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Illusion of linearity in area and volume problems: Do metacognitive and visual scaffolds help university students?

NINA PAVLIN-BERNARDIĆ and BARBARA MATEŠA

When solving geometry problems, students are prone to the illusion of linearity – a tendency to believe that when one side of a geometrical figure is increased or decreased by a factor *k*, its area and volume are also changed by that same factor. The aim of this study was to examine how different types of help provided to university students influence their achievement in mathematical problems involving the enlargement or reduction of geometrical figures. The participants, 122 undergraduate psychology students, were divided into four groups. One group solved an introductory task with visual scaffolds (help in the form of illustrations), second group received metacognitive scaffolds (help intended to provoke a cognitive conflict), third group received a combination of these, while the fourth group was the control group. All of the groups then solved a list of area, volume, and linear problems. The results show that metacognitive and visual scaffolds enhanced students' performance in volume and area problems. There were no differences in the achievement between the experimental groups. The students in all experimental groups were better in solving area problems than volume problems, while there were no differences in the control group between the achievement in these two types of problems.

Key words: illusion of linearity, area problems, volume problems, metacognitive scaffolds, visual scaffolds

Many mathematical problems in school and in everyday life can be solved using a linear model. For example, during primary and secondary education students learn that there is a proportional relation between the diameter and perimeter of a circle, between the time and the distance travelled at a constant speed, that many mathematical word problems can be solved using linear relations, etc. However, the experience of using this model, which seems self-evident and simple, can lead students to a wrong belief that it could be applied universally, i.e., that all problems can be solved by using it. This error is often referred to as an *illusion of linearity* or a *linearity trap* (De Bock, Van Dooren, Janssens, & Verschaffel, 2002, 2007).

Studies have confirmed the dominance of linearity in mathematical thinking in many areas, such as geometry, arithmetic, algebra, and probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010; Verschaffel, Greer, & De Corte, 2000). The best-known example of students' improper use of linearity occurs in the domain of geometry, where it occurs in problems that include relationships between the lengths and the area or the volume of similarly enlarged or reduced figures (e.g., De Bock, Verschaffel, & Janssens, 2002; Vlahović-Štetić, Pavlin-Bernardić, & Rajter, 2010; Paić-Antunović & Vlahović-Štetić, 2011). For example, when solving problems where the sides of a figure were doubled to produce a similar figure, many participants in the studies thought that the area and volume of that figure will also double. Of course, the enlargement or reduction of the sides of a figure with factor k enlarges or reduces the areas with factor k^2 and volumes with factor k^3. Also, this factor does not depend on the characteristics of figures (whether they are regular or irregular).

Studies with primary and secondary school students have shown that they solve non-linear problems very poorly compared to linear problems presented in the same form (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004; Vlahović-Štetić et al., 2010; Vlahović-Štetić & Zekić, 2004). For example, in one study the following non-proportional item about the area of a square was presented to students between 12 and 16 years old: "Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?", and more than 80% of the students answered incorrectly "24 hours" (De Bock, Verschaffel, & Janssens, 1998).

The interviews with students, the aim of which was to unravel the problem solving processes and reasons for

Nina Pavlin-Bernardić, Department of Psychology, Faculty of Humanities and Social Sciences, Ivana Lučića 3, 10000 Zagreb, Croatia. E-mail: nbernardi@ffzg.hr (the address for correspondence);

Barbara Mateša, Department of Psychology, Faculty of Humanities and Social Sciences, Zagreb, Croatia.

such mistakes, showed that the majority of students used the proportional model in a spontaneous, almost intuitive way, not aware of the model they chose, while others were really convinced that linear functions were applicable "everywhere" and deliberately chose a linear method (De Bock, Van Dooren, et al., 2002).

The focus of several studies was to examine which factors can help students to overcome or reduce the illusion of linearity. Vlahović-Štetić et al. (2010) investigated whether options offered in multiple choice problems which were presented to high school students make an influence on students' reasoning. One group of students solved non-linear problems where, among five answers offered, one solution could be obtained if a participant were liable to the illusion of linearity. The other group solved the same problems, but without the linear solution among the offered answers. The absence of the possibility to choose a linear answer for a non-linear problem encouraged students – particularly the older ones – to consider the problem more deeply and to develop a more adequate problem schema.

Paić-Antunović and Vlahović-Štetić (2011) gave high school students a test with linear and non-linear problems. The students were very successful in solving linear problems, but their achievement in non-linear problems was very poor. After two days, half of the students (control group) solved the same test they had on the first day, while the experimental group were first given feedback regarding their performance in the first three problems from the test. If their solutions were different from the correct one, they were given the opportunity to determine their errors and solve these problems again. After that, they solved the whole test again. The students who were given feedback solved non-linear problems better than the control group; however they had weaker results in linear problems.

De Bock, Verschaffel, and Janssens (2002) included metacognitive and visual scaffolds to area problems presented to students who were 12-13 and 15-16 years old. The scaffolds were aimed at arousing students' doubts about the appropriateness of the linear model and at helping them to find the appropriate mathematical model. There were significant effects of scaffolds on students' performance on the area problems. However, similarly to the results of previous studies, students' performance on linear problems decreased because they started to question the correctness of the linear model even when it was appropriate.

There are many more studies that examine the overgeneralization of linear models among elementary and high school students than those that examine the same thing among university students, although several qualitative studies indicate that they are also very susceptible to it (Esteley, Villarreal, & Alagia, 2004, 2010; Villarreal, Esteley, & Alagia, 2006). Thus, we were interested in finding out how strong will the predominance of linear thinking be for university students, who are older and more mathematically proficient than elementary and high school students. Also, most of the previous studies used only area problems, or if the list of the problems comprised of area and volume problems, the achievement in them was not compared.

Thus, the aim of this study was to examine how the different types of help (visual, metacognitive, and the combination of these two scaffolds), provided to university students, influence their achievement in area and volume problems involving the linear enlargement or reduction of figures. Regarding the type of help, we hypothesized that the group of students which received both metacognitive and visual scaffold would have the best performance in both types of problems, while the control group would have the worst performance.

Our second hypothesis regarded the types of problems that students solved. The relations in volume problems are more complex than in area problems, because they include three dimensions – height, length, and width, while area problems include only two dimensions – height and length. Therefore, we hypothesized that all groups of participants would be more successful in solving area problems than volume problems, although both kinds of problems were presented in the similar form and had numbers of the same order of magnitude.

METHOD

Participants

The sample in the preliminary study consisted of 24 graduate psychology students from the University of Zagreb (19 female and 5 male students). The sample in the main study consisted of 122 undergraduate and graduate psychology students (100 female and 22 male students). Their age was from 19 to 36 (M = 21.6 years).

Materials

The students in the preliminary study solved 17 word problems, three of which were linear problems, seven were area problems, and seven were volume problems. Area and volume problems involved regular figures. Half of the participants solved form A, and the other half form B, which were different only in the sequence of problems. The participants were asked to write down the solution to every problem, as well as the procedure they used to solve the problem.

After the analysis of student's answers, we excluded four non-linear problems from the list, which were too easy, too difficult, or unclear to participants. The final list of the problems, which was given to the participants in the main study, consisted of 13 word problems: three linear problems, five area problems, and five volume problems. We used the linear problems so that the participants wouldn't be focused only on the solving of non-linear problems, which could make the task easier for them. Again, two different sequences of problems were used, and in each of them the problems were ordered randomly.

Here are the examples of the three kinds of problems that were used:

Linear problem: A worker makes 10 sandwiches during the period of half an hour. How many sandwiches will he make during 8 hours if he works at the same speed?

Area problem: The restaurant has two dinning halls where dinner parties are organized. Both are square shaped, but the larger one has sides three times longer than the smaller one. If the optimal number of guests in the smaller dinning hall is 100, what is the optimal number of guests in the larger one?

Volume problem: In his box with toys Ivan has cubes of different sizes, which are made from the same material. The side of the smallest cube is 10 mm and its weight is 1 gram. What is the weight of the largest cube, with 30 mm sides?

Procedure

The preliminary study was conducted in a group. The participants needed around 35 minutes to solve the problems.

In the main study, students were divided into four groups of 30 or 31 participants. For practical reasons, we offered them the possibility to choose one of the four different times during the same day when they could participate in the study, taking into account their other faculty obligations. Three of the groups were experimental groups, and one was the control group. We randomly determined which groups were experimental and which one was the control group. All participants were told not to talk during that day with the students from the other groups about the problems they solved, so that they wouldn't think about the problems in advance.

Similar to De Bock, Verschaffel, and Janssens (2002), the experimental groups differed from each other in scaffolds given to participants before solving of the problems. There were three kinds of scaffolds: metacognitive, visual, and the combination of metacognitive and visual.

In the group with metacognitive scaffold, prior to the solving of problems the students were given an introductory task with one area problem and one volume problem. They had to write down the solution to each problem and then turn the second page and read two different solution strategies for each problem presented as answers of two fictitious students. One solution represented incorrect, linear thinking, while the other represented correct, non-linear thinking. The participants had to select the correct answer. This confrontation with two alternative solutions was expected to provoke a cognitive conflict (De Bock, Verschaffel, & Janssens, 2002). After the introductory task, the experimenter shortly discussed the correct answers with the group to make sure they all understood them and told them to proceed with solving the list of problems.

The problems in the introductory task and the answers of fictitious students were as follows:

Problem 1: 50 apple trees can grow in a square-shaped orchard with sides 20 meters long. How many apple trees can grow in a square-shaped orchard with sides of 200 meters?

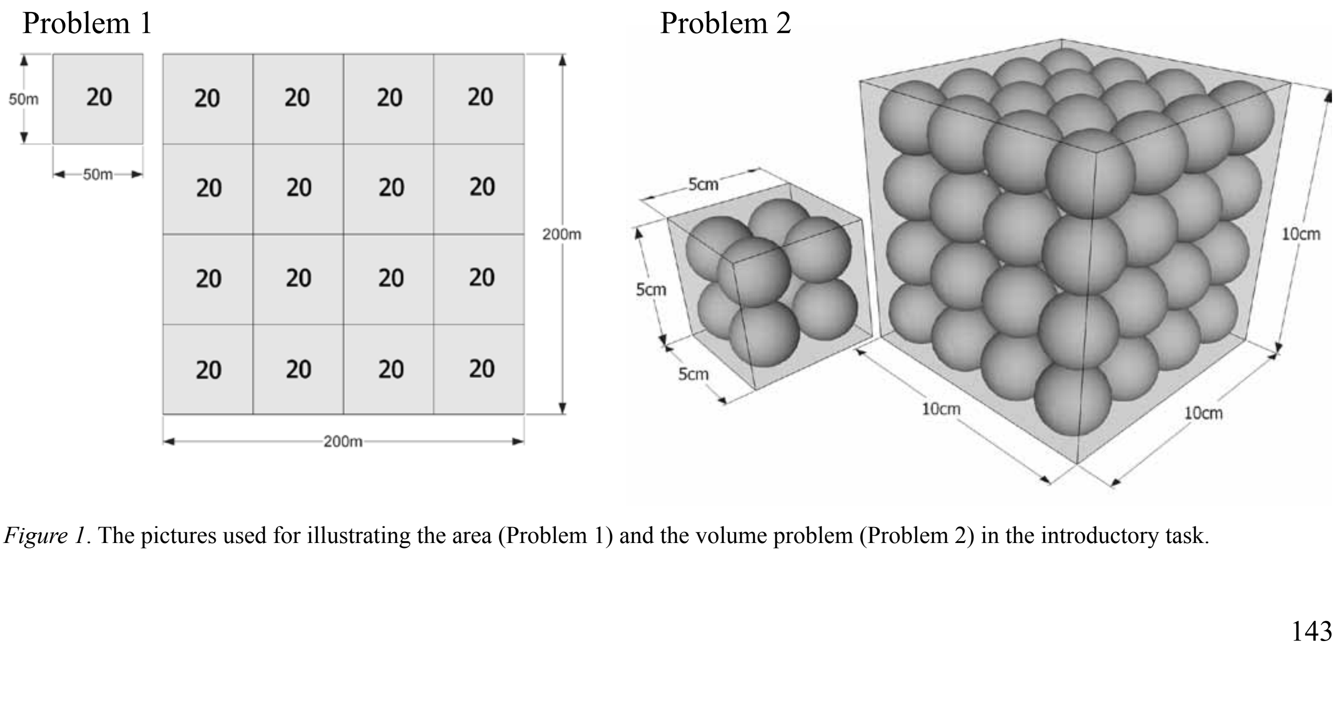

Figure 1. The pictures used for illustrating the area (Problem 1) and the volume problem (Problem 2) in the introductory task.

Problem 2: A cube-shaped box with a side of 5 cm is filled with 8 balls. How many balls of the same size can be put in a cube-shaped box with a side of 10 cm?

Two students, Ivana and Marija, solved both problems. Please, read their solutions for each of the problems and select the one you agree with.

Problem 1:

a) Ivana says: “An orchard with the side of 200 meters has the side four times longer than the orchard the side of which is 50 meters. Thus, I have to multiply the number of apple trees by 4, so the answer is: 20 × 4 = 80 apple trees.”

b) Marija says: “A square with the side of 200 meters can contain 4 × 4 squares with the side of 50 meters. Thus, I have to multiply the number of apple trees by 16, so the answer is: 20 × 16 = 320 apple trees.”

Problem 2:

a) Ivana says: “A box with the side of 10 cm has a side twice longer than a box with the side of 5 cm. Thus, I have to multiply the number of balls by 2, so the answer is: 8 × 2 = 16 balls.”

b) Marija says: “A cube with the side of 10 cm can contain 8 cubes with sides of 5 cm. Thus, I have to multiply the number of balls by 8, so the answer is: 8 × 8 = 64 balls.”

In the group with visual scaffold, both introductory problems came with the drawing of the problem situation (Figure 1). The students had to solve both introductory problems and then proceed with solving the list of problems.

Finally, in the group with both metacognitive and visual scaffolds, both kinds of help were combined. The pictures shown in Figure 1 accompanied answers of fictitious students. In all of the groups, the testing lasted between 40 and 50 minutes.

RESULTS

Prior to the analysis of the differences between experimental groups in the achievement in problems, we examined whether there were differences between the groups taking into account the number of students who attended mathematics-program secondary schools or secondary technical schools (which both have more mathematics classes in a week and deal with more complex mathematics materials than general-program or languages-program secondary schools) and students’ average grades in mathematics. Each participant’s grade in mathematics was calculated as the average of four grades in mathematics in high school. In every group, there were between four and seven students who attended mathematics-program secondary schools or secondary technical schools, and the average mathematics grade in different groups ranged between 3.70 and 4.00. The results showed that there were no differences between the groups in the number of students who attended mathematics-program or technical schools ($\chi^2(3) = 1.03$, $p = .79$) nor in average math grades ($F(3) = 1.04$, $p = .38$), which shows that neither of the groups had the initial advantage in the knowledge of mathematics, therefore the groups were comparable before the solving of problems.

Table 1
Mean scores and standard deviations in linear problems for different experimental groups

Group	*M*	*SD*	*N*
Visual scaffold	2.87	0.34	31
Metacognitive scaffold	2.87	0.35	30
Visual + metacognitive scaffold	2.97	0.18	31
Control group	2.83	0.38	30
Total	2.89	0.32	122

Besides non-linear problems, the participants also solved three linear problems. The average numbers of correctly solved linear problems are presented in Table 1. There were no differences between different groups ($F(3) = 1.03$, $p = .39$).

All of the participants solved five area and five volume problems. Table 2 shows descriptive data for the correctly solved area and volume problems in every group. We used mixed-model ANOVA to examine whether there were differences in students’ achievement in problems regarding the type of problems and the type of help provided. The main effect of the type of problems was significant, $F(1/118) = 40.28$, $p < .001$, $\eta_p^2 = .25$. The participants were more successful in solving area problems than volume problems. The main effect of the type of help was also significant, $F(3/118) = 10.12$, $p < .001$, $\eta_p^2 = .21$. Scheffé’s post-hoc tests showed that the results of the control group were significantly different from the other groups’ results ($p < .01$). The control group had the worst results, while there were no differences between other groups ($p > .05$).

The interaction between the type of problems and the type of help was also statistically significant, $F(3/118) = 3.05$, $p < .05$, $\eta_p^2 = .07$. Namely, the difference between the

Table 2
Mean scores and standard deviations in non-linear problems for different experimental groups

		Area problems		Volume problems	
Group	*N*	*M*	*SD*	*M*	*SD*
Visual scaffold	31	2.48	1.59	1.87	1.43
Metacognitive scaffold	30	2.37	1.47	1.70	1.26
Visual + metacognitive scaffold	31	3.32	1.16	2.10	1.45
Control group	30	1.10	1.49	0.80	1.42
Total	122	2.33	1.63	1.62	1.46

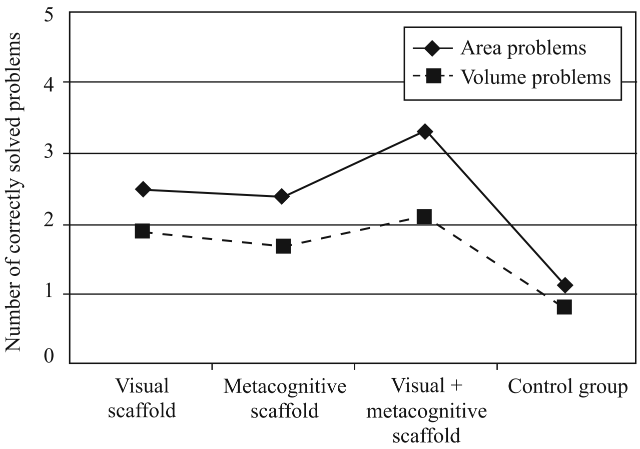

Figure 2. The interaction between the type of problems and the type of help provided to participants.

achievement in area and volume problems was not statistically significant only in the control group ($t(29) = 1.66$, $p = .107$). Figure 2 shows this interaction, where it can also be seen that the greatest difference in solving area and volume problems was in the group that had both visual and metacognitive scaffolds.

DISCUSSION

The aim of this study was to examine how different types of scaffolds provided to university students influence their achievement in mathematical problems which involve the enlargement or reduction of geometrical figures. The results show that metacognitive and visual scaffolds enhanced students' performance in volume and area problems. We hypothesized that the group of students who received both metacognitive and visual scaffolds will have the best performance in both types of problems. The control group had the worst results in both types of problems; however there were no differences between experimental groups. De Bock, Verschaffel, and Janssens (2002) who used metacognitive and visual scaffolds with 12-13 and 15-16 year old students also did not find an additive effect of these scaffolds. However, in our study the largest difference in solving area and volume problems was in the group that had both visual and metacognitive scaffolds.

We also hypothesized that students in all groups will have better achievement in area problems than in volume problems, since volume problems include more complex, three-dimensional objects. Indeed, the students in all experimental groups solved area problems better than volume problems; however there were no differences in the control group between the achievement in these two types of problems. Obviously, students in the control group were very susceptible to the illusion of linearity and their achievement in both area and volume problems was very low. The results of our sample of university students are in this respect very similar to the results of primary and secondary students from previous studies, who were also prone to this illusion (e.g., Van Dooren et al., 2004; Vlahović-Štetić et al., 2010; Vlahović-Štetić & Zekić, 2004). Also, consistent with the findings of previous studies, our intervention reduced the illusion of linearity only to some extent.

In the study performed by De Bock, Verschaffel, and Janssens (2002), the metacognitive scaffold referred to volume, while the non-linear test items referred to area. Therefore, the students' task somewhat differed from what they encountered in the scaffold, and the scaffold only alarmed them that not all problems are linear. In this study, the scaffolds consisted of both area and volume problems, which were kinds of problems used in the test. Because of that, the scaffolds we used were even more direct in hinting how the problems should be solved, however the students were still very susceptible to the illusion of linearity. There were no differences between visual and metacognitive scaffolds in reducing the illusion of linearity. The students obviously needed to be reminded in some way how the problems should be solved, and metacognitive and visual scaffold were probably equally straightforward and clear to students for that purpose.

Our findings also suggest that in studies which try to find factors that are helping students to overcome the illusion in linearity the results in area and volume problems should be treated separately, because the effect of intervention may be different for these types of problems. In our study visual and metacognitive scaffolds helped students to overcome the illusion of linearity to some extent, however it should be noted that for younger students using these strategies may not be that useful or may require more time. For example, in the study conducted by Kalogirou, Kattou, Thanasia, & Gagatsis (2009) sixth grade elementary school students solved stereometry problems even worse when they were accompanied with informative pictures and nets. De Bock, Verschaffel, and Janssens (2002) showed that there was a small but significant positive effect of the visual scaffold, but the metacognitive scaffold helped only 15-16-year-olds.

It should also be noted that the problems we used in this study involved only regular figures. Thus, it is possible that our participants would be less successful if the materials contained irregular figures also (see De Bock, Van Dooren, et al., 2002).

This study was conducted in just one point in time, therefore it would be interesting to examine the effects of longer interventions. Van Dooren et al. (2004) conducted a teaching experiment aimed at remedying 13-14 year old students' illusion of linearity. One class had 10 experimental lessons within a two-week period, while the other class was a control group. The experimental group's results in non-linear problems significantly improved, and that improvement persisted over several months. However, the authors reported that the progress was not as high as they had hoped, and that the number of linear problems solved correctly decreased. They concluded that a teaching intervention should be conducted over a longer period of time and that it should be focused much more on intentional learning. Some of the methods that can be used in such a longer intervention with university students or older secondary school students may include various visual and metacognitive scaffolds similar to those we used in this study. Our results show that reducing the illusion of linearity in problems that include relationships between the lengths and the volume of similarly enlarged or reduced figures is even harder than in problems which include relationships between the lengths and the area. Thus, in future interventions, more attention should be given to volume problems. For example, students should solve more kinds of volume problems with visual and metacognitive scaffolds (in this study, only one example for each kind of the problem was used) and then have more practice in solving these kinds of problems during a longer period of time.

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