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Uniform distribution of sequences involving divisor function

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Abstract. We modify the recent method of J.-M. Deshouillers and H. Iwaniec in the theory of uniform distribution to show that the sequence with general term $a_n = \frac{1}{n}\sum_{m\leq n}\sigma(m)$ is uniformly distributed modulo 1. We also study uniform distribution modulo 1 of some sequences related by the functions σ and φ.

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1. Introduction

In 2008, J.-M. Deshouillers and H. Iwaniec [1] introduced a method for studying uniform distribution modulo 1 of means of some certain Euler function - type sequences. Their method implies uniform distribution modulo 1 of the sequence $(\frac{1}{n}\sum_{m\leq n}\varphi(m))_{n\geq 1}$, where φ is the Euler function. Our main goal in this paper is to prove a similar result for the sequence with general term

$$a_n = \frac{1}{n}\sum_{m\leq n}\sigma(m), \qquad (1)$$

where $\sigma(m) = \sum_{d|m} d$ is the sum of positive divisors of m. More precisely, we show the following result.

Theorem 1. *The sequence $(a_n)_{n\geq 1}$ with a general term defined by (1) is uniformly distributed modulo 1.*

In comparison with the Euler function, we note that $\varphi(m)/m$ is strongly multiplicative, but $\sigma(m)/m$ is not. Also, in the case of the Euler function, because of the connection to the Möbius μ function, Deshouillers and Iwaniec use the prime number theorem with error term, but for the σ function we do not need such tools, albeit the method of Deshouillers and Iwaniec is applicable in this case, too.

Before starting the proof of Theorem 1, let us explain non-triviality of its truth. We write

$$a_n = \frac{\pi^2}{12}n + R(n).$$

If we may reduce the remainder term $R(n)$ up to $o(1)$, then uniform distribution mod-

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ulo 1 of a_n becomes trivial by means of Weyl criterion [4]. But, the best known [3] approximation for $R(n)$ is $O(\log^{2/3} n)$, and it is known [2] that $R(n) \neq o(\log\log n)$. Therefore, we need a careful analysis on remainder term $R(n)$.

2. Analysis of the remainder term

Let ψ be the saw function defined by $\psi(x) = \{x\} - 1/2$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. For real $z \geq 2$, we set

$$P(z) = \prod_{p<z} p^{\lfloor (\log z)/(\log p) \rfloor},$$

which will be simply also shown by P. We take D with $P < D < n$ and we let

$$\rho_n(z) = \sum_{d|P} \frac{1}{d}\psi\Big(\frac{n}{d}\Big), \tag{2}$$

and

$$\rho_n(D, z) = \sum_{d \leq D} \frac{1}{d}\psi\Big(\frac{n}{d}\Big) - \rho_n(z). \tag{3}$$

Also, we take

$$\alpha = \frac{\pi^2}{12}.$$

Lemma 1. *We have*

$$a_n = \alpha n - \rho_n(z) - \rho_n(D, z) - \frac{1}{2} + O\Big(\frac{D}{n} + \frac{n^2}{D^3}\Big). \tag{4}$$

Proof. We write

$$a_n = \frac{1}{n}\sum_{ad \leq n} a = \frac{1}{n}\sum_{d \leq D}\sum_{a \leq \frac{n}{d}} a + \frac{1}{n}\sum_{a \leq \frac{n}{D}}\sum_{D<d \leq \frac{n}{a}} a := A_1 + A_2,$$

say. We have

$$\begin{aligned} A_1 = \frac{1}{n}\sum_{d \leq D}\sum_{a \leq \frac{n}{d}} a &= \frac{1}{2n}\sum_{d \leq D} \Big\lfloor \frac{n}{d} \Big\rfloor \Big(\Big\lfloor \frac{n}{d} \Big\rfloor + 1\Big) \\ &= \frac{1}{2n}\sum_{d \leq D} \Big(\frac{n}{d} - \psi\Big(\frac{n}{d}\Big) - \frac{1}{2}\Big)\Big(\frac{n}{d} - \psi\Big(\frac{n}{d}\Big) + \frac{1}{2}\Big) \\ &= \frac{n}{2}\sum_{d \leq D}\frac{1}{d^2} - \sum_{d \leq D}\frac{1}{d}\psi\Big(\frac{n}{d}\Big) + O\Big(\frac{D}{n}\Big). \end{aligned}$$

Using the Euler–Maclaurin summation formula, we have

$$-\frac{n}{2}\sum_{d>D}\frac{1}{d^2} = -\frac{n}{2D} + O\Big(\frac{n}{D^2}\Big).$$

Thus

$$A_1 = \alpha n - \rho_n(z) - \rho_n(D,z) - \frac{n}{2D} + O\Big(\frac{D}{n} + \frac{n}{D^2}\Big).$$

To approximate A_2, first we change the order of summation to get

$$A_2 = \frac{1}{n} \sum_{D<d\le n} \sum_{a\le \frac{n}{d}} a.$$

Now, we observe that since we do not take D small, thus d is large and there are many values of d for which $\lfloor \frac{n}{d} \rfloor$ takes the same values. Let indeed k be an integer. The inequality $\frac{n}{k+1} < d \le \frac{n}{k}$ holds if and only if $\lfloor \frac{n}{d} \rfloor = k$. We let $K = \lfloor \frac{n}{D} \rfloor$, and we consider the following splitting

$$(D,n] = \Big(D, \frac{n}{K}\Big] \cup \Big(\frac{n}{K}, \frac{n}{K-1}\Big] \cup \cdots \cup \Big(\frac{n}{2}, n\Big],$$

to write

$$A_2 = \frac{1}{n} \sum_{D<d\le \frac{n}{K}} \sum_{a\le K} a + \frac{1}{n} \sum_{k=1}^{K-1} \sum_{\frac{n}{k+1}<d\le \frac{n}{k}} \sum_{a\le k} a := A_2' + A_2'',$$

say. We have

$$\begin{aligned} A_2' &= \frac{1}{n} \sum_{D<d\le \frac{n}{K}} \sum_{a\le K} a = \frac{K(K+1)}{2n}\Big(\Big\lfloor \frac{n}{K} \Big\rfloor - \lfloor D \rfloor\Big) \\ &= \frac{K(K+1)}{2n}\Big(\frac{n}{K} - D + O(1)\Big) = \frac{K+1}{2} - \frac{K(K+1)D}{2n} + O\Big(\frac{K^2}{n}\Big). \end{aligned}$$

Since $\frac{K^2}{n} \le \frac{n}{D^2} < \frac{n^2}{D^3}$, and also, since $K = \lfloor \frac{n}{D} \rfloor = \frac{n}{D} - \{\frac{n}{D}\}$, we finally have

$$A_2' = \frac{1}{2}\Big\{\frac{n}{D}\Big\} + O\Big(\frac{D}{n} + \frac{n^2}{D^3}\Big).$$

We now treat the second term in A_2. We have

$$\begin{aligned} A_2'' = \frac{1}{n} \sum_{k=1}^{K-1} \sum_{\frac{n}{k+1}<d\le \frac{n}{k}} \sum_{a\le k} a &= \frac{1}{2n} \sum_{k=1}^{K-1} k(k+1)\Big(\Big\lfloor \frac{n}{k} \Big\rfloor - \Big\lfloor \frac{n}{k+1} \Big\rfloor\Big) \\ &= \frac{K-1}{2} + O\Big(\frac{n^2}{D^3}\Big) \\ &= \frac{n}{2D} - \frac{1}{2}\Big\{\frac{n}{D}\Big\} - \frac{1}{2} + O\Big(\frac{n^2}{D^3}\Big). \end{aligned}$$

This gives

$$A_2 = \frac{n}{2D} - \frac{1}{2} + O\Big(\frac{D}{n} + \frac{n^2}{D^3}\Big).$$

We put approximations of A_1 and A_2 together to obtain (4), hence completing the proof. □

3. Proof of Theorem 1

Recall that $e(x) = \mathrm{e}^{2\pi i x}$. We use Weyl criterion to prove uniform distribution modulo 1 of the sequence a_n. Indeed, we show validity of

$$\sum_{n\le X} e(ha_n) = o(X), \qquad \text{as } X\to\infty, \tag{5}$$

for any positive integer h. We choose arbitrary small constant $\varepsilon > 0$, and we consider the trivial estimate

$$\Big|\sum_{n\le X} e(ha_n)\Big| \le \varepsilon X + \Big|\sum_{\varepsilon X<n\le X} e(ha_n)\Big|,$$

from which we observe that to get (5), it is enough to prove

$$\Big|\sum_{\varepsilon X<n\le X} e(ha_n)\Big| \le \varepsilon X,$$

for X sufficiently large. To do this, we write

$$\sum_{\varepsilon X<n\le X} e(ha_n) = S(\varepsilon, z; X) + E, \tag{6}$$

where

$$S(\varepsilon, z; X) = \sum_{\varepsilon X<n\le X} e\Big(h\Big(\alpha n - \rho_n(z) - \frac{1}{2}\Big)\Big),$$

and

$$E \ll \sum_{\varepsilon X<n\le X}\Big\{|\rho_n(D,z)| + \frac{D}{n} + \frac{n^2}{D^3}\Big\} \ll \sum_{n\le X}|\rho_n(D,z)| + D\log X + X^3D^{-3}.$$

We estimate $\sum_{n\le X}|\rho_n(D,z)|$ by using Lemma 5 of [1], which is an important part of the method of Deshouillers and Iwaniec and asserts that if $\tau_\nu(d)$ denote the generalized divisor function, then for any complex number $c(d)$ with

$$|c(d)| \le \tau_\nu(d), \tag{7}$$

we have

$$\sum_{|n|\le T}\Big|\sum_{z\le d\le D}\frac{c(d)}{d}\psi\Big(\frac{n}{d}\Big)\Big|^2 \ll_\nu Tz^{-1}(\log z)^{B(\nu)} + D(\log D)^{B(\nu)}, \tag{8}$$

where the symbol $B(\nu)$ denotes a constant that depends only on the number ν, the value which may change from one occurrence to the other one. As an immediate and useful consequence of (8), we use the Cauchy's inequality to obtain

$$\sum_{|n|\le T}\Big|\sum_{z\le d\le D}\frac{c(d)}{d}\psi\Big(\frac{n}{d}\Big)\Big| \ll_\nu Tz^{-\frac{1}{2}}(\log z)^{B(\nu)} + \sqrt{TD}(\log D)^{B(\nu)}. \tag{9}$$

We use (9) by taking

$$c(d) = \begin{cases} 0 & \text{if } d|P(z), \\ 1 & \text{if } d\nmid P(z). \end{cases} \tag{10}$$

Let us verify that $c(d)$ defined by (10) satisfies the required condition (7). For $d < z$ we let $d = \prod_p p^{v_p(d)}$, where $p^{v_p(d)}\|d$, and so $p^{v_p(d)} \le d < z$. Thus $v_p(d) < (\log z)/(\log p)$, and we imply that $d|P(z)$. This gives $c(d) = 0$ for $d < z$. On the other hand, for $z \le d \le D$ we have $|c(d)| \le 1$. Therefore, we may apply (9) to obtain

$$\sum_{n\le X} |\rho_n(D,z)| \ll Xz^{-\frac{1}{2}}(\log z)^{B_1} + \sqrt{XD}(\log X)^{B_2},$$

for some real numbers B_1 and B_2, and for $D < X$. Thus, we get

$$E \ll Xz^{-\frac{1}{2}}(\log z)^{B_1} + \sqrt{XD}(\log X)^{B_2} + D\log X + X^3D^{-3}.$$

We put $2 \le z \le \log X$ and we take $D = X(\log X)^{-c}$, where c is some positive constant satisfying the condition $\min\{-c/2 + B_2, 1 - c\} < -1/2$. Now, we have $D < n$ provided X is sufficiently large and $\varepsilon X < n \le X$. Considering all of these, implies that

$$E \ll Xz^{-\frac{1}{2}}(\log z)^{B_1} \ll Xz^{-\frac{1}{3}}. \tag{11}$$

To approximate $S(\varepsilon, z; X)$ we use the irrationality of α and the fact that $\rho_n(z)$ defined by (2) is periodic in n with period P, hence we obtain

$$\begin{aligned} |S(\varepsilon,z;X)| &= \Big|\sum_{b=0}^{P-1} \sum_{\substack{\varepsilon X<n\le X\\ n\equiv b\ [P]}} e\Big(h\big(\alpha n - \rho_n(z) - \frac{1}{2}\big)\Big)\Big| \\ &\le \sum_{b=0}^{P-1}\Big|\sum_{\substack{\varepsilon X<n\le X\\ n\equiv b\ [P]}} e\left(h\alpha n\right)\Big| = \sum_{b=0}^{P-1}\Big|\sum_{\frac{\varepsilon X-b}{P}<k\le\frac{X-b}{P}} e\left(h\alpha(b+kP)\right)\Big|. \end{aligned}$$

The inner sum is indeed a geometric sum, which is bounded by $\frac{2}{|e(h\alpha P)-1|}$. Thus, by considering the identity $|e(x) - 1| = 2|\sin(\pi x)|$, we get

$$|S(\varepsilon,z;X)| \le \frac{P}{|\sin(h\alpha P\pi)|}. \tag{12}$$

Now, we consider (6) and use approximations (11) and (12) to obtain the estimate

$$\Big|\sum_{\varepsilon X<n\le X} e(ha_n)\Big| \le \frac{P}{|\sin(h\alpha P\pi)|} + R(z;X),$$

with $R(z;X) = O(Xz^{-\frac{1}{3}})$. We choose z to depend only on ε in such a way that

$$|R(z;X)| \le \frac{\varepsilon}{2}X.$$

Moreover, we imply that $P(z)$ is dependent only on ε, and we can find $X_0 = X_0(h,\varepsilon)$ such that

$$\frac{P}{|\sin(h\alpha P\pi)|} \le \frac{\varepsilon}{2}X, \qquad \text{for } X > X_0.$$

By combining the last two estimates we complete the proof of Theorem 1.

4. Some remarks

Remark 1. *For any real number η, the sequence $(a_\eta(n))_n$ defined by*

$$a_\eta(n) = n^{1-2\eta}\Big(\sum_{m\le n}\sigma(m)\Big)^\eta$$

is uniformly distributed modulo 1, provided the number α^η is irrational.

Proof. We write $a_n = \alpha n + R(n)$. So, we have

$$\sum_{m\le n}\sigma(m) = na_n = \alpha n^2\left(1+\frac{R(n)}{\alpha n}\right).$$

This implies that

$$\begin{aligned} a_\eta(n) &= \alpha^\eta n\left(1+\frac{R(n)}{\alpha n}\right)^\eta \\ &= \alpha^\eta n\left(1+\frac{\eta R(n)}{\alpha n}+O\Big(\frac{\log^2 n}{n^2}\Big)\right) \\ &= \alpha^\eta n + \eta\alpha^{\eta-1}R(n) + O\Big(\frac{\log^2 n}{n}\Big). \end{aligned}$$

We apply the truth of Lemma 1 to get

$$a_\eta(n) = \alpha^\eta n - \eta\alpha^{\eta-1}\rho_n(z) - \eta\alpha^{\eta-1}\rho_n(D,z) - \frac{\eta\alpha^{\eta-1}}{2} + O\Big(\frac{D}{n}+\frac{n^2}{D^3}+\frac{\log^2 n}{n}\Big).$$

Using this relation and the following similar argument as in the proof of Theorem 1, we obtain $\sum_{n\le X} e(ha_\eta(n)) = o(X)$ as $X\to\infty$, for any positive integer h. This completes the proof. □

Corollary 1. *Sequences with general terms*

$$s_n = \sqrt{\sum_{m\le n}\sigma(m)}, \qquad w_n = \frac{n^2}{s_n}, \qquad r_n = \frac{n^2}{a_n},$$

are uniformly distributed modulo 1.

Proof. We apply the truth of Remark 1 by taking $\eta = \frac{1}{2}, -\frac{1}{2}$ and -1, respectively, and we note that α^η is irrational in each case. □

We easily obtain the following similar results for the Euler function in the work of J.-M. Deshouillers and H. Iwaniec [1].

Remark 2. *Let $\beta = 3/\pi^2$. Then, for any real number η, the sequence with general term*

$$b_\eta(n) = n^{1-2\eta}\Big(\sum_{m\le n}\varphi(m)\Big)^\eta,$$

is uniformly distributed modulo 1, provided the number β^η is irrational.

Corollary 2. *Let* $b_n = \frac{1}{n}\sum_{m\le n}\varphi(m)$. *Sequences with general terms*

$$s_n = \sqrt{\sum_{m\le n}\varphi(m)}, \qquad w_n = \frac{n^2}{s_n}, \qquad r_n = \frac{n^2}{b_n},$$

are uniformly distributed modulo 1.

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