$(\beta+\gamma)yz + (\gamma+\alpha)zx + (\alpha+\beta)xy = 0 \quad \text{or} \quad a^2yz + b^2zx + c^2xy = 0.$$

For any point $P = (x, y, z)$ (except A, B, C) on this circle equalities (10) hold.

Point $P = (x, y, z)$ is collinear with two different points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ iff there is a number $\lambda \in \mathbb{R}$ such that $\overrightarrow{P_1P} = \lambda\overrightarrow{P_1P_2}$, i.e. $\boldsymbol{P} - \boldsymbol{P}_1 = \lambda(\boldsymbol{P}_2 - \boldsymbol{P}_1)$ or $\boldsymbol{P} = (1-\lambda)\boldsymbol{P}_1 + \lambda\boldsymbol{P}_2$. With $\kappa = 1 - \lambda$ we conclude that point P lies on straight line P_1P_2 iff two numbers κ and λ exist such that $\kappa + \lambda = 1$ and

$$x = \kappa x_1 + \lambda x_2, \quad y = \kappa y_1 + \lambda y_2, \quad z = \kappa z_1 + \lambda z_2. \tag{11}$$

Numbers (11) satisfy the equation

$$Xx + Yy + Zz = 0, \tag{12}$$

where

$$X = k(y_1z_2 - z_1y_2), \quad Y = k(z_1x_2 - x_1z_2), \quad Z = k(x_1y_2 - y_1x_2). \tag{13}$$

and $k \in \mathbb{R} \setminus \{0\}$. Indeed, we obtain an obvious equality

$$(y_1z_2 - z_1y_2)(\kappa x_1 + \lambda x_2) + (z_1x_2 - x_1z_2)(\kappa y_1 + \lambda y_2) + (x_1y_2 - y_1x_2)(\kappa z_1 + \lambda z_2) = 0.$$

Conversely, if numbers x, y, z satisfy equation (12), where (13) holds, then this equation (12) can be written in the form

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0. \tag{14}$$

Therefore, there are the numbers κ and λ such that equalities (11) are valid. Adding these equalities it follows $\kappa + \lambda = 1$ because of $x + y + z = 1$ and $x_i + y_i + z_i = 1$ $(i = 1, 2)$. We have the following theorem.

Theorem 3. *Point $P = (x, y, z)$ is collinear with points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ iff equality (12) holds, where numbers X, Y, Z are given by (13), where $k \in \mathbb{R} \setminus \{0\}$.*

Theorem 3 implies that the coordinates of any point of the given straight line $\mathcal{P}$ satisfy an equation of the form (12), the equation of this line $\mathcal{P}$, where numbers X, Y, Z are determined up to proportionality. These numbers are baricentric coordinates of line $\mathcal{P}$ and we write $\mathcal{P} = (X : Y : Z)$. As $P_1 \neq P_2$, so (13) implies $(X : Y : Z) \neq (0 : 0 : 0)$. The equality (12) is the necessary and sufficient condition for the incidency of point $P = (x : y : z)$ and line $\mathcal{P} = (X : Y : Z)$.

The equality $x + y + z = 0$ characterizes the points at infinity. Therefore, all these points lie on a line $\mathcal{N} = (1 : 1 : 1)$, the line at infinity.

Corollary 10. *Three points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2, 3)$ are collinear iff*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Specially, two points $P_i = (x_i : y_i : z_i)$ $(i = 1, 2)$ are collinear with point A iff $y_1 : z_1 = y_2 : z_2$, with point B iff $z_1 : x_1 = z_2 : x_2$ and with point C iff $x_1 : y_1 = x_2 : y_2$.

Corollary 11. *The join of two different points $P_i = (x_i : y_i : z_i)$ is the straight line*

$$P_1P_2 = \left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} : \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} : \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right).$$

As we have $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, so by *Corollary 11* it follows $BC = (1 : 0 : 0)$, $CA = (0 : 1 : 0)$, $AB = (0 : 0 : 1)$ and these three lines have the equations $x = 0$, $y = 0$, $z = 0$ respectively. If $P = (x : y : z)$, then $AP = (0 : -z : y)$, $BP = (z : 0 : -x)$, $CP = (-y : x : 0)$.

To be honest, we must say that the statement of *Corollary 10* in the proof of *Theorem 3* is proved only in the case of finite points P_1, P_2 and $P_3 = P$. Three points P_1, P_2, P at infinity obviously satisfy equation (14) because of $x + y + z = 0$ and $x_i + y_i + z_i = 0$ $(i = 1, 2)$. Conversely, from (14) and $x_i + y_i + z_i = 0$ $(i = 1, 2)$ if follows $x + y + z = 0$. We must prove the statement for the finite points P_1, P_2 and point P at infinity. The point at infinity of line P_1P_2 is the point

$$(x : y : z) = ((x_1 - x_2) : (y_1 - y_2) : (z_1 - z_2))$$

and it obviously satisfies equation (14). Conversely, let point $P = (x : y : z)$ at infinity satisfy equation (14). Then there are two numbers κ and λ such that (11) holds. Adding these equations we obtain $0 = \kappa + \lambda$ because of $x + y + z = 0$ and $x_i + y_i + z_i = 1$ $(i = 1, 2)$. Therefore, $\lambda = -\kappa$ and equalities (11) obtain the form $x = \kappa(x_1 - x_2)$, $y = \kappa(y_1 - y_2)$, $z = \kappa(z_1 - z_2)$, i.e. P is the point at infinity of line P_1P_2.

From *Corollary 10* it follows that point P is collinear with two different points P_1 and P_2 iff there are two numbers μ and ν such that $x = \mu x_1 + \nu x_2$, $y = \mu y_1 + \nu y_2$, $z = \mu z_1 + \nu z_2$ for the coordinates of these points. We shall write $P = \mu P_1 + \nu P_2$ in this case.

Theorem 4. *Let points $P_i = (x_i : y_i : z_i)$ $(i = 1, 2)$ have the sums $s_i = x_i + y_i + z_i$ of coordinates. If point $P = (x : y : z)$ satisfies the equality*

$$P = \mu P_1 + \nu P_2, \tag{15}$$

then these three points have the ratio

$$(P_1P_2P) = -\frac{\nu}{\mu} \cdot \frac{s_2}{s_1}. \tag{16}$$

Proof. We pass onto absolute coordinates. Then the right-hand side of (15) is of the form $\mu s_1 P_1 + \nu s_2 P_2$ and because of equality of coordinate sums of both sides we must take $(\mu s_1 + \nu s_2)P$ on the left-hand side of (15). The obtained equality has the vector form $(\mu s_1 + \nu s_2)\boldsymbol{P} = \mu s_1 \boldsymbol{P}_1 + \nu s_2 \boldsymbol{P}_2$, i.e. $\mu s_1(\boldsymbol{P} - \boldsymbol{P}_1) = -\nu s_2(\boldsymbol{P} - \boldsymbol{P}_2)$ or $\mu s_1 \cdot \overrightarrow{P_1P} = -\nu s_2 \cdot \overrightarrow{P_2P}$. The last equality is equivalent to (16). □

For the point $A = (1, 0, 0)$, the midpoint $D = (0 : 1 : 1)$ of side $\overline{BC}$ and for centroid $G = (1 : 1 : 1)$ of triangle ABC we have the equality $G = A + D$ and *Theorem 4* implies the equality $(ADG) = -2$.

Equality (13) is symmetrical in variables x, y, z and X, Y, Z. Therefore, for the sets of points and lines (finite ones and at infinity) there holds the principle of duality. The following theorem is dual of *Theorem 3*.

Theorem 5. *Straight line $\mathcal{P} = (X : Y : Z)$ is incident with the intersection of two different lines $\mathcal{P}_1 = (X_1 : Y_1 : Z_1)$ and $\mathcal{P}_2 = (X_2 : Y_2 : Z_2)$ iff the equality (12) holds, where numbers x, y, z are given by*

$$x = K(Y_1Z_2 - Z_1Y_2), \quad y = K(Z_1X_2 - X_1Z_2), \quad z = K(X_1Y_2 - Y_1X_2),$$

where $K \in \mathbb{R} \setminus \{0\}$.

Corollary 12. *Three straight lines $\mathcal{P}_i = (X_i : Y_i : Z_i)$ $(i = 1, 2, 3)$ are concurrent iff*

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0.$$

Corollary 13. *The intersection of two different lines $\mathcal{P}_i = (X_i : Y_i : Z_i)$ $(i = 1, 2)$ is the point*

$$\mathcal{P}_1 \cap \mathcal{P}_2 = \left(\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix} : \begin{vmatrix} Z_1 & X_1 \\ Z_2 & X_2 \end{vmatrix} : \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} \right).$$

If $P = (x : y : z)$, then we have $AP = (0 : -z : y)$, $BC = (1 : 0 : 0)$ and therefore $AP \cap BC = (0 : y : z)$. Analogously $BP \cap CA = (x : 0 : z)$ and $CP \cap AB = (x : y : 0)$.

The point at infinity of a line is its intersection with the line $\mathcal{N} = (1 : 1 : 1)$ at infinity. Hence, *Corollary 13* implies:

Corollary 14. *The line $\mathcal{P} = (X : Y : Z)$ has the point $\mathcal{P} \cap \mathcal{N} = ((Y - Z) : (Z - X) : (X - Y))$ at infinity.*

Two lines are parallel iff they have the same intersection with the line at infinity. Therefore, *Corollary 12* implies:

Corollary 15. *Lines $\mathcal{P}_i = (X_i : Y_i : Z_i)$ $(i = 1, 2)$ are parallel iff*

$$\begin{vmatrix} 1 & 1 & 1 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0.$$

Any line parallel to the line $(X : Y : Z)$ has a form $((X + K) : (Y + K) : (Z + K))$ for some $K \in \mathbb{R}$.

From *Theorem 1* we obtain:

Corollary 16. *Two vectors $\boldsymbol{v}_i = [x_i, y_i, z_i]$ $(i = 1, 2)$ are ortogonal iff*

$$\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2 = 0, \tag{17}$$

where numbers α, β, γ are given by (7). Equality (17) is the condition for ortogonality of two lines with the points $(x_i : y_i : z_i)$ $(i = 1, 2)$ at infinity.

Theorem 6. *The lines ortogonal to the line with the point $(x : y : z)$ at infinity (where $x + y + z = 0$) have the point at infinity*

$$((\beta y - \gamma z) : (\gamma z - \alpha x) : (\alpha x - \beta y)). \tag{18}$$

Proof. Point (18) is obviously a point at infinity and the ortogonality follows by *Corollary 16* because of $\alpha x(\beta y-\gamma z)+\beta y(\gamma z-\alpha x)+\gamma z(\alpha x-\beta y)=0$.

□

Lines BC, CA, AB have the points $(0:-1:1)$, $(1:0:-1)$, $(-1:1:0)$ at infinity and *Theorem 6* implies:

Corollary 17. *The lines orthogonal to lines BC, CA, AB, respectively, have the points at infinity*

$$\begin{aligned} N_a &= (-(\beta+\gamma):\gamma:\beta),\\ N_b &= (\gamma:-(\gamma+\alpha):\alpha),\\ N_c &= (\beta:\alpha:-(\alpha+\beta)). \end{aligned} \tag{19}$$

The line $(0:-\beta:\gamma)$ passes through point $A=(1:0:0)$ and point N_a from (19). Therefore, this line is the altitude through vertex A. Analogously, the altitudes through vertices B and C are $(\alpha:0:-\gamma)$ and $(-\alpha:\beta:0)$. All three altitudes obviously pass through point $H=(\beta\gamma:\gamma\alpha:\alpha\beta)$, the orthocenter of triangle ABC. Line $((\beta-\gamma):-(\beta+\gamma):(\beta+\gamma))$ passes through midpoint $(0:1:1)$ of side $\overline{BC}$, through point N_a from (19) and through the point

$$O=(\alpha(\beta+\gamma):\beta(\gamma+\alpha):\gamma(\alpha+\beta)). \tag{20}$$

Indeed, we have without common factor $\beta+\gamma$ the equalities $-(\beta-\gamma)-\gamma+\beta=0$ and $\alpha(\beta-\gamma)-\beta(\gamma+\alpha)+\gamma(\alpha+\beta)=0$. Therefore, this line is the perpendicular bisector of side $\overline{BC}$, and for sides $\overline{CA}$ and $\overline{AB}$ we have analogous perpendicular bisectors. We have the following theorem.

Theorem 7. *The fundamental triangle ABC has the altitudes $AH=(0:-\beta:\gamma)$, $BH=(\alpha:0:-\gamma)$, $CH=(-\alpha:\beta:0)$, the orthocenter $H=(\beta\gamma:\gamma\alpha:\alpha\beta)$, the perpendicular bisectors of the sides are*

$$\begin{aligned} &((\beta-\gamma):-(\beta+\gamma):(\beta+\gamma)),\\ &((\gamma+\alpha):(\gamma-\alpha):-(\gamma+\alpha)),\\ &(-(\alpha+\beta):(\alpha+\beta):(\alpha-\beta)) \end{aligned}$$

and the circumcenter O of triangle ABC is given by (20).

According to equalities

$$-\alpha=-\cot A=\cot(\pi-A)=\cot(B+C)=\frac{\cot B\cot C-1}{\cot B+\cot C}=\frac{\beta\gamma-1}{\beta+\gamma}$$

we have the fundamental identity

$$\beta\gamma+\gamma\alpha+\alpha\beta=1. \tag{21}$$

Therefore, we have more precise equalities $H=(\beta\gamma,\gamma\alpha,\alpha\beta)$ and

$$O=\left(\frac{1}{2}\alpha(\beta+\gamma),\frac{1}{2}\beta(\gamma+\alpha),\frac{1}{2}\gamma(\alpha+\beta)\right)=\left(\frac{1}{2}(1-\beta\gamma),\frac{1}{2}(1-\gamma\alpha),\frac{1}{2}(1-\alpha\beta)\right).$$

Lines $\mathcal{P}_i = (X_i : Y_i : Z_i)$ $(i = 1, 2)$ have points $(x_i : y_i : z_i)$ at infinity, where $x_i = Y_i - Z_i$, $y_i = Z_i - X_i$, $z_i = X_i - Y_i$. Vectors $[x_i, y_i, z_i]$ are parallel with lines $\mathcal{P}_i$ for $i = 1, 2$. Therefore, angle ϑ of lines $\mathcal{P}_1$ and $\mathcal{P}_2$ is given by *Corollary 2* in the form

$$\cos\vartheta = \frac{1}{\Omega_1\Omega_2}|\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2|, \tag{22}$$

where

$$\Omega_i^2 = \alpha x_i^2 + \beta y_i^2 + \gamma z_i^2 \quad (i = 1, 2). \tag{23}$$

We obtain

$$\begin{aligned} y_1 z_2 - z_1 y_2 &= (Z_1 - X_1)(X_2 - Y_2) - (X_1 - Y_1)(Z_2 - X_2) \\ &= Y_1 Z_2 - Z_1 Y_2 + Z_1 X_2 - X_1 Z_2 + X_1 Y_2 - Y_1 X_2 = k \end{aligned}$$

and analogously $z_1 x_2 - x_1 z_2 = k$ and $x_1 y_2 - y_1 x_2 = k$, where

$$k = \begin{vmatrix} 1 & 1 & 1 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}. \tag{24}$$

Further

$$\sin^2\vartheta = 1 - \cos^2\vartheta = \frac{1}{\Omega_1^2\Omega_2^2}[\Omega_1^2\Omega_2^2 - (\alpha x_2 + \beta y_2 + \gamma z_2)^2].$$

and as we have

$$\begin{aligned} &\Omega_1^2\Omega_2^2 - (\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)^2 \\ &\quad = (\alpha x_1^2 + \beta y_1^2 + \gamma z_1^2)(\alpha x_2^2 + \beta y_2^2 + \gamma z_2^2) - (\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)^2 \\ &\quad = \beta\gamma(y_1 z_2 - z_1 y_2)^2 + \gamma\alpha(z_1 x_2 - x_1 z_2)^2 + \alpha\beta(x_1 y_2 - y_1 x_2)^2 \\ &\quad = (\beta\gamma + \gamma\alpha + \alpha\beta)k^2 = k^2, \end{aligned}$$

so it follows

$$\sin\vartheta = \frac{|k|}{\Omega_1\Omega_2}$$

and (22) implies

$$\cot\vartheta = \frac{1}{|k|}|\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2|.$$

Substitutions $X_1 \to -X_1$, $Y_1 \to -Y_1$, $Z_1 \to -Z_1$ imply substitutions $x_1 \to -x_1$, $y_1 \to -y_1$, $z_1 \to -z_1$ and $k \to -k$, $\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2 \to -(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)$. Therefore, the number $\frac{1}{k}(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)$ does not change the sign. The same is true for substitutions $X_2 \to -X_2$, $Y_2 \to -Y_2$, $Z_2 \to -Z_2$. Substitution $1 \leftrightarrow 2$ implies substitutions $k \to -k$ and $\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2 \to \alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2$. Therefore, the number $\frac{1}{k}(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2)$ changes the sign in this case. We conclude that the equalities

$$\cos\vartheta = \frac{1}{\Omega_1\Omega_2}(\alpha x_1 x_2 + \beta y_1 y_2 + \gamma z_1 z_2), \quad \sin\vartheta = \frac{k}{\Omega_1\Omega_2}, \tag{25}$$

$$\cot\vartheta = \frac{1}{k}(\alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2) \tag{26}$$

give the oriented angle ϑ of the ordered pair of oriented lines $\mathcal{P}_1$ and $\mathcal{P}_2$. We have:

Theorem 8. *The oriented angle ϑ of the oriented lines $\mathcal{P}_i = (X_i : Y_i : Z_i)$ $(i = 1, 2)$ is given by (25) and (26), where $x_i = Y_i - Z_i$, $y_i = Z_i - X_i$, $z_i = X_i - Y_i$ $(i = 1, 2)$ and where numbers Ω_1, Ω_2, k are given by (23) and (24).*

For $\mathcal{P}_1 = BC = (1 : 0 : 0)$, $\mathcal{P}_2 = \mathcal{P}(X : Y : Z)$ we have $x_1 = 0$, $y_1 = -1$, $z_1 = 1$, $x_2 = Y - Z$, $y_2 = Z - X$, $z_2 = X - Y$ and then by (8)

$$\Omega_1 = \sqrt{\beta+\gamma} = \frac{a}{\sqrt{2\Delta}}, \quad \Omega_2 = \Omega = \sqrt{\alpha x^2 + \beta y^2 + \gamma z^2},$$

$$k = Y - Z = x, \quad \alpha x_1x_2 + \beta y_1y_2 + \gamma z_1z_2 = \gamma z - \beta y.$$

Analogous equalities we have for $\mathcal{P}_1 = CA$ or $\mathcal{P}_1 = AB$. Therefore, *Theorem 8* implies:

Corollary 18. *The oriented angles φ, χ, ψ of lines BC, CA, AB with line $\mathcal{P} = (X : Y : Z)$ with point $(x : y : z) = ((Y - Z) : (Z - X) : (X - Y))$ at infinity are given by equalities*

$$\begin{aligned}
a\cos\varphi &= \frac{\sqrt{2\Delta}}{\Omega}(\gamma z - \beta y), \quad a\sin\varphi = \frac{\sqrt{2\Delta}}{\Omega}x, \quad \cot\varphi = \frac{\gamma z - \beta y}{x},\\
b\cos\chi &= \frac{\sqrt{2\Delta}}{\Omega}(\alpha x - \gamma z), \quad b\sin\chi = \frac{\sqrt{2\Delta}}{\Omega}y, \quad \cot\chi = \frac{\alpha x - \gamma z}{y},\\
c\cos\psi &= \frac{\sqrt{2\Delta}}{\Omega}(\beta y - \alpha x), \quad c\sin\psi = \frac{\sqrt{2\Delta}}{\Omega}z, \quad \cot\psi = \frac{\beta y - \alpha x}{z},
\end{aligned}$$

where Δ is the area of triangle ABC and $\Omega^2 = \alpha x^2 + \beta y^2 + \gamma z^2$.

Every line $\mathcal{P}$, which passes through the point $A = (1 : 0 : 0)$, has a form $\mathcal{P} = (0 : Y : Z)$ and has the point $(x : y : z) = ((Y - Z) : Z : -Y)$ at infinity. By *Corollary 18* we obtain

$$\begin{aligned}
\cot\varphi &= \frac{-\gamma Y - \beta Z}{Y - Z} = \frac{\gamma Y + \beta Z}{Z - Y},\\
\cot\chi &= \frac{\alpha(Y - Z) + \gamma Y}{Z} = (\gamma + \alpha)\frac{Y}{Z} - \alpha,\\
\cot\psi &= \frac{\beta Z - \alpha(Y - Z)}{Y} = \alpha - (\alpha + \beta)\frac{Z}{Y}
\end{aligned}$$

and analogous statements for the lines through points B and C. Therefore, we have the following corollary.

Corollary 19. *Lines BC, CA, AB make angles φ, χ, ψ with a line $(0 : Y : Z)$ through point A resp. a line $(X : 0 : Z)$ through point B resp. a line $(X : Y : 0)$ through point C such that*

$$\cot\varphi = \frac{\gamma Y + \beta Z}{Z - Y}, \quad \cot\chi = (\gamma + \alpha)\frac{Y}{Z} - \alpha, \quad \cot\psi = \alpha - (\alpha + \beta)\frac{Z}{Y}$$

resp.

$$\cot\varphi = \beta - (\beta + \gamma)\frac{X}{Z}, \quad \cot\chi = \frac{\gamma X + \alpha Z}{X - Z}, \quad \cot\psi = (\alpha + \beta)\frac{Z}{X} - \beta$$

resp.

$$\cot\varphi = (\beta+\gamma)\frac{X}{Y} - \gamma, \quad \cot\chi = \gamma - (\gamma+\alpha)\frac{Y}{X}, \quad \cot\psi = \frac{\alpha Y + \beta X}{Y - X}.$$

Theorem 9. *If ϑ is the oriented angle between line $\mathcal{P} = (X : Y : Z)$ and line $\mathcal{P}'$ with point $(x : y : z)$ at infinity and if $\tau = \cot\vartheta$, then*

$$\begin{aligned} x &= (\beta+\gamma)X + (\tau-\gamma)Y - (\tau+\beta)Z, \\ y &= -(\tau+\gamma)X + (\gamma+\alpha)Y + (\tau-\alpha)Z, \\ z &= (\tau-\beta)X - (\tau+\alpha)Y + (\alpha+\beta)Z. \end{aligned} \tag{27}$$

Proof. Obviously we have $x+y+z=0$ and by (27) a point at infinity is given. Line $\mathcal{P}$ has the point $((Y-Z) : (Z-X) : (X-Y))$ at infinity and by *Theorem 8* we obtain

$$\begin{aligned} \cot\angle(\mathcal{P},\mathcal{P}') &= \frac{\alpha(Y-Z)x + \beta(Z-X)y + \gamma(X-Y)z}{(Z-X)y - (X-Y)z} \\ &= \frac{X(\gamma z - \beta y) + Y(\alpha x - \gamma z) + Z(\beta y - \alpha x)}{-(y+z)X + yY + zZ} \\ &= -\frac{(\beta y - \gamma z)X + (\gamma z - \alpha x)Y + (\alpha x - \beta y)Z}{xX + yY + zZ}. \end{aligned}$$

However, by (27) and (21) we get e.g.

$$\begin{aligned} \beta y - \gamma z = {} & [-\beta(\tau+\gamma) - \gamma(\tau-\beta)]X + [\beta(\gamma+\alpha) + \gamma(\tau+\alpha)]Y \\ & +[\beta(\tau-\alpha) - \gamma(\alpha+\beta)]Z = -(\beta+\gamma)\tau X + (\gamma\tau+1)Y + (\beta\tau-1)Z \end{aligned}$$

and analogously

$$\gamma z - \alpha x = (\gamma\tau - 1)X - (\gamma+\alpha)\tau Y + (\alpha\tau+1)Z$$

and

$$\alpha x - \beta y = (\beta\tau+1)X + (\alpha\tau-1)Y - (\alpha+\beta)\tau Z.$$

So we obtain further

$$\begin{aligned} -[(\beta y - \gamma z)X &+ (\gamma z - \alpha x)Y + (\alpha x - \beta y)Z] \\ &= [(\beta+\gamma)\tau X - (\gamma\tau+1)Y - (\beta\tau-1)Z]X \\ &\quad +[-(\gamma\tau-1)X + (\gamma+\alpha)\tau Y - (\alpha\tau+1)Z]Y \\ &\quad +[-(\beta\tau+1)X - (\alpha\tau-1)Y + (\alpha+\beta)\tau Z]Z \\ &= (\beta+\gamma)\tau X^2 + (\gamma+\alpha)\tau Y^2 + (\alpha+\beta)\tau Z^2 \\ &\quad -2\alpha\tau YZ - 2\beta\tau ZX - 2\gamma\tau XY, \end{aligned}$$

$$\begin{aligned} xX+yY+zZ &= [(\beta+\gamma)X+(\tau-\gamma)Y-(\tau+\beta)Z]X \\ &\quad +[-(\tau+\gamma)X+(\gamma+\alpha)Y+(\tau-\alpha)Z]Y \\ &\quad +[(\tau-\beta)X-(\tau+\alpha)Y+(\alpha+\beta)Z]Z \\ &= (\beta+\gamma)X^2+(\gamma+\alpha)Y^2+(\alpha+\beta)Z^2 \\ &\quad -2\alpha YZ-2\beta ZX-2\gamma XY \end{aligned}$$

and finally

$$\cot\angle(\mathcal{P},\mathcal{P}^{'})=\tau=\cot\vartheta.$$

Corollary 20. *The oriented angle ϑ of the line $(X:Y:Z)$ and a line with point $(x:y:z)$ at infinity is given by*

$$\cot\vartheta=-\frac{(\beta y-\gamma z)X+(\gamma z-\alpha x)Y+(\alpha x-\beta z)Z}{xX+yY+zZ}.$$

If $\mathcal{P}=BC=(1:0:0)$, then in *Theorem 9* we have $X=1$, $Y=Z=0$ and (27) implies $x=\beta+\gamma$, $y=-(\tau+\gamma)$, $z=\tau-\beta$. Analogous equalities can be obtained if $\mathcal{P}=CA$ or $\mathcal{P}=AB$. Hence, we have:

Corollary 21. *If ϑ is the oriented angle between line BC resp. CA resp. AB and the line $\mathcal{P}^{'}$, then line $\mathcal{P}^{'}$ has the point at infinity $((\beta+\gamma):-(\tau+\gamma):(\tau-\beta))$ resp. $((\tau-\gamma):(\gamma+\alpha):-(\tau+\alpha))$ resp. $(-(\tau+\beta):(\tau-\alpha):(\alpha+\beta))$, where $\tau=\cot\vartheta$.*

For three collinear points B,C,D and any point A ratio (DCB) is equal to the ratio of oriented areas of triangles ABD and ABC. Therefore

$$\text{area}ABD=(DCB)\cdot\text{area}ABC. \tag{28}$$

If $P=(x,y,z)$ then point $D=AP\cap BC$ is given by $D=(0:y:z)$ and has the sum $y+z$ of coordinates. We have equalities $xA=P-D$ and $yB=D-zC$. Therefore, *Theorem 4* implies the equalities

$$(PDA)=y+z,\quad (DCB)=\frac{z}{y+z}.$$

Using (28) and analogous equality $\text{area}ABP=(PDA)\cdot\text{area}ABD$ we obtain

$$\text{area}ABD=\frac{z}{y+z},\quad \text{area}ABP=(y+z)\cdot\text{area}ABD=\Delta z,$$

where $\Delta=\text{area}ABC$. We have analogous results for another vertices of triangle ABC, i.e. the following theorem holds.

Theorem 10. *For any point $P=(x,y,z)$ triangles BCP, CAP, ABP have the oriented areas $\Delta x,\Delta y,\Delta z$, where $\Delta=areaABC$.*

Corollary 22. *For any points A,B,C,P there holds the equality*

$$areaABC=areaBCP+areaCAP+areaABP.$$

Theorem 10 justifies the name areal coordinates for barycentric coordinates of a point.

Now, let $\mathcal{P}_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ be any two points. Then for the points $D_i = AP_i \cap BC$ we have equalities

$$(P_iD_iA) = y_i + z_i, \quad \text{area}ABD_i = \frac{z_i}{y_i + z_i} \quad (i = 1, 2).$$

Therefore, we obtain successively

$$\begin{aligned}\text{area}AD_1D_2 &= \text{area}ABD_2 - \text{area}ABD_1 = \Delta\left(\frac{z_2}{y_2 + z_2} - \frac{z_1}{y_1 + z_1}\right)\\ &= \Delta\frac{y_1z_2 - z_1y_2}{(y_1 + z_1)(y_2 + z_2)},\\ \text{area}AP_1P_2 &= (P_1D_1A) \cdot \text{area}AD_1P_2 = (P_1D_1A)(P_2D_2A) \cdot \text{area}AD_1D_2\\ &= (y_1 + z_1)(y_2 + z_2)\text{area}AD_1D_2 = \Delta(y_1z_2 - z_1y_2).\end{aligned}$$

Finally, we can give a probably new proof of a well-known formula for the oriented area of a triangle.

Theorem 11. *Oriented area of any triangle with vertices* $\mathcal{P}_i = (x_i, y_i, z_i)$ $(i = 1, 2, 3)$ *is given by the formula*

$$\text{area}P_1P_2P_3 = \Delta \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \tag{29}$$

Proof. The proven formula for areaAP_1P_2 and analogous formulas for areaAP_2P_3 and areaAP_3P_1 and *Corollary 22* imply

$$\begin{aligned}\text{area}P_1P_2P_3 &= \text{area}AP_2P_3 + \text{area}AP_3P_1 + \text{area}AP_1P_2\\ &= \Delta(y_2z_3 - z_2y_3 + y_3z_1 - z_3y_1 + y_1z_2 - z_1y_2)\\ &= \Delta \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix},\end{aligned}$$

wherefrom (29) follows because of the equalities $1 - y_i - z_i = x_i$ $(i = 1, 2, 3)$.

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