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Ultrametrization of pro*-morphism sets

Nikica Uglešić[1,*]

[1] *University of Zadar, Pavlinovićeva bb, HR-23 000 Zadar, Croatia*

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Abstract. For every pair of inverse systems $\boldsymbol{X}$, $\boldsymbol{Y}$ in a category $\mathcal{A}$, where $\boldsymbol{Y}$ is cofinite, there exists a complete ultrametric structure on the set pro^*-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. The corresponding hom-bifunctor is the internal and invariant Hom of a subcategory, containing tow^*-$\mathcal{A}$, in the category of complete metric spaces. Several applications to the shapes (ordinary, coarse and weak) are considered.

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1. Introduction

In seeking a natural structure of the shape morphism sets ([3, 4, 13 − 17, 20, 21] and some others) it has become clear that, in general, there is no unique topological structure on those sets. The original idea was to consider the shape morphisms as certain classes of Cauchy sequences, i.e., to obtain the shape as a Cantor completion process. However, their starting point was *not* (except [20, 21]) a (pseudo)metric on a pro-morphism set. Although not unique, the obtained (ultra)metric and topological structures on the shape morphism sets yield some interesting and useful results. For instance, they permit relations between rather distant theories and the shape theory. Further, they admit constructions of some new shape invariants, in addition to simpler expressions of the old ones by means of the new technique.

In this paper (similarly to the previous two, [20, 21]) we obtain, by metric tools, a better view into coarse and weak shape type classifications, especially for metrizable compacta. Our starting point is a naturally existing countable decreasing family of equivalence relations on a set inv^*-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$, where $\mathcal{A}$ is an arbitrary category. It induces a pseudoultrametric whenever the codomain inverse system $\boldsymbol{Y}$ is cofinite, [22]. By passing to the quotient set pro^*-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ we obtain the ultrametric space $(pro^*\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y}), d^*)$, denoted by $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$, which is complete (Theorem 1). Moreover, this metric structure is an extension of the known one on pro-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$, [20], such that the canonical injection of pro-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ into pro^*-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ is an isometric closed embedding of $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ into $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ (Theorem 2).

Further, we consider the hom-bifunctor

$$\hom : (pro^*\text{-}\mathcal{A})^{op} \times (pro^*\text{-}\mathcal{A}) \to Set,$$

*Corresponding author. *Email address:* nuglesic@unizd.hr (N. Uglešić)

and establish the sufficient condition for hom to be an internal *Hom*, i.e., to be continuous with respect to the category M_c of complete metric spaces (Lemma 4 and Theorem 4). Especially, hom is (uniformly) continuous for inverse sequences (Corollary 2), i.e., there exists

$$Hom : (tow\text{-}\mathcal{A})^{op} \times (tow\text{-}\mathcal{A}) \to M_c.$$

Furthermore, we have found the sufficient condition for hom to be invariant (Theorem 5). Especially, hom, i.e., *Hom* is invariant (and uniformly continuous) for all inverse sequences (Theorem 6).

At the end, in Section 5, there follows several applications of the new results and technique to get a better view into classifications by shapes, especially, by the coarse shape. We define the coarse equivalence and uniform coarse equivalence (as the analogues and improvements of the Borsuk quasi-equivalence, [1, 2, 19]) - both of which are coarser than the coarse shape type. At the very end we have constructed a new complete ultrametric structure on a set $pro^{\sim}_{*}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ which is naturally comparable to that on $pro^{*}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. It is obtained by slightly changing the known one of [21]. Namely, the old one was incomparable to that on $pro\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ as well as to that constructed hereby on $pro^{*}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. In the new setting the injections of pro-sets induce the isometric closed embeddings of the spaces $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ into $(\boldsymbol{Y}^{\boldsymbol{X}}_{*}, d_*)$ (Theorem 11). Since $pro^{\sim}_{*}\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ is a realizing set for the weak shape morphisms, we may conclude that in some categories, especially for compact metrizable spaces, there exist canonical complete ultrametric structures on the shape, coarse shape and weak shape morphism sets, $Sh(X,Y)$, $Sh^*(X,Y)$ and $Sh_*(X,Y)$ respectively, such that the natural (functorial) injections

$$Sh(X,Y) \to Sh^*(X,Y) \to Sh_*(X,Y)$$

are isometric closed embeddings (Corollary 8).

2. Ultrametric on a pro*-morphism set

First of all, recall the complete ultrametric structure on a set $pro\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$, [20]. Let $\mathcal{A}$ be a category (our category terminology is based on [6]) and let $inv\text{-}\mathcal{A}$ be the corresponding inv-category of $\mathcal{A}$, [11], i.e., the objects of $inv\text{-}\mathcal{A}$ are all the inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{A}$, and $inv\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ is the set of all morphisms

$$(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y} = (\boldsymbol{Y}_\mu, q_{\mu\mu'}, M),$$

defined by the following condition

$$(\forall \mu \le \mu')(\exists \lambda \ge f(\mu)) \quad f(\mu') f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

The composition is defined by $(g, g_\nu)(f, f_\mu) = (fg, g_\nu f_{g(\nu)})$, and the identity on an $\boldsymbol{X}$ is $(1_\Lambda, 1_{X_\lambda})$.

Two morphisms $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ of $inv\text{-}\mathcal{A}$ are said to be equivalent (homotopic), denoted by $(f, f_\mu) \simeq (f', f'_\mu)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \ge f(\mu), f'(\mu)$, such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

This relation is an equivalence relation that is compatible with composition in *inv*-$\mathcal{A}$. Therefore, there exists the corresponding quotient category (pro-category) *inv*-$\mathcal{A}/(\simeq) \equiv$ *pro*-$\mathcal{A}$. A morphism $[(f, f_\mu)]$ of *pro*-$\mathcal{A}$ is denoted by $\boldsymbol{f}$.

Given two morphisms $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ of *inv*-$\mathcal{A}$, and $\mu \in M$, (f, f_μ) is said to be *μ-homotopic to* (f', f'_μ), denoted by $(f, f_\mu) \simeq_\mu (f', f'_\mu)$, provided there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

Then the relation $\simeq_\mu$ is an equivalence relation on each set *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, $\simeq_{\mu'}$ implies $\simeq_\mu$ whenever $\mu \leq \mu'$, and $\simeq_\mu$ for all $\mu \in M$ is equivalent to $\simeq$ (Lemma 2.2 of [20]). Further, if $(f, f_\mu) \simeq_\mu (f', f'_\mu)$, then $(f, f_\mu)(h, h_\lambda) \simeq_\mu (f', f'_\mu)(h, h_\lambda)$, while $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ implies $(g, g_\nu)(f, f_\mu) \simeq_\nu (g, g_\nu)(f', f'_\mu)$ whenever $g(\nu) \leq \mu$.

Given a $\lambda \in \Lambda$, let $|\lambda|$ denote the cardinal of the set of all the predecessors λ' of λ in Λ, $\lambda' < \lambda$ (i.e., $\lambda' \leq \lambda$ and $\lambda' \neq \lambda$). In the case of a cofinite inverse system (indexing set), for every $\lambda \in \Lambda$, $|\lambda|$ is finite, i.e., $|\lambda| = m - 1$ for some $m \in \mathbb{N}$.

Let $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of *inv*-$\mathcal{A}$, and let κ be a cardinal. Then (f, f_μ) is said to be *κ-homotopic to* (f', f'_μ), denoted by $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$, provided for every $\mu \in M$, such that $|\mu| < \kappa$, $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ holds. Clearly, in the case of a cofinite $\boldsymbol{Y}$, those cardinals (representatives - numbers) κ range over the set of non-negative integers. Furthermore, if $\boldsymbol{Y}$ is an inverse sequence, the relations $\simeq_m$ and $\simeq_\mu$ coincide ($\mu = |\mu| + 1 = m$).

According to Lemma 2.4 of [20], the relation $\simeq_\kappa$ is an equivalence relation on each set *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, $\simeq_{\kappa'}$ implies $\simeq_\kappa$ whenever $\kappa \leq \kappa'$, and, if $\boldsymbol{Y}$ is cofinite, $\simeq$ is equivalent to $\simeq_m$ for all $m \in \mathbb{N}$. Further,

(i) $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$ and $(f', f'_\mu) \simeq_{\kappa'} (f'', f''_\mu)$ imply $(f, f_\mu) \simeq_{\kappa''} (f'', f''_\mu)$, where $\kappa'' = \min\{\kappa, \kappa'\}$;

(ii) $(f, f_\mu) \simeq_\kappa (g, g_\mu)$, $(f', f'_\mu) \simeq_{\kappa'} (g', g'_\mu)$ and $(f, f_\mu) \simeq_\eta (f', f'_\mu)$ imply $(g, g_\mu) \simeq_{\eta'} (g', g'_\mu)$, where $\eta' = \min\{\kappa, \kappa', \eta\}$;

(iii) $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$ implies $(f, f_\mu)(h, h_\lambda) \simeq_\kappa (f', f'_\mu)(h, h_\lambda)$; .

(iv) $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$ implies $(g, g_\nu)(f, f_\mu) \simeq_{\kappa'} (g, g_\nu)(f', f'_\mu)$, provided for every $\nu \in N$ $\mid \nu \mid < \kappa'$ implies $\mid g(\nu) \mid < \kappa$.

Given a pair of inverse systems $\boldsymbol{X}, \boldsymbol{Y}$, where $\boldsymbol{Y}$ is cofinite, the function $\rho :$ *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \times$ *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}$,

$$\rho((f, f_\mu), (f', f'_\mu)) = \begin{cases} \inf\{\frac{1}{m+1} \mid (f, f_\mu) \simeq_m (f', f'_\mu),\ m \in \mathbb{N}\} \\ \qquad 1, \text{ otherwise} \end{cases},$$

defines a pseudoultrametric on the set *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}).\rho)$ (Lemma 2.5 of [20]).

Finally, since ρ is invariant with respect to the relation $\simeq$, the function $d :$ *pro*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \times$ *pro*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}$,

$$d(\boldsymbol{f}, \boldsymbol{f}') = \rho((f, f_\mu), (f', f'_\mu)),$$

where $(f, f_\mu) \in \boldsymbol{f}$, $(f', f'_\mu) \in \boldsymbol{f'}$ is any pair of representatives, is well defined. Let us denote $\boldsymbol{Y}^{\boldsymbol{X}} \equiv pro\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. Then, for every $\boldsymbol{X}$ and every cofinite $\boldsymbol{Y}$, the ordered pair $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is a complete ultrametric space (Theorem 2.6 of [20]).

Consider now a set $pro^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. We firstly recall the definition of a pro*-category (see [8]). In the first step we define the category $inv^*\text{-}\mathcal{A}$. The object class $Ob(inv^*\text{-}\mathcal{A}) = Ob(inv\text{-}\mathcal{A})$, i.e., it consists of all inverse systems in $\mathcal{A}$, while the morphisms are all so called $*$-morphisms. Given a pair of inverse systems in $\mathcal{A}$, $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, a *$*$-morphism* (originally an *S^*-morphism*) of inverse systems, $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, consists of a function $f : M \to \Lambda$ (the *index function*) and, for each $\mu \in M$, of a sequence of $\mathcal{A}$-morphisms $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, such that, for every related pair $\mu \leq \mu'$ in M, there exists a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f(\mu')$, and there exists an $n \in \mathbb{N}$ so that, for every $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}.$$

If $\left(f, f_\mu^n\right) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\nu^n) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$ are $*$-morphisms, then we *compose* them by the rule

$$(g, g_\nu^n)\left(f, f_\mu^n\right) = (h, h_\nu^n)\,,$$

where $h = fg : N \to \Lambda$ and $h_\nu^n = g_\nu^n f_{g(\nu)}^n$, $n \in \mathbb{N}$, $\nu \in N$. One readily verifies that $(h, h_\nu^n) : \boldsymbol{X} \to \boldsymbol{Z}$ is a $*$-morphism and that the composition is associative.
Given an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{A}$, let $\left(1_\Lambda, 1_{X_\lambda}^n\right)$ consist of the identity function 1_Λ and of the identity morphisms $1_{X_\lambda}^n = 1_{X_\lambda}$ of $\mathcal{A}$, for every $n \in \mathbb{N}$ and every $\lambda \in \Lambda$. Then $(1_\Lambda, 1_{X_\lambda}^n) : \boldsymbol{X} \to \boldsymbol{X}$ is the identity $*$-morphism on $\boldsymbol{X}$. In this way, given any category $\mathcal{A}$, the corresponding inv*-category $inv^*\text{-}\mathcal{A}$ is obtained.

Now we define (see [8], Definitions 3.8 and 3.19) that a $*$-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is *equivalent to* a $*$-morphism $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu^n) \sim (f', f_\mu'^n)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, and an $n \in \mathbb{N}$, such that, for every $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation on each set $inv^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. The equivalence class $[(f, f_\mu^n)]$ of a $*$-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$. This equivalence relation is compatible with the composition in $inv^*\text{-}\mathcal{A}$, i.e., if

$$(f, f_\mu^n) \sim (f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y} \quad \text{and} \quad (g, g_\nu^n) \sim (g', g_\nu'^n) : \boldsymbol{Y} \to \boldsymbol{Z},$$

then

$$(g, g_\nu^n)(f, f_\mu^n) \sim (g', g_\nu'^n)(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Z}.$$

Therefore, one may compose the equivalence classes of $*$-morphisms by putting

$$\boldsymbol{g}^* \boldsymbol{f}^* = \boldsymbol{h}^* \equiv [(h, h_\nu^n)],$$

where

$$(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) = (fg, g_\nu^n f_{g(\nu)}^n).$$

Finally, the desired pro*-category pro^*-$\mathcal{A}$ is the corresponding quotient category, i.e.,

$$pro^*\text{-}\mathcal{A} = (inv^*\text{-}\mathcal{A})/_{(\sim)}.$$

Let us add a few words about the corresponding functors, [8]. First, for every category $\mathcal{A}$, there exists a (faithful) functor

$$\underline{J} : pro\text{-}\mathcal{A} \to pro^*\text{-}\mathcal{A}$$

defined by $\underline{J}(\boldsymbol{X}) = \boldsymbol{X}$ and, for an $\boldsymbol{f} = [(f, f_\mu)]$, $\underline{J}(\boldsymbol{f}) = \boldsymbol{f}^* = [(f, f_\mu^n)]$, where, for every $n \in \mathbb{N}$, $f_\mu^n = f_\mu$, for all $\mu \in M$. (Such an $(f, f_\mu^n = f_\mu)$ is called an *induced* or *commutative* $*$-morphism.) Especially, for a pro-reflective subcategory $\mathcal{D} \subseteq \mathcal{C}$ there exists a faithful functor $\underline{J} : pro\text{-}\mathcal{D} \to pro^*\text{-}\mathcal{D}$ described above. It induces a faithful functor

$$J : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})}$$

(relating the shape and coarse shape category), which keeps the objects fixed. Further, if one puts $S^*(X) = X$, $X \in Ob\mathcal{C}$, and $S^*(f) = F^* = \langle \boldsymbol{f}^* \rangle$, $f \in Mor\mathcal{C}$, where $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f})$ and $\boldsymbol{f}$ is induced by f, then

$$S^* : \mathcal{C} \to Sh^*_{(\mathcal{C},\mathcal{D})}$$

becomes a functor, called the (*abstract*) *coarse shape functor*. Its relationship with the (abstract) shape functor $S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$ ([11], I. 2) is given by the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{C} & \\ S \swarrow & & \searrow S^* \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{J}{\to} & Sh^*_{(\mathcal{C},\mathcal{D})} \end{array},$$

where the functor J is faithful keeping the objects fixed ([8], Section 4).

In order to endow a set pro^*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ with a metric structure, we follow the pattern for pro-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$.

Definition 1. *Let $(f, f_\mu^n), (f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of inv^*-$\mathcal{A}$, and let $\mu \in M$. Then (f, f_μ^n) is said to be μ-**homotopic to** $(f', f_\mu'^n)$, denoted by $(f, f_\mu^n) \sim_\mu (f', f_\mu'^n)$, provided there exist a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, and an $n \in \mathbb{N}$, such that for every $n' \geq n$,*

$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}.$$

Lemma 1. *(i) The relation $\sim_\mu$ is an equivalence relation on each set inv^*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$.*

(ii) If $(f, f_\mu^n) \sim_{\mu'} (f', f_\mu'^n)$ and $\mu \leq \mu'$, then $(f, f_\mu^n) \sim_\mu (f', f_\mu'^n)$.

(iii) If $(f, f_\mu^n) \sim_\mu (f', f_\mu'^n)$, then $(f, f_\mu^n)(h, h_\lambda^n) \sim_\mu (f', f_\mu'^n)(h, h_\lambda^n)$.

(iv) If $(f, f_\mu^n) \sim_\mu (f', f_\mu'^n)$, then $(g, g_\nu^n)(f, f_\mu^n) \sim_\nu (g, g_\nu^n)(f', f_\mu'^n)$, whenever $g(\nu) \leq \mu$.

(v) $(f, f_\mu^n) \sim (f', f_\mu'^n)$ if and only if $(f, f_\mu^n) \sim_\mu (f', f_\mu'^n)$ for every $\mu \in M$.

Proof. All the claims obviously follow by the definition. □

Definition 2. *Let $(f, f_\mu^n), (f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of inv***-*$\mathcal{A}$*, and let* κ *be a cardinal. Then* (f, f_μ^n) *is said to be* ***κ-homotopic to*** $(f', f_\mu'^n)$*, denoted by* $(f, f_\mu^n) \sim_\kappa (f', f_\mu'^n)$*, provided* $(f, f_\mu^n) \sim_\mu (f', f_\mu'^n)$ *holds for every* $\mu \in M$ *such that* $|\mu| < \kappa$,.

Clearly, if $\boldsymbol{Y}$ is cofinite, then those cardinals (representatives - numbers) κ range over the set of non-negative integers. Moreover, in the case of an inverse sequence $\boldsymbol{Y}$, the relations $\sim_m$ and $\sim_\mu$ coincide ($\mu = |\mu| + 1 = m \in \mathbb{N}$). It is obvious that Definitions 1 and 2 and Lemma 1 imply the following facts.

Lemma 2. *(i) The relation* $\sim_\kappa$ *is an equivalence relation on each set inv***-*$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$.

(ii) If $(f, f_\mu^n) \sim_{\kappa'} (f', f_\mu'^n)$ *and* $\kappa \leq \kappa'$*, then* $(f, f_\mu^n) \sim_\kappa (f', f_\mu'^n)$.

(iii) If $(f, f_\mu^n) \sim_\kappa (f', f_\mu'^n)$ *and* $(f', f_\mu'^n) \sim_{\kappa'} (f'', f_\mu''^n)$*, then* $(f, f_\mu^n) \sim_{\kappa''} (f'', f_\mu''^n)$*, where* $\kappa'' = \min\{\kappa, \kappa'\}$.

(iv) If $(f, f_\mu^n) \sim_\kappa (g, g_\mu^n)$*,* $(f', f_\mu'^n) \sim_{\kappa'} (g', g_\mu'^n)$ *and* $(f, f_\mu^n) \sim_\eta (f', f_\mu'^n)$*, then* $(g, g_\mu^n) \sim_{\eta'} (g', g_\mu'^n)$*, where* $\eta' = \min\{\kappa, \kappa', \eta\}$.

(v) If $(f, f_\mu^n) \sim_\kappa (f', f_\mu'^n)$*, then* $(f, f_\mu^n)(h, h_\lambda^n) \sim_\kappa (f', f_\mu'^n)(h, h_\lambda^n)$.

(vi) If $(f, f_\mu^n) \sim_\kappa (f', f_\mu'^{]n})$*, then* $(g, g_\nu^n)(f, f_\mu^n) \sim_{\kappa'} (g, g_\nu^n)(f', f_\mu'^n)$*, provided, for every* $\nu \in N$*,* $| \nu | < \kappa'$ *implies* $| g(\nu) | < \kappa$.

(vii) If $\boldsymbol{Y}$ *is cofinite, then* $(f, f_\mu^n) \sim (f', f_\mu'^n)$ *if and only if* $(f, f_\mu) \sim_m (f', f_\mu')$ *for every* $m \in \mathbb{N}$.

Given a pair of inverse systems $\boldsymbol{X}, \boldsymbol{Y}$ in $\mathcal{A}$, where $\boldsymbol{Y}$ is cofinite, let us define the function

$$\rho^* : inv^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \times inv^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}$$

by putting

$$\rho^*((f, f_\mu^n), (f', f_\mu'^n)) = \begin{cases} \inf\{\frac{1}{m+1} \mid (f, f_\mu^n) \sim_m (f', f_\mu'^n),\ m \in \mathbb{N}\} \\ \qquad\qquad 1, \text{otherwise} \end{cases}.$$

Lemma 3. *For every* $\boldsymbol{X}$ *and every cofinite* $\boldsymbol{Y}$*, the ordered pair* $(inv^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}), \rho^*)$ *is a pseudoultrametric space.*

Proof. (See also Theorem 2 of [22].) Clearly, $\rho^*((f, f_\mu^n), (f', f_\mu'^n)) \geq 0$, $\rho^*((f, f_\mu^n), (f, f_\mu^n)) = 0$ and $\rho^*((f, f_\mu^n), (f', f_\mu'^n)) = \rho^*((f', f_\mu'^n), (f, f_\mu^n))$. It remains to prove that

$$\rho^*((f, f_\mu^n), (f'', f_\mu''^n)) \leq \max\{\rho^*((f, f_\mu^n), (f', f_\mu'^n)), \rho^*((f', f_\mu'^n), (f'', f_\mu''^n))\}$$

holds true. If $\rho^*((f, f_\mu^n), (f', f_\mu'^n)) = 1$ or $\rho^*((f', f_\mu'^n), (f'', f_\mu''^n)) = 1$, the statement is obviously true. Further, the inequality holds in the case of $\rho^*((f, f_\mu^n), (f', f_\mu'^n)) = 0$ or $\rho^*((f', f_\mu'^n), (f'', f_\mu''^n)) = 0$ as well. Namely, in that case

$$\rho^*((f, f_\mu^n), (f'', f_\mu''^n)) = \rho^*((f', f_\mu'^n), (f'', f_\mu''^n))$$

or

$$\rho^*((f,f^n_\mu),(f'',f''^n_\mu)) = \rho^*((f,f^n_\mu),(f',f'^n_\mu))$$

hold, respectively. Let

$$\rho^*((f,f^n_\mu),(f',f'^n_\mu)) = \frac{1}{m+1} \quad \text{and} \quad \rho^*((f',f'^n_\mu),(f'',f''^n_\mu)) = \frac{1}{m'+1}$$

for a pair $m, m' \in \mathbb{N}$. It means that

$$(f,f^n_\mu) \sim_m (f',f'^n_\mu) \wedge (f,f^n_\mu) \not\sim_{m+1} (f',f'^n_\mu)$$
$$(f',f'^n_\mu) \sim_{m'} (f'',f''^n_\mu) \wedge (f',f'^n_\mu) \not\sim_{m'+1} (f'',f''^n_\mu)$$

Then, by Lemma 2 (iii), $(f,f^n_\mu) \sim_{m''} (f'',f''^n_\mu)$, where $m'' = \min\{m,m'\}$. Thus,

$$\rho^*((f,f^n_\mu),(f'',f''^n_\mu)) \leq \frac{1}{m''+1} = \max\{\frac{1}{m+1},\frac{1}{m'+1}\},$$

and the conclusion follows. □

Let us briefly denote $pro^*\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y}) \equiv \boldsymbol{Y}^{\boldsymbol{X}*}$. Observe that, by Lemma 2, (iii) and (vii), if $(f,f^n_\mu) \sim (g,g^n_\mu)$ and $(f',f'^n_\mu) \sim (g',g'^n_\mu)$ (all of $\boldsymbol{X}$ to $\boldsymbol{Y}$), then $\rho^*((f,f^n_\mu),(f',f'^n_\mu)) = \rho^*((g,g^n_\mu),(g',g'^n_\mu))$. Thus, for every cofinite $\boldsymbol{Y}$, the function $d^* : \boldsymbol{Y}^{\boldsymbol{X}*} \times \boldsymbol{Y}^{\boldsymbol{X}*} \to \mathbb{R}$ is well defined by putting

$$d^*(\boldsymbol{f}^*,\boldsymbol{f}'^*) = \rho^*((f,f^n_\mu),(f',f'^n_\mu)),$$

where $(f,f^n_\mu) \in \boldsymbol{f}^*$, $(f',f'^n_\mu) \in \boldsymbol{f}'^*$ is any pair of representatives.

Theorem 1. *For every $\boldsymbol{X}$ and every cofinite $\boldsymbol{Y}$, the ordered pair $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ is a complete ultrametric space. Consequently, the space $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ is totally disconnected and the (covering) dimension*

$$\dim(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) = 0.$$

Proof. (See also Theorem 2 and Remark 2 of [22].) By Lemma 3, it suffices to prove that $d^*(\boldsymbol{f}^*,\boldsymbol{f}'^*) = 0$ implies $\boldsymbol{f}^* = \boldsymbol{f}'^*$, and the completeness. Let $d^*(\boldsymbol{f}^*,\boldsymbol{f}'^*) = 0$. Then,

$$\rho^*((f,f^n_\mu),(f',f^{n\prime}_\mu)) = 0$$

for any pair of the representatives. By the definition of ρ^* and Lemma 2 (vii), it is equivalent to $(f,f^n_\mu) \sim (f',f'^n_\mu)$, i.e., $\boldsymbol{f}^* = \boldsymbol{f}'^*$.

Let $(\boldsymbol{f}^*_k)$, $\boldsymbol{f}^*_k = ((f_k,f^n_{\mu,k}))$ be a Cauchy sequence in $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$. Then, for every $m \in \mathbb{N}$, there exists a $k_m \in \mathbb{N}$ such that, for every pair $k,k' \in \mathbb{N}$, $k,k' \geq k_m$,

$$d^*(\boldsymbol{f}^*_k,\boldsymbol{f}^*_{k'}) \leq \frac{1}{m+1},$$

i.e.,

$$(f_k,f^n_{\mu,k}) \sim_m (f_{k'},f^n_{\mu,k'}),$$

whenever $k, k' \geq k_m$.
Without loss of generality, one may assume that the sequence (k_m) is increasing and unbounded. Since $\boldsymbol{Y}$ is cofinite, given a $\mu \in M$, denote $|\mu|+1 \equiv m(\mu) \in \mathbb{N}$. Observe that there exists a unique function

$$f_0 : M \to \Lambda, \quad f_0(\mu) = f_{k_{m(\mu)}}(\mu).$$

Now, for every $\mu \in M$ and every $n \in \mathbb{N}$, we put

$$f^n_{\mu,0} = f^n_{\mu,k_{m(\mu)}} : X_{f_0(\mu)} = X_{f_{k_{m(\mu)}}(\mu)} \to Y_\mu.$$

(An equivalent choice can be $f^n_{\mu,0} = f^{k_{m(\mu)}}_{\mu,k_{m(\mu)}}$, for $n = 1, \ldots, k_{m(\mu)}$, and $f^n_{\mu,0} = f^n_{\mu,k_{m(\mu)}}$, for $n > k_{m(\mu)}$.) In this way we have obtained the ordered pair $(f_0, (f^n_{\mu,0})_{\mu \in M, n \in \mathbb{N}})$, briefly $(f_0, f^n_{\mu,0})$, where all the $f^n_{\mu,0} : X_{f_0(\mu)} \to Y_\mu$, are $\mathcal{A}$-morphisms. We are to show that $(f_0, f^n_{\mu,0})$ is a $*$-morphism of $\boldsymbol{X}$ to $\boldsymbol{Y}$. Given a pair $\mu \leq \mu'$ in M, we have to prove that there exist a $\lambda \geq f_0(\mu), f_0(\mu')$ and an $n \in \mathbb{N}$, such that, for every $n' \geq n$,

$$f^{n'}_{\mu,0} p_{f_0(\mu)\lambda} = q_{\mu\mu'} f^{n'}_{\mu',0} p_{f_0(\mu')\lambda}.$$

Denote, as before, $|\mu| + 1 \equiv m(\mu) = m$ and $|\mu'| + 1 \equiv m(\mu') = m'$. Then, $m \leq m'$ and $k_m \leq k_{m'}$. Therefore,

$$d^*(\boldsymbol{f}^*_{k_m}, \boldsymbol{f}^*_{k_{m'}}) \leq \frac{1}{m+1}, \quad \text{i.e.,} \quad (f_{k_m}, f^n_{\mu,k_m}) \sim_m (f_{k_{m'}}, f^n_{\mu,k_{m'}}).$$

This means that there exist a $\lambda_1 \geq f_{k_m}(\mu), f_{k_{m'}}(\mu)$ and an $n_1 \in \mathbb{N}$, such that for every $n' \geq n_1$,

(1) $f^{n'}_{\mu,k_m} p_{f_{k_m}(\mu)\lambda_1} = f^{n'}_{\mu,k_{m'}} p_{f_{k_{m'}}(\mu)\lambda_1}$.

Since $(f_{k_{m'}}, f^n_{\mu,k_{m'}})$ is a $*$-morphism, there exist a $\lambda_2 \geq f_{k_{m'}}(\mu), f_{k_{m'}}(\mu')$ and an $n_2 \in \mathbb{N}$, such that for every $n' \geq n_2$,

(2) $f^{n'}_{\mu,k_{m'}} p_{f_{k_{m'}}(\mu)\lambda_2} = q_{\mu\mu'} f^{n'}_{\mu',k_{m'}} p_{f_{k_{m'}}(\mu')\lambda_2}$.

Since Λ is directed, there exists a $\lambda \geq \lambda_1, \lambda_2$, and thus, $\lambda \geq f_0(\mu) = f_{k_m}(\mu)$ and $\lambda \geq f_0(\mu') = f_{k_{m'}}(\mu')$. Put $n = \max\{n_1, n_2\}$, and let $n' \geq n$. Then, by means of (1) and (2), one straightforwardly obtains that

$$f^{n'}_{\mu,0} p_{f_0(\mu)\lambda} = f^{n'}_{\mu,k_m} p_{f_{k_m}(\mu)\lambda} = \cdots = q_{\mu\mu'} f^{n'}_{\mu',k_{m'}} p_{f_{k_{m'}}(\mu')\lambda} = q_{\mu\mu'} f^{n'}_{\mu',0} p_{f_0(\mu')\lambda}.$$

Hence, $(f_0, f^n_{\mu,0}) : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism of *inv**-$\mathcal{A}$. Let $\boldsymbol{f}^*_0 = [(f_0, f^n_{\mu,0})]$ be the corresponding morphism of *pro**-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. Notice that for every m and every $k \geq k_m$,

$$(f_k, f^n_{\mu,k}) \sim_m (f_0, f^n_{\mu,0}), \quad \text{i.e.,} \quad d^*(\boldsymbol{f}^*_k, \boldsymbol{f}^*_0) \leq \frac{1}{m+1},$$

holds. Namely, by our construction, for every $\mu \in M$ and every $n \in \mathbb{N}$,

$$f_0(\mu) = f_{k_m}(\mu) \quad \text{and} \quad f^n_{\mu,0} = f^n_{\mu,k_m}, \ m = |\mu| + 1.$$

Thus, $\lim(\boldsymbol{f}^*_k) = \boldsymbol{f}^*_0$, i.e., the Cauchy sequence $(\boldsymbol{f}^*_k)$ converges to $\boldsymbol{f}^*_0$ in $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$. The last statement follows by the main result of [5] or by [22] (Theorem 4 and Remark 2). □

Recall that there is the *canonical* injection of *pro*-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ into *pro**-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$,

$$\boldsymbol{f}=[(f,f_\mu)]\mapsto i(\boldsymbol{f})=\boldsymbol{f}^*=[(f,f_\mu^n=f_\mu)].$$

Theorem 2. *The canonical injection of pro-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ into pro*-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ is an isometric closed embedding, $i:(\boldsymbol{Y}^{\boldsymbol{X}},d)\to(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$.*

Proof. It is obvious by the appropriate definitions that

$$\rho^*((f,f_\mu^n=f_\mu),(f',f_\mu'^n=f_\mu'))=\rho((f,f_\mu),(f',f_\mu')).$$

Thus, if $\boldsymbol{f}^*=i(\boldsymbol{f})$ and $\boldsymbol{f}'^*=i(\boldsymbol{f}')$, then $d^*(\boldsymbol{f}^*,\boldsymbol{f}'^*)=d(\boldsymbol{f},\boldsymbol{f}')$. Consequently, i maps the space $(\boldsymbol{Y}^{\boldsymbol{X}},d)$ isometrically onto $(i[\boldsymbol{Y}^{\boldsymbol{X}}],d^*)$. Since $(\boldsymbol{Y}^{\boldsymbol{X}},d)$ is complete (Theorem 2.6 of [20]), the conclusion follows. □

Consider the subspace $(\boldsymbol{Y}_\omega^{\boldsymbol{X}*},d^*)\subseteq(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ consisting of all the morphisms $\boldsymbol{f}^*$ having a commutative representative (f,f_μ^n), i.e., for each $n_0\in\mathbb{N}$ (fixed), $(f,f_\mu^{n_0}):\boldsymbol{X}\to\boldsymbol{Y}$ is a morphism of *inv*-$\mathcal{A}$. Clearly, according to Theorem 2 and the proof of Theorem 1, the canonical injection (restricted to the smaller codomain) $i:(\boldsymbol{Y}^{\boldsymbol{X}},d)\to(\boldsymbol{Y}_\omega^{\boldsymbol{X}*},d^*)$ $(\subseteq(\boldsymbol{Y}^{\boldsymbol{X}*},d^*))$ is an isometric closed embedding as well. However, the subspace $(\boldsymbol{Y}_\omega^{\boldsymbol{X}*},d^*)\subseteq(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ is *not* closed. Namely, the proof of Theorem 1 shows that $(\boldsymbol{Y}_\omega^{\boldsymbol{X}*},d^*)$ is *not* complete. Indeed, given an $n_0\in\mathbb{N}$ and a pair $\mu_0\le\mu_0'$, the term $f_{\mu_0,0}^{n_0}$ of $(f_0,f_{\mu,0}^n)$ belongs to $(f_{k_{m(\mu_0)}},f_{\mu,k_{m(\mu_0)}}^n)\in\boldsymbol{f}_{k_{m(\mu_0)}}^*$, while the term $f_{\mu_0',0}^{n_0}$ of $(f_0,f_{\mu,0}^n)$ belongs to $(f_{k_{m(\mu_0')}},f_{\mu,k_{m(\mu_0')}}^n)\in\boldsymbol{f}_{k_{m(\mu_0')}}^*$, which, in general, do *not mutually commute* unless n_0 is large enough. Further, $(\boldsymbol{Y}_\omega^{\boldsymbol{X}*},d^*)$ is not open in $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ since, in general, a commutative morphism admits an arbitrarily close noncommutative one. Finally, in general, $\boldsymbol{Y}_\omega^{\boldsymbol{X}*}$ is not dense in $\boldsymbol{Y}^{\boldsymbol{X}*}$ (consider, for instance, polyhedral inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ associated via its limits with a pair of solenoids).

Remark 1. *If $\boldsymbol{Y}=(Y_\mu=Y,q_{\mu\mu'}=1_Y,M)\in Ob(pro\text{-}\mathcal{A})$ is cofinite, then one readily sees that for every $\boldsymbol{X}\in Ob(pro\text{-}\mathcal{A})$ the space $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ is discrete. However, by Example 2.8 of [20] and our Theorem 2, there exist spaces $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ which are not discrete. Especially, there exist inverse sequences $\boldsymbol{X}$ such that $(\boldsymbol{X}^{\boldsymbol{X}*},d^*)$ are not discrete. Clearly, according to Theorem 2, if $(\boldsymbol{Y}^{\boldsymbol{X}},d)$ is a nondiscrete space then so is $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$, while if $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ is discrete, then such is $(\boldsymbol{Y}^{\boldsymbol{X}},d)$.*

The next theorem diminishes technical difficulties in manipulating with Cauchy sequences (compare Theorem 2.10 of [20]).

Theorem 3. *For every $\boldsymbol{X}$ and every cofinite $\boldsymbol{Y}$, every Cauchy sequence in $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ admits a representing sequence having a unique increasing index function.*

Proof. First, every sequence $(\boldsymbol{f}_k^*)$ in $(\boldsymbol{Y}^{\boldsymbol{X}*},d^*)$ admits a representing sequence $((f_k,f_{\mu,k}^n))$ such that all the index functions are increasing and $f_1\le\cdots\le f_k\le\cdots$. (This can be achieved by a straightforward inductive construction.) Let $(\boldsymbol{f}_k^*)$ be a Cauchy sequence. Recall the proof of Theorem 1, i.e., the construction of the limit $\boldsymbol{f}_0^*=\lim(\boldsymbol{f}_k^*)$. The constructed representative $(f_0,f_{\mu,0}^n)\in\boldsymbol{f}_0^*$ has been defined by means of a subsequence $((f_{k_m}, f^n_{\mu,k_m}))$, where $k_1 \leq \cdots \leq k_m \leq \cdots$ is unbounded such that $f_0(\mu) = f_{k_m}(\mu)$ and $f^n_{\mu,0} = f^n_{\mu,k_m}$, $m = |\mu| + 1$. This implies that $f_0 : M \to \Lambda$ is an increasing function. Let $\mu \in M$, $|\mu| = 0$. Since $f_1 \leq \cdots \leq f_{k_1}$, one can, for every $k = 1, \ldots, k_1$ and every n, replace $f_k(\mu)$ with $f'_k(\mu) = f_0(\mu) = f_{k_1}(\mu)$ and $f^n_{\mu,k}$ with $f'^n_{\mu,k} = f^n_{\mu,k} p_{f_k(\mu) f_0(\mu)}$. In the next step, since $f_1 \leq \cdots \leq f_{k_1} \leq f_{k_1+1} \leq \cdots \leq f_{k_2}$, given a $\mu \in M$, $|\mu| = 1$, one can, for every $k = 1, \ldots, n_2$, replace $f_k(\mu)$ with $f'_k(\mu) = f_0(\mu) = f_{k_2}(\mu)$ and $f^n_{\mu,k}$ with $f'^n_{\mu,k} = f^n_{\mu,k} p_{f_k(\mu) f_0(\mu)}$. Moreover, for every $\mu' \in M$, $|\mu'| = 0$, and every $k = k_1 + 1, \ldots, k_2$, one can replace $f_k(\mu')$ with $f'_k(\mu') = f_0(\mu') = f_{k_2}(\mu')$ and $f^n_{\mu',k}$ with $f'^n_{\mu',k} = f^n_{\mu',k} p_{f_k(\mu') f_0(\mu')}$.

The construction proceeds in an obvious way by induction on $|\mu| + 1 = m \in \mathbb{N}$ through the sequence (k_m). Thus, in the inductive step $m \mapsto m + 1$, one also must correctly move the values of every f_k, $k = k_m + 1, \ldots, k_{m+1}$, for all $\mu \in M$, $|\mu| \leq m$. Observe that for every $k \in \mathbb{N}$, $(f'_k, f'^n_{\mu,k}) \sim (f_k, f^n_{\mu,k})$. Clearly, by construction, the new representing sequence $((f'_k, f'^n_{\mu,k}))$ has the unique increasing index function, namely, $f_0 = f'_k$ for all k. □

3. The hom-bifunctor

Given a category $\mathcal{K}$, let us consider the hom-bifunctor (see [6])

$$\hom : \mathcal{K}^{op} \times \mathcal{K} \to Set$$

defined by $\hom(X, Y) = \mathcal{K}(X, Y)$ and $\hom(u, v)(f) = vfu$. More precisely, for each pair (of pairs) of objects (X, Y), $(X', Y') \in Ob(\mathcal{K}^{op} \times \mathcal{K}) = Ob\mathcal{K}^{op} \times Ob\mathcal{K} = Ob\mathcal{K} \times Ob\mathcal{K}$,

$$\begin{aligned} &\hom^{X,Y}_{X',Y'} : (\mathcal{K}^{op} \times \mathcal{K})((X, Y), (X', Y')) \\ &\quad (= \mathcal{K}^{op}(X, X') \times \mathcal{K}(Y, Y') = \mathcal{K}(X', X) \times \mathcal{K}(Y, Y')) \to Set(\mathcal{K}(X, Y), \mathcal{K}(X', Y')), \\ &\quad (u, v) \mapsto (\hom^{X,Y}_{X',Y'}(u, v) : \mathcal{K}(X, Y) \to \mathcal{K}(X', Y')) \end{aligned}$$

is defined by the composition, i.e., $\hom^{X,Y}_{X',Y'}(u, v)(f) = vfu$.

If the sets $\mathcal{K}(X, Y)$ are endowed with a structure, and if the hom-bifunctor preserves the structure, then notation hom is usually changed into Hom (the "internal" Hom-bifunctor), having an appropriate codomain category (instead of Set).

Let us now consider the case $\mathcal{K} = pro^*\text{-}\mathcal{A}$ for an arbitrary category $\mathcal{A}$, i.e., $\hom : (pro^*\text{-}\mathcal{A})^{op} \times (pro^*\text{-}\mathcal{A}) \to Set$,

$$\hom(\boldsymbol{X}, \boldsymbol{Y}) = pro^*\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \equiv \boldsymbol{Y}^{\boldsymbol{X}*}$$

and $\hom^{\boldsymbol{X},\boldsymbol{Y}}_{\boldsymbol{X}',\boldsymbol{Y}'} : \boldsymbol{X}^{\boldsymbol{X}'*} \times \boldsymbol{Y}'^{\boldsymbol{Y}*} \to Set(\boldsymbol{Y}^{\boldsymbol{X}*}, \boldsymbol{Y}'^{\boldsymbol{X}'*})$, where the function $\hom^{\boldsymbol{X},\boldsymbol{Y}}_{\boldsymbol{X}',\boldsymbol{Y}'}(\boldsymbol{u}^*, \boldsymbol{v}^*) : \boldsymbol{Y}^{\boldsymbol{X}*} \to \boldsymbol{Y}'^{\boldsymbol{X}'*}$ is defined by

$$\hom^{\boldsymbol{X},\boldsymbol{Y}}_{\boldsymbol{X}',\boldsymbol{Y}'}(\boldsymbol{u}^*, \boldsymbol{v}^*)(\boldsymbol{f}^*) = \boldsymbol{v}^* \boldsymbol{f}^* \boldsymbol{u}^*,$$

i.e.,

$$\begin{array}{ccc} \boldsymbol{X} & \overset{\boldsymbol{u}^*}{\leftarrow} & \boldsymbol{X}' \\ \boldsymbol{f}^* \downarrow & \overset{\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)}{\mapsto} & \downarrow \boldsymbol{v}^* \boldsymbol{f}^* \boldsymbol{u}^* \\ \boldsymbol{Y} & \underset{\boldsymbol{v}^*}{\rightarrow} & \boldsymbol{Y}' \end{array} .$$

We assume in the sequel that all inverse systems are *cofinite*. As in the case of $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ in [20], the natural question is: Does the hom-bifunctor preserve the added complete ultrametric structure of the sets $\boldsymbol{Y}^{\boldsymbol{X}*}$? In other words: Is the function $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*) : (\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) \to (\boldsymbol{Y}'^{\boldsymbol{X}'*}, d^*)$ continuous for all (some) $\boldsymbol{u}^* : \boldsymbol{X}' \to \boldsymbol{X}$ and $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$?

Since the restriction $\hom(\boldsymbol{u}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{X}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$ (of $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)$) is, in general, not continuous (see Example 3.2, Lemma 3.3 and Theorem 3.4 of [20] as well as Example 2 in Section 5 below), the answer is negative. Nevertheless, there is a certain subcategory, containing tow^*-$\mathcal{A}$, such that the corresponding hom-bifunctor is (uniformly) continuous (compare Lemma 3.5 and Theorem 3.6 of [20]). Similarly to the pro-case, the continuity depends only on a specific "uniformity" property of the morphism $\boldsymbol{v}^*$ (relating the codomain systems $\boldsymbol{Y}$ and $\boldsymbol{Y}'$).

Lemma 4. *Let $\boldsymbol{u}^* : \boldsymbol{X}' \to \boldsymbol{X}$ and $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$ be morphisms of pro*-$\mathcal{A}$. If $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ is discrete or $\boldsymbol{v}^*$ satisfies the following "uniformity" condition:*

$$(U) \quad (\exists (v, v^n_{\mu'}) \in \boldsymbol{v}^*)(\forall m \in \mathbb{N})(\exists s_m \in \mathbb{N})(\forall \mu' \in M') \quad |\mu'| < m \Rightarrow |v(\mu')| < s_m,$$

then the function

$$\hom(\boldsymbol{u}^*, \boldsymbol{v}^*) : (\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) \to (\boldsymbol{Y}'^{\boldsymbol{X}'*}, d^*)$$

is uniformly continuous.

Proof. Clearly, it is enough to prove the statement when $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ is not discrete. First, to prove the continuity it suffices to show that the function $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)$ preserves convergent sequences. Let $\lim(\boldsymbol{f}^*_k) = \boldsymbol{f}^*_0$ in $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$. We are to prove that the sequence $(\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)(\boldsymbol{f}^*_k)) = (\boldsymbol{v}^* \boldsymbol{f}^*_k \boldsymbol{u}^*)$ in $(\boldsymbol{Y}'^{\boldsymbol{X}'*}, d^*)$ converges to $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)(\boldsymbol{f}^*_0) = \boldsymbol{v}^* \boldsymbol{f}^*_0 \boldsymbol{u}^*$. Let $(f_k, f^n_{\mu,k}) \in \boldsymbol{f}^*_k$, $k \in \mathbb{N}$, $(f_0, f^n_{\mu,0}) \in \boldsymbol{f}^*_0$ and $(u, u^n_\lambda) \in \boldsymbol{u}^*$ be chosen arbitrarily, and let $(v, v^n_{\mu'}) \in \boldsymbol{v}^*$ be a representative according to condition (U). By Lemma 2, (v) and (vi), if $(f_k, f^n_{\mu,k}) \sim_m (f_0, f^n_{\mu,0})$, then

$$(f_k, f^n_{\mu,k})(u, u^n_\lambda) \sim_m (f_0, f^n_{\mu,0})(u, u^n_\lambda), \quad \text{and} \quad (v, v^n_{\mu'})(f_k, f^n_{\mu,k}) \sim_{m'} (v, v^n_{\mu'})(f_0, f^n_{\mu,0})$$

provided $|\mu'| < m'$ implies $|v(\mu')| < m$. Since $\lim(\boldsymbol{f}^*_k) = \boldsymbol{f}^*_0$,

$$d^*(\boldsymbol{f}^*_k, \boldsymbol{f}^*_0) = \rho^*((f_k, f^n_{\mu,k}), (f_0, f^n_{\mu,0}))$$

becomes arbitrarily small when k increases, i.e., for every $m \in \mathbb{N}$, there exists a $k_m \in \mathbb{N}$ such that, for every $k \geq k_m$,

$$\rho^*((f_k, f^n_{\mu,k}), (f_0, f^n_{\mu,0})) \leq \frac{1}{m+1},$$

Hence, $(f_k, f^n_{\mu,k}) \sim_m (f_0, f^n_{\mu,0})$, $k \geq k_m$, and thus,

$$(v, v^n_{\mu'})(f_k, f^n_{\mu,k})(u, u^n_\lambda) \sim_{m'} (v, v^n_{\mu'})(f_0, f^n_{\mu,0})(u, u^n_\lambda), \; k \geq k_m,$$

provided, for every $\mu' \in M'$, $|\mu'| < m'$ implies $|v(\mu')| < m$. Since, by condition (U), for every m there exists an s_m such that for every $\mu' \in M'$ $|\mu'| < m$ implies $|v(\mu')| < s_m$, we infer tha, for every m and every $k \geq k_{s_m}$

$$(v, v^n_{\mu'})(f_k, f^n_{\mu,k})(u, u^n_\lambda) \sim_m (v, v^n_{\mu'})(f_0, f^n_{\mu,0})(u, u^n_\lambda)$$

holds. Thus,

$$d^*(\boldsymbol{v}^*\boldsymbol{f}_k^*\boldsymbol{u}^*, \boldsymbol{v}^*\boldsymbol{f}_0^*\boldsymbol{u}^*) = \rho^*((v, v_{\mu'}^n)(f_k, f_{\mu,k}^n)(u, u_\lambda^n), (v, v_{\mu'}^n)(f_0, f_{\mu,0}^n)(u, u_\lambda^n)) \leq \frac{1}{m+1},$$

for every $k \geq k_{s_m}$. This means that $\lim(\boldsymbol{v}^*\boldsymbol{f}_k^*\boldsymbol{u}^*) = \boldsymbol{v}^*\boldsymbol{f}_0^*\boldsymbol{u}^*$, which proves the continuity of $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)$. In addition, notice that a $\delta > 0$ (for continuity of $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)$) does not depend on any particular point $\boldsymbol{f}^* \in \boldsymbol{Y}^{\boldsymbol{X}*}$. Namely, given any $\varepsilon = \frac{1}{m+1} > 0$, one may put $\delta = \frac{1}{s_m+1} > 0$. Therefore, $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is uniformly continuous. □

Problem 1. *Does the converse of Lemma 4 hold when $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ is not discrete and $\boldsymbol{v}^*$ is a "nontrivial" morphism? (The same question concerning Lemma 3.5 of [20], because the corresponding part of its proof is not correct! Consequently, the proof of the "only if" part of Theorem 4.1 of [20] is not correct!)*

Remark 2. *Let $\boldsymbol{X}$ be a system over an infinite Λ, and let $\boldsymbol{Y}$ be a system over an M such that there are infinitely many $\mu \in M$ with $|\mu| = m$, for some $m \in \{0\} \cup \mathbb{N}$. Then every morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ admits a representative $(f', f_\mu'^n)$ which* does not have the property *of condition(U) (by shifting the index function). Nevertheless, this fact does not contradict the definition of the metric d^*. Further, notice that every representative of every morphism of inverse* sequences has the property *of condition (U).*

Observe that the property of condition (U) of some morphisms of *pro**-$\mathcal{A}$ is preserved by composition. Since each identity morphism $\mathbf{1}_{\boldsymbol{X}}^*$ obviously satisfies condition (U), there exists a certain subcategory pro_{U}^*-$\mathcal{A} \subseteq pro^*$-$\mathcal{A}$, which shares the same object class, while $Mor(pro_{\mathrm{U}}^*$-$\mathcal{A})$ is a proper subclass of $(pro^*$-$\mathcal{A})$. Clearly, by Remark 2, tow^*-$\mathcal{A} \subseteq pro_{\mathrm{U}}^*$-$\mathcal{A}$. Let us briefly denote pro_{U}^*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \equiv \boldsymbol{Y}_{\mathrm{U}}^{\boldsymbol{X}*} \subseteq \boldsymbol{Y}^{\boldsymbol{X}*}$. By assuming the restriction to all *cofinite* inverse systems, the following theorem holds (compare Theorem 3.6 of [20]).

Theorem 4. *The hom-bifunctor for the subcategory pro_U^*-$\mathcal{A}$ is a structure preserving (continuous) one, i.e., it is*

$$Hom : (pro_U^*\text{-}\mathcal{A})^{op} \times (pro_U^*\text{-}\mathcal{A}) \to M_c,$$

where M_c is the category of complete metric spaces.

Proof. According to Theorem 1 and Lemma 4, it suffices to prove that $(\boldsymbol{Y}_{\mathrm{U}}^{\boldsymbol{X}*}, d^*) \subseteq (\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ is a closed subspace. If $\boldsymbol{Y}_{\mathrm{U}}^{\boldsymbol{X}*} = \varnothing$, then there is nothing to prove. Thus, let $\boldsymbol{Y}_{\mathrm{U}}^{\boldsymbol{X}*} \neq \varnothing$. Suppose that a sequence $(\boldsymbol{f}_k^*)$ in $\boldsymbol{Y}_{\mathrm{U}}^{\boldsymbol{X}*}$ converges to an $\boldsymbol{f}_0^*$ in $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$. We have to prove that $\boldsymbol{f}_0^* \in \boldsymbol{Y}_{\mathrm{U}}^{\boldsymbol{X}*}$. Recall the construction of the limit morphism $\boldsymbol{f}_0^*$ in the proof of Theorem 1. Given any representing sequence $((f_k, f_{\mu,k}^n))$ of $(\boldsymbol{f}_k^*)$, the representing $*$-morphism $(f_0, f_{\mu,0}^n)$ of $\boldsymbol{f}_0^*$ has been defined by means of

$$f_{\mu,0}^n = f_{\mu,k_{m(\mu)}}^n : X_{f_0(\mu)} = X_{f_{k_{m(\mu)}}(\mu)} \to Y_\mu,$$

where $m(\mu) = |\mu| + 1$ and (k_m) assure the relation $\sim_m$. In this case, however, we can choose a representing sequence $(f_k, f_{\mu,k}^n)$ so that

$$(\forall k \in \mathbb{N})(\forall m \in \mathbb{N})(\exists s_m^k \in \mathbb{N})(\forall \mu \in M) \quad |\mu| < m \Rightarrow |f_k(\mu)| < s_m^k.$$

Then the obtained unique limit morphism $\boldsymbol{f}_0^* = [(f_0, f_{\mu,0}^n)]$ satisfies condition (U). Indeed, given an $m \in \mathbb{N}$, put $s_m = s_m^{k_m}$ (depending on m only!), and then $|\mu| < m$ implies that

$$|f_0(\mu)| = |f_{k_m}(\mu)| < s_m^{k_m} = s_m.$$

□

An inverse system $\boldsymbol{X}$ is said to have property (F) provided, for every $m \in \mathbb{N}$, the subset

$$\Lambda_{m-1} \equiv \{\lambda \in \Lambda \mid |\lambda| = m-1\} \subseteq \Lambda$$

is finite. For instance, every inverse sequence $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ has property (F) since in this case $|\Lambda_{m-1}| = m$. Let $pro_{\mathrm{F}}^*\text{-}\mathcal{A} \subseteq pro^*\text{-}\mathcal{A}$ be the full subcategory containing all the cofinite objects which have property (F).

Corollary 1. *The hom-bifunctor for the subcategory $pro_F^*\text{-}\mathcal{A} \subseteq pro^*\text{-}\mathcal{A}$ is structure preserving (continuous), i.e. it is*

$$Hom : (pro_F^*\text{-}\mathcal{A})^{op} \times (pro_F^*\text{-}\mathcal{A}) \to M_c.$$

Proof. Observe that $pro_{\mathrm{F}}^*\text{-}\mathcal{A} \subseteq pro_{\mathrm{U}}^*\text{-}\mathcal{A}$ is a full subcategory, because every morphism of $pro_{\mathrm{F}}^*\text{-}\mathcal{A}$ satisfies condition (U) (s_m is maximal among finitely many values). Hence, the conclusion follows by Theorem 4. □

Corollary 2. *The hom-bifunctor for the tower*-category $tow^*\text{-}\mathcal{A}$ is structure preserving (continuous), i.e., it is*

$$Hom : (tow^*\text{-}\mathcal{A})^{op} \times (tow^*\text{-}\mathcal{A}) \to M_c.$$

Proof. Every inverse sequence has property (F), i.e., $tow^*\text{-}\mathcal{A} \subseteq pro_{\mathrm{F}}^*\text{-}\mathcal{A}$ is a full subcategory (see also Remark 2). Thus, the conclusion follows by Corollary 1. □

Let

$$(\boldsymbol{Y}^{\boldsymbol{X}*} \times \boldsymbol{Z}^{\boldsymbol{Y}*}, d') = (\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) \times (\boldsymbol{Z}^{\boldsymbol{Y}*}, d^*)$$

be the product space endowed with an appropriate metric d' (for instance, d_2, d_1 or d_∞ with respect to the metrics on the factors). Then the function

$$\omega : (\boldsymbol{Y}^{\boldsymbol{X}*} \times \boldsymbol{Z}^{\boldsymbol{Y}*}, d') \to (\boldsymbol{Z}^{\boldsymbol{X}*}, d'),$$

defined by the composition, $(\boldsymbol{f}^*, \boldsymbol{g}^*) \mapsto \boldsymbol{g}^* \boldsymbol{f}^*$, naturally arises. According to preceding results, ω cannot be continuous in general. However, the following fact holds as a consequence of Theorem 4.

Corollary 3. *The function (restriction)*

$$\omega : (\boldsymbol{Y}^{\boldsymbol{X}*} \times \boldsymbol{Z}_U^{\boldsymbol{Y}*}, \quad d') \to (\boldsymbol{Z}^{\boldsymbol{X}*}, d^*), \omega(\boldsymbol{f}^*, \boldsymbol{g}^*) = \boldsymbol{g}^* \boldsymbol{f}^*,$$

is uniformly continuous. Especially, for all inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ *and* $\boldsymbol{Z}$ *in* $\mathcal{A}$, *the function* $\omega : (\boldsymbol{Y}^{\boldsymbol{X}*} \times \boldsymbol{Z}^{\boldsymbol{Y}*}, d') \to (\boldsymbol{Z}^{\boldsymbol{X}*}, d^*)$ *is uniformly continuous. Moreover, for every section* $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$, *the hom-bifunctor commutes with* ω, *i.e., the diagram*

$$\begin{array}{ccc} \boldsymbol{Y}^{\boldsymbol{X}*} \times \boldsymbol{Z}^{\boldsymbol{Y}*} & \overset{\hom(\boldsymbol{u}^*,\boldsymbol{v}^*)\times\hom(\boldsymbol{v}'^*,\boldsymbol{w}^*)}{\to} & \boldsymbol{Y}'^{\boldsymbol{X}'*} \times \boldsymbol{Z}'^{\boldsymbol{Y}*'} \\ \omega \downarrow & & \downarrow \omega' \\ \boldsymbol{Z}^{\boldsymbol{X}*} & \overset{\hom(\boldsymbol{u}^*,\boldsymbol{w}^*)}{\to} & \boldsymbol{Z}'^{\boldsymbol{X}'*} \end{array}$$

is commutative. More precisely,

$$\omega' \circ (\hom(\boldsymbol{u}^*, \boldsymbol{v}^*) \times \hom(\boldsymbol{v}'^*, \boldsymbol{w}^*)) = \hom(\boldsymbol{u}^*, \boldsymbol{w}^*) \circ \omega,$$

where $\boldsymbol{v}'^* : \boldsymbol{Y}' \to \boldsymbol{Y}$ *is a left inverse of* $\boldsymbol{v}^*$, $\boldsymbol{v}'^*\boldsymbol{v}^* = \boldsymbol{1}^*_{\boldsymbol{Y}}$.

Proof. It suffices to prove that $\lim(\boldsymbol{f}^*_k) = \boldsymbol{f}^*_0$ in $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ and $\lim(\boldsymbol{g}^*_k) = \boldsymbol{g}^*_0$ in $(\boldsymbol{Z}^{\boldsymbol{Y}*}_{\mathrm{U}}, d^*)$ imply $\lim(\boldsymbol{g}^*_k\boldsymbol{f}^*_k) = \boldsymbol{g}^*_0\boldsymbol{f}^*_0 = \lim(\boldsymbol{g}^*_k)\lim(\boldsymbol{f}^*_k)$ in $(\boldsymbol{Z}^{\boldsymbol{X}*}, d^*)$. Since $\lim(\boldsymbol{g}^*_k) = \boldsymbol{g}^*_0$, Lemma 2 (v) implies that, for each $k \in \mathbb{N}$,

$$\lim_{k'}(\boldsymbol{g}^*_{k'}\boldsymbol{f}^*_k) = \boldsymbol{g}^*_0\boldsymbol{f}^*_k.$$

Thus, $d^*(\boldsymbol{g}^*_k\boldsymbol{f}^*_k, \boldsymbol{g}^*_{k'}\boldsymbol{f}^*_k)$ and $d^*(\boldsymbol{g}^*_{k'}\boldsymbol{f}^*_k, \boldsymbol{g}^*_0\boldsymbol{f}^*_k)$ are arbitrarily small, for $k, k' \in \mathbb{N}$ large enough. Since $\boldsymbol{g}^*_0 \in \boldsymbol{Z}^{\boldsymbol{Y}*}_{\mathrm{U}}$, the function $\hom(\boldsymbol{1}^*_{\boldsymbol{X}}, \boldsymbol{g}^*_0)$ is (uniformly) continuous. Thus,

$$\begin{aligned} \lim(\boldsymbol{g}^*_0\boldsymbol{f}_k) &= \lim(\hom(\boldsymbol{1}^*_{\boldsymbol{X}}, \boldsymbol{g}^*_0)(\boldsymbol{f}^*_k)) = \hom(\boldsymbol{1}^*_{\boldsymbol{X}}, \boldsymbol{g}^*_0)(\lim(\boldsymbol{f}^*_k)) \\ &= \hom(\boldsymbol{1}^*_{\boldsymbol{X}}, \boldsymbol{g}^*_0)(\boldsymbol{f}^*_0)) = \boldsymbol{g}^*_0\boldsymbol{f}^*_0. \end{aligned}$$

Finally, since d^* is a(n) (ultra)meric, the conclusion follows. (Observe that we have only needed $\boldsymbol{g}^*_0 \in \boldsymbol{Z}^{\boldsymbol{Y}*}_{\mathrm{U}}$!) The commutativity of the diagram goes as follows:

$$\begin{aligned} &(\omega' \circ (\hom(\boldsymbol{u}^*, \boldsymbol{v}^*) \times \hom(\boldsymbol{v}'^*, \boldsymbol{w}^*)))(\boldsymbol{f}^*, \boldsymbol{g}^*) \\ &\qquad = \omega'(\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)(\boldsymbol{f}^*), \hom(\boldsymbol{v}'^*, \boldsymbol{w}^*)(\boldsymbol{g}^*)) \\ &\qquad = \omega'(\boldsymbol{v}^*\boldsymbol{f}^*\boldsymbol{u}^*, \boldsymbol{w}^*\boldsymbol{g}^*\boldsymbol{v}'^*) = (\boldsymbol{w}^*\boldsymbol{g}^*\boldsymbol{v}'^*)(\boldsymbol{v}^*\boldsymbol{f}^*\boldsymbol{u}^*) \\ &\qquad = \boldsymbol{w}^*\boldsymbol{g}^*(\boldsymbol{v}'^*\boldsymbol{v}^*)\boldsymbol{f}^*\boldsymbol{u}^* = \boldsymbol{w}^*(\boldsymbol{g}^*\boldsymbol{f}^*)\boldsymbol{u}^* \\ &\qquad = \hom(\boldsymbol{u}^*, \boldsymbol{w}^*)(\boldsymbol{g}^*\boldsymbol{f}^*) \\ &\qquad = \hom(\boldsymbol{u}^*, \boldsymbol{w}^*)(\omega(\boldsymbol{f}^*, \boldsymbol{g}^*)) \\ &\qquad = (\hom(\boldsymbol{u}^*, \boldsymbol{w}^*) \circ \omega)(\boldsymbol{f}^*, \boldsymbol{g}^*). \end{aligned}$$

□

4. Invariance of the hom-bifunctor

Consider now the invariance problem for the hom-bifunctor, i.e., under what conditions, $\boldsymbol{X} \cong \boldsymbol{X}'$ and $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in *pro**-$\mathcal{A}$ imply that the spaces $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ and $(\boldsymbol{Y}'^{\boldsymbol{X}'*}, d^*)$ are homeomorphic. Clearly, every pair of isomorphisms $\boldsymbol{u}^* : \boldsymbol{X}' \to \boldsymbol{X}$, $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$ yields a set bijection $\boldsymbol{Y}^{\boldsymbol{X}*} \to \boldsymbol{Y}'^{\boldsymbol{X}'*}$, $\boldsymbol{f}^* \mapsto \boldsymbol{v}^*\boldsymbol{f}^*\boldsymbol{u}^*$, having the inverse function $\boldsymbol{Y'}^{\boldsymbol{X'}*} \to \boldsymbol{Y}^{\boldsymbol{X}*}$, $\boldsymbol{f}'^* \mapsto (\boldsymbol{v}^*)^{-1}\boldsymbol{f}'^*(\boldsymbol{u}^*)^{-1}$. Therefore, for every such a pair of isomorphisms, the function $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*) : \boldsymbol{Y}^{\boldsymbol{X}*} \to \boldsymbol{Y'}^{\boldsymbol{X'}*}$ is a bijection with the inverse $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)^{-1} = \hom((\boldsymbol{u}^*)^{-1}, (\boldsymbol{v}^*)^{-1})$. According to Lemma 4, the following theorem holds.

Theorem 5. *Let $\boldsymbol{u}^* : \boldsymbol{X'} \to \boldsymbol{X}$ and $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y'}$ be isomorphisms of pro*-$\mathcal{A}$. If $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ and $(\boldsymbol{Y'}^{\boldsymbol{X'}*}, d^*)$ are discrete spaces or $\boldsymbol{v}^*$ and $(\boldsymbol{v}^*)^{-1}$ satisfy condition (U), then $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*) : (\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) \to (\boldsymbol{Y'}^{\boldsymbol{X'}*}, d^*)$ is a uniform homeomorphism (of complete ultrametric spaces).*

Proof. In the special case of discrete spaces the statement is trivial. In the general case, by Lemma 4, $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)$ and $\hom(\boldsymbol{u}^*, \boldsymbol{v}^*)^{-1} = \hom((\boldsymbol{u}^*)^{-1}, (\boldsymbol{v}^*)^{-1})$ are (uniformly) continuous whenever $\boldsymbol{v}^*$ and $(\boldsymbol{v}^*)^{-1}$ satisfy condition (U), respectively. The conclusion follows. □

Theorem 6. *For every category $\mathcal{A}$, the hom-bifunctor for pro*-$\mathcal{A}$ is invariant (and continuous into Met_c) with respect to the object isomorphisms in the following subcategories: tow*-$\mathcal{A}$, pro$^*_F$-$\mathcal{A}$ and pro$^*_U$-$\mathcal{A}$.*

Proof. Apply Theorem 5 together with Corollary 2, Corollary 1 and Theorem 4 respectively. □

Remark 3. *(a) By Theorem 5, for every (cofinite) $\boldsymbol{Y}$ and every pair $\boldsymbol{X} \cong \boldsymbol{X'}$ in pro*-$\mathcal{A}$, $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) \approx (\boldsymbol{Y}^{\boldsymbol{X'}*}, d^*)$ in M_c holds via the hom-bifunctor. Moreover, it is readily seen that for every isomorphism $\boldsymbol{u}^* : \boldsymbol{X'} \to \boldsymbol{X}$ the homeomorphism $\hom(\boldsymbol{u}^*, \boldsymbol{1}^*_{\boldsymbol{Y}})$ is an* isometry. *On the other hand (by Example 3.2 and Theorem 3.4 of [20] and our Theorem 2), there exist an inverse sequence $\boldsymbol{Y}$ and a (countable and cofinite) inverse system $\boldsymbol{Y'}$ isomorphic to $\boldsymbol{Y}$, $\boldsymbol{Y} \cong \boldsymbol{Y'}$ in pro-$\mathcal{A}$ (and, thus, in pro*-$\mathcal{A}$), such that, for every isomorphism $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y'}$, the bijection $\hom(\boldsymbol{1}^*_{\boldsymbol{Y}}, \boldsymbol{v}^*) : (\boldsymbol{Y}^{\boldsymbol{Y}*}, d^*) \to (\boldsymbol{Y'}^{\boldsymbol{Y}*}, d^*)$ is not continuous. Moreover, there is such a pair of complete ultrametric spaces which are* not *homeomorphic (see Example 1 below). An important implication of this fact is that, in general, there is* no unique *canonical complete ultrametrization of the coarse shape (shape as well) morphism sets. Nevertheless, in some special cases (for instance, compact metrizable spaces, by using only sequential HcANR- or HcPol-expansions) a unique canonical complete ultrametrization of the coarse shape (shape as well) morphism sets is possible (see Section 5 below).*
(b) As we have mentioned in Introduction, in the last decade several papers dealing with (ultra)metric and topological structures on the shape morphism sets were written: [3, 4, 14 – 17, 20, 21, ...]. Looking for the basic idea, one readily sees that it is the notion of being μ-homotopic. However, the germ of this idea goes back to 1976 when K. Borsuk [2] introduced the notion of quasi-equivalence of metric compacta. This becomes quite clear after seeing the characterization (reinterpretation) of the quasi-equivalence in terms of sequences of morphisms of inverse sequences, [19].

Example 1. *Let $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$ be an inverse sequence in a category $\mathcal{A}$, and let $\boldsymbol{Y'} = (Y'_\mu, q'_{\mu\mu'}, M)$ be associated with $\boldsymbol{Y}$ by the "Mardešić trick" (see also Example*

3.2 of [20]), i.e.,

$$\begin{aligned} M &= \{\mu \subseteq \mathbb{N} \mid \varnothing \neq \mu \textit{ is finite}\}, \\ \mu \leq \mu' &\Leftrightarrow \mu \subseteq \mu', \\ Y'_\mu &= Y_j, j = \max(\mu), \\ \textit{and} \quad q'_{\mu\mu'} &= q_{jj'} : Y'_{\mu'} = Y_{j'} \to Y_j = Y'_\mu, \mu \leq \mu'. \end{aligned}$$

Then, $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro-$\mathcal{A}$ (via isomorphisms yielded by the bonding morphisms and identities on the terms), and thus, $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro*-$\mathcal{A}$ as well. We shall show that the space $(\boldsymbol{Y}'^{\boldsymbol{Y}*}, d^*)$ is discrete (see the proof below). Therefore, by choosing a $\boldsymbol{Y}$ of tow–$\mathcal{A}$ (for instance, $\mathcal{A} = HcANR$) such that $(\boldsymbol{Y}^{\boldsymbol{Y}*}, d^*)$ is not discrete (see Remark 1), one provides an example with $\boldsymbol{Y} \cong \boldsymbol{Y}'$ such that the spaces $(\boldsymbol{Y}^{\boldsymbol{Y}*}, d^*)$ and $(\boldsymbol{Y}'^{\boldsymbol{Y}*}, d^*)$ are not *homeomorphic.*

Let us prove that the space $(\boldsymbol{Y}'^{\boldsymbol{Y}*}, d^*)$ of Example 1 is discrete. Moreover, we will show that

$$d^*(\boldsymbol{f}^*, \boldsymbol{f}'^*) = \begin{cases} 1, \ \boldsymbol{f}^* \neq \boldsymbol{f}'^* \\ 0, \ \boldsymbol{f}^* = \boldsymbol{f}'^* \end{cases}.$$

If $d^*(\boldsymbol{f}^*, \boldsymbol{f}'^*) = 1$ for every pair $\boldsymbol{f}^*, \boldsymbol{f}'^* \in \boldsymbol{Y}^{\boldsymbol{Y}*}$, $\boldsymbol{f}^* \neq \boldsymbol{f}'^*$, the conclusion follows. Thus, since $diam(\boldsymbol{Y}'^{\boldsymbol{Y}*}, d^*) \leq 1$, let us assume that there is a pair $\boldsymbol{f}^*, \boldsymbol{f}'^* \in \boldsymbol{Y}'^{\boldsymbol{Y}*}$ such that $d^*(\boldsymbol{f}^*, \boldsymbol{f}'^*) < 1$, or equivalently, $d^*(\boldsymbol{f}^*, \boldsymbol{f}'^*) \leq \frac{1}{2}$ (because d^* takes its values in $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$). We are to prove that $d^*(\boldsymbol{f}^*, \boldsymbol{f}'^*) = 0$, i.e., that $\boldsymbol{f}^* = \boldsymbol{f}'^*$. Let $(f, f^n_\mu) \in \boldsymbol{f}^*$ and $(f', f'^n_\mu) \in \boldsymbol{f}'^*$ be any pair of representatives. Then

$$\rho^*((f, f^n_\mu), (f', f'^n_\mu)) \leq \frac{1}{2},$$

which implies $(f, f^n_\mu) \simeq_1 (f', f'^n_\mu)$, i.e., $(f, f^n_\mu) \simeq_\mu (f', f'^n_\mu)$ for every $\mu \in M$, $|\mu| = 0$. By construction of $\boldsymbol{Y}'$, $|\mu| = 0$ means

$$\mu = \{j\} \in M_0 \subseteq M = \sqcup_{k \in \mathbb{N}} M_{k-1}$$

($M_{k-1} = \{\mu \in M \mid |\mu| = k - 1\}$, see the proof of Lemma 3.3 of [20]) and $Y'_\mu = Y_j$, $j \in \mathbb{N}$. Thus,

$$(\forall j \in \mathbb{N})(\exists i_j \geq f(\{j\}), f'(\{j\}))(\exists n(j) \in \mathbb{N})(\forall n' \geq n(j))(\forall i \geq i_j) \quad f^{n'}_{\{j\}} q_{f(\{j\})i} = f'^{n'}_{\{j\}} q_{f'(\{j\})i}.$$

Since $M_1 = \varnothing$, consider any $\mu = \{j, j'\} \in M_2 \subseteq M$, $j < j'$. Then $\{j\}, \{j'\} < \mu$, $Y'_\mu = Y_{j'}$, $q'_{\{j\}\mu} = q_{jj'}$ and $q'_{\{j'\}\mu} = 1_{Y_{j'}}$. Since $q'_{\{j'\}\mu} = 1_{Y_{j'}}$, the above relation and properties of morphisms (f, f^n_μ) and (f', f'^n_μ) of inv^*-$\mathcal{A}$, with respect to $\{j\}, \{j'\} \leq \mu$, imply that there exist an $i_\mu \geq i_j, i_{j'}, f(\mu), f'(\mu)$ and an $n(\mu) \geq n(j), n(j')$ such that for every $n' \geq n(\mu)$ and every $i \geq i_\mu$,

$$f^{n'}_\mu q_{f(\mu)i} = f'^{n'}_\mu q_{f'(\mu)i}$$

holds. This shows that $(f, f_\mu^n) \simeq_\mu (f', f_\mu'^n)$ for every $\mu \in M$, $|\mu| \le 2$, i.e., $(f, f_\mu^n) \simeq_3 (f', f_\mu'^n)$, and thus,

$$\rho^*((f, f_\mu^n), (f', f_\mu'^n)) \le \frac{1}{4}.$$

Now, by induction on $m \in \mathbb{N}$, assuming that

$$\rho^*((f, f_\mu^n), (f', f_\mu'^n)) \le \frac{1}{m+1},$$

one can prove in the same way as above that

$$\rho^*((f, f_\mu^n), (f', f_\mu'^n)) \le \frac{1}{m+1+k_m},$$

holds for some $k_m \in \mathbb{N}$. Therefore, $d^*(\boldsymbol{f}^*, \boldsymbol{f}'^*) = \rho^*((f, f_\mu^n), (f', f_\mu'^n)) = 0$, i.e., $\boldsymbol{f}^* = \boldsymbol{f}'^*$, which completes the proof.

5. Applications

5.1. The coarse equivalence

We want to show how the introduced ultrametric structure on the sets $\boldsymbol{Y}^{\boldsymbol{X}*}$ yields a new equivalence relation on the cofinite object subclass of *pro*-$\mathcal{A}$ - strictly coarser than isomorphiness on *pro*$^*_{\mathrm{U}}$-$\mathcal{A}$, especially, on *tow*$^*_\omega$-$\mathcal{A}$. First, a sequence $(\boldsymbol{f}_k^*)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$ is said to be a *U-sequence*, if it admits a representing sequence $((f_k = f, f_{\mu,k}^n))$, with a unique index function $f_k = f$ for all k, having the property of condition (U), i.e.,

$$(\forall m \in \mathbb{N})(\exists s_m \in \mathbb{N})(\forall \mu \in M) \quad |\mu| < m \Rightarrow |f(\mu)| < s_m.$$

Notice that if $\boldsymbol{X}$ is cofinite and $\boldsymbol{Y}$ is an inverse sequence, then every sequence $(\boldsymbol{f}_k^*)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$, having a representative $((f_k = f, f_{\mu,k}^n))$, is a U-sequence (compare Example 2 below). Thus, in the case of inverse sequences, i.e., in *tow**-$\mathcal{A}$, a unique index function is all one needs.

Definition 3. *Let $\mathcal{A}$ be a category, and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be cofinite systems in $\mathcal{A}$. Then $\boldsymbol{X}$ is said to be* ***coarse equivalent to $\boldsymbol{Y}$****, denoted by $\boldsymbol{X} \sim^* \boldsymbol{Y}$, if there exist U-sequences $(\boldsymbol{f}_k^*)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$ and $(\boldsymbol{g}_k^*)$ in $\boldsymbol{X}^{\boldsymbol{Y}*}$ such that* $\lim(\boldsymbol{g}_k^* \boldsymbol{f}_k^*) = \boldsymbol{1}_{\boldsymbol{X}}^*$ *in* $(\boldsymbol{X}^{\boldsymbol{X}*}, d^*)$ *and* $\lim(\boldsymbol{f}_k^* \boldsymbol{g}_k^*) = \boldsymbol{1}_{\boldsymbol{Y}}^*$ *in* $(\boldsymbol{Y}^{\boldsymbol{Y}*}, d^*)$.

Lemma 5. *The coarse equivalence $\sim^*$ is an equivalence relation on the cofinite object subclass of Ob(pro-$\mathcal{A}$).*

Proof. Since the relation $\sim^*$ is obviously reflexive and symmetric, it remains to prove that $\sim^*$ is transitive. Let $\boldsymbol{X} \sim^* \boldsymbol{Y}$ be realized via an $(\boldsymbol{f}_k^*)$ and a $(\boldsymbol{g}_k^*)$, and let $\boldsymbol{Y} \sim^* \boldsymbol{Z}$ be realized via an $(\boldsymbol{f}_k'^*)$ and a $(\boldsymbol{g}_k'^*)$ - each of them admitting a required representing sequence. Then, for every $s \in \mathbb{N}$ there exists a $k_s \in \mathbb{N}$ such that for every $k \ge k_s$,

$$d^*(\boldsymbol{g}_k^* \boldsymbol{f}_k^*, \boldsymbol{1}_{\boldsymbol{X}}^*) \le \frac{1}{s+1}, \quad d^*(\boldsymbol{f}_k^* \boldsymbol{g}_k^*, \boldsymbol{1}_{\boldsymbol{Y}}^*) \le \frac{1}{s+1},$$
$$d^*(\boldsymbol{g}_k'^* \boldsymbol{f}_k'^*, \boldsymbol{1}_{\boldsymbol{Y}}^*) \le \frac{1}{s+1}, \quad d^*(\boldsymbol{f}_k'^* \boldsymbol{g}_k'^*, \boldsymbol{1}_{\boldsymbol{Z}}^*) \le \frac{1}{s+1}$$

By Lemma 2 (v), for every $k \geq k_s$,

$$d^*(\boldsymbol{f}_k^*\boldsymbol{g}_k^*\boldsymbol{g}_k'^*, \boldsymbol{g}_k'^*) \leq \frac{1}{s+1} \quad \text{and} \quad d^*(\boldsymbol{g}_k'^*\boldsymbol{f}_k'^*\boldsymbol{f}_k^*, \boldsymbol{f}_k^*) \leq \frac{1}{s+1}$$

hold as well. Now, given any $m \in \mathbb{N}$, choose $s_m, s'_m \in \mathbb{N}$ according to condition (U) for the sequences $(\boldsymbol{g}_k^*)$, $(\boldsymbol{f}_k'^*)$, respectively. Then, for every $k \geq k_{s_m}, k_{s'_m}$,

$$d^*(\boldsymbol{g}_k^*\boldsymbol{g}_k'^*\boldsymbol{f}_k'^*\boldsymbol{f}_k^*, \boldsymbol{g}_k^*\boldsymbol{f}_k^*) \leq \frac{1}{m+1} \quad \text{and} \quad d^*(\boldsymbol{f}_k'^*\boldsymbol{f}_k^*\boldsymbol{g}_k^*\boldsymbol{g}_k'^*, \boldsymbol{f}_k'^*\boldsymbol{g}_k'^*) \leq \frac{1}{m+1},$$

respectively. Since d^* is an ultrametric, it must hold

$$d^*(\boldsymbol{g}_k^*\boldsymbol{g}_k'^*\boldsymbol{f}_k'^*\boldsymbol{f}_k^*, \boldsymbol{1}_{\boldsymbol{X}}^*) \leq \frac{1}{m+1}, k \geq k_{s_m} \quad \text{and} \quad d^*(\boldsymbol{f}_k'^*\boldsymbol{f}_k^*\boldsymbol{g}_k^*\boldsymbol{g}_k'^*, \boldsymbol{1}_{\boldsymbol{Z}}^*) \leq \frac{1}{m+1}, k \geq k_{s'_m}.$$

Put, for every $k \in \mathbb{N}$,

$$\boldsymbol{u}_k^* = \boldsymbol{f}_k'^*\boldsymbol{f}_k^* : \boldsymbol{X} \to \boldsymbol{Z} \quad \text{and} \quad \boldsymbol{v}_k^* = \boldsymbol{g}_k^*\boldsymbol{g}_k'^* : \boldsymbol{Z} \to \boldsymbol{X}.$$

Since condition (U) is preserved by the coordinatewise composition of sequences with unique index functions, the conclusion follows. □

Remark 4. *The relation $\sim^*$ is a "uniform" analogue and a generalization of Borsuk's quasi-equivalence (of inverse sequences of compact ANR's, [2, 2, 19]). However, the quasi-equivalence is not an equivalence relation (see [7]). The reason why is "too much freedom for the index functions" - because its uniformization (by controlling the index-functions), called the $\bar{q}$-equivalence (see [8, 19]), is an equivalence relation. So we have to use the sequences of morphisms having unique index functions. If not, the counterexample of [7] would work herein as well.*

Corollary 4. *The $\bar{q}$-equivalence strictly implies the coarse equivalence (for inverse sequences), i.e., if $\boldsymbol{X} \overset{\bar{q}}{\simeq} \boldsymbol{Y}$ in tow-$\mathcal{A}$ then $\boldsymbol{X} \sim^* \boldsymbol{Y}$, but not conversely. In particular, $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow$_\omega^*$-$\mathcal{A}$ implies $\boldsymbol{X} \sim^* \boldsymbol{Y}$, but not conversely.*

Proof. Both relations are defined via sequences of the appropriate morphisms having unique index functions. Hence, every $\boldsymbol{f}_k = [(f, f_{j,k})]$, $k \in \mathbb{N}$, of *tow*-$\mathcal{A}$ yields the corresponding $\boldsymbol{f}_k^* = [(f, f_{j,k}^n = f_{j,k})]$, and similarly for $\boldsymbol{g}_k$ and $\boldsymbol{g}_k^*$. Thus, $\overset{\bar{q}}{\simeq}$ implies $\sim^*$ (see also subsection 5.2 of [20]). The converse does not hold because of the same counterexample (due to J. Keesling and S. Mardešić, [9]) used in Corollary 5.7 of [23] and Corollary 5.2 of [8]. □

Notice that $\boldsymbol{X} \cong \boldsymbol{Y}$ in *pro*$_{\mathrm{U}}^*$-$\mathcal{A}$ ($\supseteq$ *tow**-$\mathcal{A}$) implies $\boldsymbol{X} \sim^* \boldsymbol{Y}$ (via an appropriate pair of constant sequences consisting of isomorphisms satisfying condition (U)). The restriction to *pro*$_{\mathrm{U}}^*$-$\mathcal{A}$ is essential because it does not hold in general - as the next example shows.

Example 2. *Let $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$ be an inverse* sequence *in $\mathcal{A}$, and let $\boldsymbol{Y}' = (Y'_\mu, q'_{\mu\mu'}, M)$ be the inverse* system *associated with $\boldsymbol{Y}$ by the Mardešić trick (see Example 1). Then, $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro-$\mathcal{A}$ and, consequently, $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro*-$\mathcal{A}$ as* *well. However (see Example 3.2 and Lemma 3.3 of [20]), if* $\boldsymbol{Y}$ *is* not semi-stable, *then there is* no section *of* $\boldsymbol{Y}$ *to* $\boldsymbol{Y}'$ *satisfying condition (U). Therefore,* $\boldsymbol{Y}$ *is* not *isomorphic to* $\boldsymbol{Y}'$ *in pro*$^*_U$*-*$\mathcal{A}$*. Moreover, such a* $\boldsymbol{Y}$ *is* not *coarse equivalent to* $\boldsymbol{Y}'$ *(see the proof below).*

First, recall the notion of *semi-stability* (the complementary part of strong movability; [19], Definition 3 and Lemma 4) of an inverse sequence $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$:

$$(\exists i_0 \in \mathbb{N})(\forall i \geq i_0)(\forall i' \geq i)(\exists r : X_i \to X_{i'})(\exists i_1 \geq i')(\forall i'' \geq i_1) \quad rp_{ii''} = p_{i'i''}.$$

It is readily seen that an $\boldsymbol{X}$ of *tow*-$\mathcal{A} \subseteq$ *pro*-$\mathcal{A}$ is semi-stable if and only if every morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro*-$\mathcal{A}$ admits an $i_0 \in \mathbb{N}$ such that $\boldsymbol{f} = [(c_{i_0}, f_\mu)]$.

Assume to the contrary, i.e., that $\boldsymbol{Y} \sim^* \boldsymbol{Y}'$. Then there exists a pair of U-sequences $(\boldsymbol{f}^*_k)$ and a $(\boldsymbol{g}^*_k)$ such that $\lim(\boldsymbol{g}^*_k \boldsymbol{f}^*_k) = \boldsymbol{1}^*_{\boldsymbol{Y}}$ and $\lim(\boldsymbol{f}^*_k \boldsymbol{g}^*_k) = \boldsymbol{1}^*_{\boldsymbol{Y}'}$. Let $((f', f'^n_{\mu,k}))$ be any appropriate representing sequence of $(\boldsymbol{f}^*_k)$, and let (s_m) be a corresponding integer sequence existing by condition (U). Since $\boldsymbol{Y}$ is an inverse sequence, there exists a representing sequence $((f, f^n_{\mu,k}))$ of $(\boldsymbol{f}^*_k)$ such that $f(\mu) = f(\mu')$ whenever $|\mu| = |\mu'|$. (The construction is by induction on $|\mu| = m - 1 \in \{0\} \cup \mathbb{N}$; put $f(\mu) = s_m$ and $f^n_{\mu,k} = f'^n_{\mu,k} q_{f'(\mu)s_m}$.). Let $((g, g^n_{j,k}))$ be any appropriate representing sequence of $(\boldsymbol{g}^*_k)$. We may assume, without loss of generality, that g is increasing. Since $\lim(\boldsymbol{f}^*_k \boldsymbol{g}^*_k) = \boldsymbol{1}^*_{\boldsymbol{Y}'}$, there exists a strictly increasing sequence (k_m) in $\mathbb{N}$ such that

$$d^*(\boldsymbol{f}^*_k \boldsymbol{g}^*_k, \boldsymbol{1}^*_{\boldsymbol{Y}'}) \leq \frac{1}{m+1}, \ k \geq k_m.$$

By the definition of d^*, it means that for every $\mu \in M$ with $|\mu| < m$ and every $k \geq k_m$

$$(f, f^n_{\mu,k})(g, g^n_{j,k}) \sim_\mu (1_M, 1^n_{Y'_\mu})$$

holds. Choose $\{1\} \in M$. Then $f(\{1\}) = s_1 = f(\{j\})$, for all $j \in \mathbb{N}$, because $|\mu| = 0$ (i.e., $|\mu| < 1$) if and only if $\mu = \{j\}$ for some $j \in \mathbb{N}$. Put $\mu_0 = g(s_1)$ and $j_0 = \max(\mu_0)$. Then $Y'_{\mu_0} = Y_{j_0}$. Let $j' \geq j \geq j_0$. Put $\mu_* = \mu_0 \cup \{j\}$ and $\mu' = \mu_* \cup \{j'\}$. Then $\mu_*, \mu' \in M$, $\{j'\} \leq \mu'$, $\mu_0 \leq \mu_* \leq \mu'$, $\max(\mu_*) = j$ and $\max(\mu') = j'$. Thus, $Y'_{\mu_*} = Y_j$ and $Y'_{\mu'} = Y_{j'}$. Since $\boldsymbol{Y}$ is cofinite, there is an m' such that $|\mu'| < m'$, and we may apply $(k = k_{m'})$

$$(f, f^n_{\mu,k_{m'}})(g, g^n_{j,k_{m'}}) \sim_{\mu'} (1_M, 1^n_{Y'_\mu}).$$

This means that there exist a $\mu_1 \geq \mu', gf(\mu')$ and an $n_1 \in \mathbb{N}$ such that for every $\mu'' \geq \mu_1$ and every $n \geq n_1$

$$f^n_{\mu',k_{m'}} g^n_{f(\mu'),k_{m'}} q'_{gf(\mu')\mu''} = q'_{\mu'\mu''}. \tag{1}$$

On the other hand, by the $*$-morphism property of

$$(f, f^n_{\mu,k_{m'}})(g, g^n_{j,k_{m'}}) : \boldsymbol{Y} \to \boldsymbol{Y},$$

for $\{j'\} \leq \mu'$ there exist a $\mu_2 \geq gf(\{\mu'\})$ $(\geq gf(\{j'\}) = \mu_0)$ and an $n_2 \in \mathbb{N}$ such that for every $\mu'' \geq \mu_2$ and every $n \geq n_2$

$$f^n_{\{j'\},k_{m'}} g^n_{s_1,k_{m'}} q'_{g(s_1)\mu''} = q'_{\{j'\}\mu'} f^n_{\mu',k_{m'}} g^n_{f(\mu'),k_{m'}} q'_{gf(\mu')\mu''}. \tag{2}$$

Choose a $\mu_3 \geq \mu_1, \mu_2$, and put $n_3 = \max\{n_1, n_2\}$. Then (1) and (2) hold for every $\mu'' \geq \mu_3$ and every $n \geq n_3$. Put $j_1 = \max(\mu_3)$ and let $j'' \geq j_1$. Then $j'' = \max(\mu'')$ for some $\mu'' \geq \mu_3$. By construction of $\boldsymbol{Y}'$, (1) implies ($j_* = \max(gf(\mu'))$) that

$$f^n_{\mu',k_{m'}} g^n_{f(\mu'),k_{m'}} q_{j_* j''} = q_{j'j''} : Y_{j''} \to Y_{j'},\ n \geq n_3, \tag{3}$$

while (2) implies ($q'_{\{j'\}\mu'} = 1_{Y_{j'}}$) that

$$f^n_{\{j'\},k_{m'}} g^n_{s_1,k_{m'}} q_{j_0 j''} = f^n_{\mu',k_{m'}} g^n_{f(\mu'),k_{m'}} q_{j_* j''} : Y_{j''} \to Y_{j'},\ n \geq n_3. \tag{4}$$

Therefore,

$$f^n_{\{j'\},k_{m'}} g^n_{s_1,k_{m'}} q_{j_0 j} q_{jj''} = f^n_{\{j'\},k_{m'}} g^n_{s_1,k_{m'}} q_{j_0 j''} = q_{j'j''},\ n \geq n_3.$$

Choose $n = n_3$ and put

$$r = f^{n_3}_{\{j'\},k_{m'}} g^{n_3}_{s_1,k_{m'}} q_{j_0 j} : Y_j \to Y_{j'}.$$

In this way we have proven that $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$ has the following property:

$$(\exists j_0)(\forall j \geq j_0)(\forall j' \geq j)(\exists r : Y_j \to Y_{j'})(\exists j_1 \geq j')(\forall j'' \geq j_1) \quad r q_{jj''} = q_{j'j''},$$

which means that $\boldsymbol{Y}$ is semi-stable - a contradiction.

Theorem 7. *Let $\boldsymbol{X} \cong \boldsymbol{X}'$ and $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in pro$^*_U$-$\mathcal{A}$, and let $\boldsymbol{X} \sim^* \boldsymbol{Y}$. Then $\boldsymbol{X}' \sim^* \boldsymbol{Y}'$.*

Proof. Let $\boldsymbol{u}^* : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{v}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$ be isomorphisms satisfying condition (U). Let $(\boldsymbol{f}^*_k)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$ and $(\boldsymbol{g}^*_k)$ in $\boldsymbol{X}^{\boldsymbol{Y}*}$be a pair of U-sequences realizing $\boldsymbol{X} \sim^* \boldsymbol{Y}$. For every $k \in \mathbb{N}$ put

$$\boldsymbol{f}'^*_k = \boldsymbol{v}^* \boldsymbol{f}^*_k (\boldsymbol{u}^*)^{-1} : \boldsymbol{X}' \to \boldsymbol{Y}' \quad \text{and} \quad \boldsymbol{g}'^*_k = \boldsymbol{u}^* \boldsymbol{g}^*_k (\boldsymbol{v}^*)^{-1} : \boldsymbol{Y}' \to \boldsymbol{X}'.$$

Since $\lim(\boldsymbol{g}^*_k \boldsymbol{f}^*_k) = \boldsymbol{1}^*_{\boldsymbol{X}}$, Lemma 2 (v) assures that $\lim((\boldsymbol{g}^*_k \boldsymbol{f}^*_k)(\boldsymbol{u}^*)^{-1}) = (\boldsymbol{u}^*)^{-1}$. In the same way, $\lim((\boldsymbol{f}^*_k \boldsymbol{g}^*_k)(\boldsymbol{v}^*)^{-1}) = (\boldsymbol{v}^*)^{-1}$.
Further, since $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ satisfy condition (U), Corollary 3 (see also its proof) assures that

$$\lim(\boldsymbol{g}'^*_k \boldsymbol{f}'^*_k) = \lim(\boldsymbol{u}^*(\boldsymbol{g}^*_k \boldsymbol{f}^*_k)(\boldsymbol{u}^*)^{-1}) = \lim(\boldsymbol{u}^*) \lim((\boldsymbol{g}^*_k \boldsymbol{f}^*_k)(\boldsymbol{u}^*)^{-1}) = \boldsymbol{u}^*(\boldsymbol{u}^*)^{-1} = \boldsymbol{1}^*_{\boldsymbol{X}'},$$
$$\lim(\boldsymbol{f}'^*_k \boldsymbol{g}'^*_k) = \lim(\boldsymbol{v}^*(\boldsymbol{f}^*_k \boldsymbol{g}^*_k)(\boldsymbol{v}^*)^{-1}) = \lim(\boldsymbol{v}^*) \lim((\boldsymbol{f}^*_k \boldsymbol{g}^*_k)(\boldsymbol{v}^*)^{-1}) = \boldsymbol{v}^*(\boldsymbol{v}^*)^{-1} = \boldsymbol{1}^*_{\boldsymbol{Y}'}.$$

Therefore, $\boldsymbol{X}' \sim^* \boldsymbol{Y}'$. □

Corollary 5. *(i) If $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow\text{-}\mathcal{A})$ such that $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow*-$\mathcal{A}$, then $\boldsymbol{X} \sim^* \boldsymbol{Y}$.*

(ii) Let $\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{Y}, \boldsymbol{Y}' \in Ob(tow\text{-}\mathcal{A})$ such that $\boldsymbol{X} \cong \boldsymbol{X}'$ and $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in tow*-$\mathcal{A}$. Then $\boldsymbol{X} \sim^* \boldsymbol{Y}$ if and only if $\boldsymbol{X}' \sim^* \boldsymbol{Y}'$.

Concerning the (coarse) shape theory ([8, 11]), by Theorem 7 and Corollary 5 (ii), we can well define the coarse equivalence for compact metrizable spaces: Two compacta X and Y are coarse equivalent, denoted by $X \sim^* Y$, if $\boldsymbol{X} \sim^* \boldsymbol{Y}$, where $\boldsymbol{X}$, $\boldsymbol{Y}$ is any pair of their *sequential* $HcPol$-expansions.

Since there are plenty of non-discrete spaces $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$, we expect that the coarse equivalence $\sim^*$ is strictly coarser than the isomorphiness in tow^*-$\mathcal{A}$, i.e., that the answer to the next question is affirmative (see also Theorem 8 below).

Problem 2. *Does there exist a pair of inverse* sequences $\boldsymbol{X}$*,* $\boldsymbol{Y}$ *in* $\mathcal{A}$ *such that* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are coarse equivalent,* $\boldsymbol{X} \sim^* \boldsymbol{Y}$*, and they are not isomorphic objects of* tow^*-$\mathcal{A}$*. (We have a pair* $\boldsymbol{X}$*,* $\boldsymbol{Y}$ *such that* $\boldsymbol{X} \sim^* \boldsymbol{Y}$ *and* $\boldsymbol{X} \ncong \boldsymbol{Y}$ *in* tow^*_ω-$HcPol$*; however, in this particular case,* $\boldsymbol{X} \cong \boldsymbol{Y}$ *in* tow^*-$HcPol$ *holds; see also Corollary 4).*

The next theorem might be a motivation to ask for the affirmative solution.

Theorem 8. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be inverse sequences in a category* $\mathcal{A}$*. Then the following claims are equivalent:*

(i) $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are isomorphic objects,* $\boldsymbol{X} \cong \boldsymbol{Y}$*, of* tow^*-$\mathcal{A}$*.*

(ii) $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are coarse equivalent,* $\boldsymbol{X} \sim^* \boldsymbol{Y}$*, and there is a pair of realizing sequences such that one (equivalently, both) of them is a Cauchy sequence.*

Proof. Since the analogue of Lemma 5.10 of [20] holds in the same way for the ultrametric d^*, the proof follows the pattern of the proof of Theorem 5.9 of [20]. □

Corollary 6. *Let* X *and* Y *be compact metrizable spaces. Then the following are equivalent:*
(i) X *and* Y *have the same coarse shape type,* $Sh^*(X) = Sh^*(Y)$*.*
(ii) X *and* Y *are coarse equivalent,* $X \sim^* Y$ *and there is a pair of realizing sequences, for* $\boldsymbol{X} \sim^* \boldsymbol{Y}$*, such that one (equivalently, both) of them is a Cauchy sequence.*

Our intention now is to show that the coarse equivalence admits a full category characterization. Let $\phi = (\boldsymbol{f}^*_k)$ be a sequence of morphisms $\boldsymbol{f}^*_k \in \boldsymbol{Y}^{\boldsymbol{X}*}$, and let $\psi = (\boldsymbol{g}^*_k)$ be a sequence of morphisms $\boldsymbol{g}^*_k \in \boldsymbol{Z}^{\boldsymbol{Y}*}$, $k \in \mathbb{N}$. Then the coordinatewise composition well defines the sequence $\chi = (\boldsymbol{g}^*_k \boldsymbol{f}^*_k)$ in $\boldsymbol{Z}^{\boldsymbol{X}*}$. Clearly, this composition is associative. Therefore, there exists a category on the object class of $Ob(pro^*\text{-}\mathcal{A}) = Ob(pro\text{-}\mathcal{A})$ having morphisms $\phi : \boldsymbol{X} \to \boldsymbol{Y}$ all the morphism sequences $(\boldsymbol{f}^*_k)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be cofinite systems, and let a pair of new morphisms $\phi = (\boldsymbol{f}^*_k) : \boldsymbol{X} \to \boldsymbol{Y}$ and $\phi' = (\boldsymbol{f}'^*_k) : \boldsymbol{X} \to \boldsymbol{Y}$ be given. Then, ϕ is said *to be equivalent to* ϕ', denoted by $\phi \sim \phi'$, if the corresponding (real) distance sequence converges to zero, i.e., $\lim(d^*(\boldsymbol{f}^*_k, \boldsymbol{f}'^*_k)) = 0$. One can easily verify that the relation $\sim$ is an equivalence relation on the set of new morphisms of an $\boldsymbol{X}$ to a $\boldsymbol{Y}$. The equivalence class $[\phi]$ of ϕ is denoted by $\underline{\phi}$

Lemma 6. *Let* $\phi = (\boldsymbol{f}^*_k) \sim (\boldsymbol{f}'^*_k) = \phi' : \boldsymbol{X} \to \boldsymbol{Y}$*, and let* $\boldsymbol{W}$ *and* $\boldsymbol{Z}$ *be cofinite systems. Then,*

(i) $(\forall \chi = (\boldsymbol{h}^*_k) : \boldsymbol{W} \to \boldsymbol{X})\ \phi\chi \sim \phi'\chi$.

(ii) $(\forall \psi = (\boldsymbol{g}_k^*) : \boldsymbol{Y} \to \boldsymbol{Z}$ *satisfying condition (U))* $\psi\phi \sim \psi\phi'$.

Proof. Statement (i) follows by Lemma 2 (v), because

$$(\forall k \in \mathbb{N})\ d^*(\boldsymbol{f}_k^*, \boldsymbol{f}_k'^*) \leq \frac{1}{m+1} \Rightarrow d^*(\boldsymbol{f}_k^* \boldsymbol{h}_k^*, \boldsymbol{f}_k'^* \boldsymbol{h}_k^*) \leq \frac{1}{m+1}.$$

On the other hand, condition (U) for $\psi = (\boldsymbol{g}_k^*)$ assures that

$$(\forall k \in \mathbb{N})\ d^*(\boldsymbol{f}_k^*, \boldsymbol{f}_k'^*) \leq \frac{1}{s_m+1} \Rightarrow d^*(\boldsymbol{g}_k^* \boldsymbol{f}_k^*, \boldsymbol{g}_k^* \boldsymbol{f}_k^*) \leq \frac{1}{m+1},$$

and statement (ii) follows. □

Since condition (U) is preserved by the composition in pro^*-$\mathcal{A}$, the coordinatewise composition of two U-sequences is a U-sequence. Further, the constant identity sequence $(\boldsymbol{1}_{\boldsymbol{X},k}^* = \boldsymbol{1}_{\boldsymbol{X}}^*)$ is obviously a U-sequence. Thus, the restriction to all the U-sequences assures the compatibility of relation $\sim$ with the coordinatewise composition. Consequently, there exists the corresponding quotient category - denoted by $\underline{pro}_{\mathrm{U}}^*$-$\mathcal{A}$ as well as its full subcategory $\underline{tow}^*$-$\mathcal{A}$. Furthermore, in the case of a pro-reflective category pair $(\mathcal{C}, \mathcal{D} = \mathcal{A})$, one can establish the corresponding "shape$_{\mathrm{U}}^*$" category "$Sh_{\mathrm{U}(\mathcal{C},\mathcal{D})}^*$" (yielding a kind of the "coarse" coarse shape theory - modeled on [11] and [8]). For instance, in the case of $\mathcal{C} = HcM$ (the homotopy category of compact metrizable spaces) and $\mathcal{D} = HcPol$ (the homotopy category of compact polyhedra), we have got the category "$(Sh^*)_{\mathrm{U}(HcM),HcPol)}^*$"denoted by $(Sh^*)^*(cM)$.

Theorem 9. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be cofinite inverse systems in a category* $\mathcal{A}$. *Then* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are coarse equivalent,* $\boldsymbol{X} \sim^* \boldsymbol{Y}$, *if and only if they are isomorphic objects of* $\underline{pro}_U^*$-$\mathcal{A}$, $\boldsymbol{X} \cong \boldsymbol{Y}$ *in* $\underline{pro}_U^*$-$\mathcal{A}$.

Proof. Let $\boldsymbol{X} \sim^* \boldsymbol{Y}$, i.e., let there exist sequences $(\boldsymbol{f}_k^*)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$ and $(\boldsymbol{g}_k^*)$ in $\boldsymbol{X}^{\boldsymbol{Y}*}$ satisfying condition (U), such that $\lim(\boldsymbol{g}_k^* \boldsymbol{f}_k^*) = \boldsymbol{1}_{\boldsymbol{X}}^*$ in $(\boldsymbol{X}^{\boldsymbol{X}*}, d^*)$ and $\lim(\boldsymbol{f}_k^* \boldsymbol{g}_k^*) = \boldsymbol{1}_{\boldsymbol{Y}}^*$ in $(\boldsymbol{Y}^{\boldsymbol{Y}*}, d^*)$. Put $\phi = (\boldsymbol{f}_k^*)$ and $\psi = (\boldsymbol{g}_k^*)$. Then $\underline{\phi} \in \underline{pro}_{\mathrm{U}}^*$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ and $\underline{\psi} \in \underline{pro}_{\mathrm{U}}^*$-$\mathcal{A}(\boldsymbol{Y}, \boldsymbol{X})$. We are to prove that $\underline{\psi\phi} = \underline{1}_{\boldsymbol{X}}$ and $\underline{\phi\psi} = \underline{1}_{\boldsymbol{Y}}$, i.e., that

$$\psi\phi \sim (\boldsymbol{1}_{\boldsymbol{X}}^*) \quad \text{and} \quad \phi\psi \sim (\boldsymbol{1}_{\boldsymbol{Y}}^*),$$

i.e., that $\lim(d^*(\boldsymbol{g}_k^* \boldsymbol{f}_k^*, \boldsymbol{1}_{\boldsymbol{X}}^*)) = 0$ and $\lim(d^*(\boldsymbol{f}_k^* \boldsymbol{g}_k^*, \boldsymbol{1}_{\boldsymbol{Y}}^*)) = 0$ hold. However, that is an immediate consequence of $\lim(\boldsymbol{g}_k^* \boldsymbol{f}_k^*) = \boldsymbol{1}_{\boldsymbol{X}}^*$ and $\lim(\boldsymbol{f}_k^* \boldsymbol{g}_k^*) = \boldsymbol{1}_{\boldsymbol{Y}}^*$. Conversely, let $\boldsymbol{X} \cong \boldsymbol{Y}$ in $\underline{pro}_{\mathrm{U}}^*$-$\mathcal{A}$, i.e., let there exist a $\underline{\phi} \in \underline{pro}_{\mathrm{U}}^*$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ and a $\underline{\psi} \in \underline{pro}_{\mathrm{U}}^*$-$\mathcal{A}(\boldsymbol{Y}, \boldsymbol{X})$ such that $\underline{\psi\phi} = \underline{1}_{\boldsymbol{X}}$ and $\underline{\phi\psi} = \underline{1}_{\boldsymbol{Y}}$. Let $\phi = (\boldsymbol{f}_k^*) \in \underline{\phi}$ and $\psi = (\boldsymbol{g}_k^*) \in \underline{\psi}$ be a pair of representatives. Then,

$$(\boldsymbol{g}_k^* \boldsymbol{f}_k^*) = \psi\phi \sim (\boldsymbol{1}_{\boldsymbol{X}}^*) \quad \text{and} \quad (\boldsymbol{f}_k^* \boldsymbol{g}_k^*) = \phi\psi \sim (\boldsymbol{1}_{\boldsymbol{Y}}^*),$$

which means that $\lim(d^*(\boldsymbol{g}_k^* \boldsymbol{f}_k^*, \boldsymbol{1}_{\boldsymbol{X}}^*)) = 0$ and $\lim(d^*(\boldsymbol{f}_k^* \boldsymbol{g}_k^*, \boldsymbol{1}_{\boldsymbol{Y}}^*)) = 0$ hold. This obviously implies that $\lim(\boldsymbol{g}_k^* \boldsymbol{f}_k^*) = \boldsymbol{1}_{\boldsymbol{X}}^*$ in $(\boldsymbol{X}^{\boldsymbol{X}*}, d^*)$ and $\lim(\boldsymbol{f}_k^* \boldsymbol{g}_k^*) = \boldsymbol{1}_{\boldsymbol{Y}}^*$ in $(\boldsymbol{Y}^{\boldsymbol{Y}*}, d^*)$. Since $(\boldsymbol{f}_k^*)$ and $(\boldsymbol{g}_k^*)$ satisfy condition (U), the conclusion follows. □

Corollary 7. *Let X and Y be compact metrizable spaces. Then the following are equivalent:*

(i) X and Y are coarse equivalent, $X \sim^* Y$.

(ii) X and Y have the same $(Sh^*)^*$-type.

5.2. The uniform coarse equivalence

In order to obtain another comparison of $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow^*-$\mathcal{A}$ to $\boldsymbol{X} \sim^* \boldsymbol{Y}$, we need a special kind of morphism sequences. A sequence $((u_k = u, u^n_{\mu,k}))$ of $*$-morphisms $(u, u^n_{\mu,k}) : \boldsymbol{X} \to \boldsymbol{Y}$ of inverse systems, $k \in \mathbb{N}$, is said to be *uniform* if, for every related pair $\mu \leq \mu'$, there exist a $\lambda \geq u(\mu), u(\mu')$ and an n, such that the appropriate condition for a $*$-morphism holds for every k, i.e.,

$$(\forall n' \geq n)(\forall k \in \mathbb{N})\ u^{n'}_{\mu,k} p_{u(\mu)\lambda} = q_{\mu\mu'} u^{n'}_{\mu',k} p_{u(\mu')\lambda}.$$

Clearly, the condition from above holds for every $\lambda' \geq \lambda$ as well. We say that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are *uniformly coarse equivalent* if they are coarse equivalent by means of a pair of uniform representing sequences. (Since we deal with inverse sequences, condition (U) is satisfied in general!).

In light of Corollaries 4 and 5, the following characterization seems to be very interesting.

Theorem 10. *Let $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow\text{-}\mathcal{A})$. Then, $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow^*-$\mathcal{A}$ if and only if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are uniformly coarse equivalent.*

Proof. Since every morphism of tow^*-$\mathcal{A}$ satisfies condition (U), and since every stationary morphism sequence in $\boldsymbol{Y}^{\boldsymbol{X}*}$ and $\boldsymbol{X}^{\boldsymbol{Y}*}$ is uniform, the necessity holds straightforwardly. Conversely, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in $\mathcal{A}$ such that $\boldsymbol{X} \sim^* \boldsymbol{Y}$ uniformly. First, let us show that every uniform sequence $((u_k = u, u^n_{\mu,k}))$ of $*$-morphisms $(u, u^n_{j,\mu}) : \boldsymbol{X}' \to \boldsymbol{Y}'$ (of systems, generally), $k \in \mathbb{N}$, induces a $*$-morphism $(u, u^n_\mu) : \boldsymbol{X}' \to \boldsymbol{Y}'$. (This has *no* analogue in the case of ordinary, i.e., commutative morphisms!). Let us apply the "diagonal procedure", i.e., put, for every $n \in \mathbb{N}$ and every $\mu \in M$,

$$u^n_\mu = u^n_{\mu,n} = X'_{u(\mu)} \to Y'_\mu.$$

Let $\mu \leq \mu'$. Since the sequence $((u, u^n_{\mu,k}))$ is uniform, there exist a $\lambda \geq u(\mu), u(\mu')$ and an $n_{\mu,\mu'}$, such that for every k every $\lambda' \geq \lambda$ and every $n \geq n_{\mu,\mu'}$,

$$u^n_{\mu,k} p_{u(\mu)\lambda'} = q_{\mu\mu'} u^n_{\mu',k} p_{u(\mu')\lambda'}.$$

By putting $k = n$, it turns into

$$u^n_{\mu,n} p_{u(\mu)\lambda'} = q_{\mu\mu'} u^n_{\mu',n} p_{u(\mu')\lambda'}, \quad \text{i.e.,} \quad u^n_\mu p_{u(\mu)\lambda'} = q_{\mu\mu'} u^n_{\mu'} p_{u(\mu')\lambda'},$$

which shows that (u, u^n_μ) is a $*$-morphism of $\boldsymbol{X}'$ to $\boldsymbol{Y}'$. Let $(\boldsymbol{f}^*_k)$ in $\boldsymbol{Y}^{\boldsymbol{X}*}$ and $(\boldsymbol{g}^*_k)$ in $\boldsymbol{X}^{\boldsymbol{Y}*}$ be a pair of appropriate sequences realizing the uniform coarse equivalence

$\boldsymbol{X} \sim^{*} \boldsymbol{Y}$. Let $((f, f^n_{j,k})), ((g, g^n_{i,k}))$ be any pair of appropriate representing sequences. Let $(f, f^n_j) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g^n_i) : \boldsymbol{Y} \to \boldsymbol{X}$ be their induced $*$-morphisms, respectively, and let $\boldsymbol{f}^* = [(f, f^n_j)]$ and $\boldsymbol{g}^* = [(g, g^n_i)]$. We are to prove that $\boldsymbol{g}^*\boldsymbol{f}^* = \boldsymbol{1}^*_{\boldsymbol{X}}$ and $\boldsymbol{f}^*\boldsymbol{g}^* = \boldsymbol{1}^*_{\boldsymbol{Y}}$. It suffices to verify that for every $m \in \mathbb{N}$

$$(fg, g^n_i f^n_{g(i)}) \sim_m (1_{\mathbb{N}}, 1^n_{X_i}) \quad \text{and}(gf, f^n_j g^n_{f(j)}) \sim_m (1_{\mathbb{N}}, 1^n_{Y_j})$$

hold. Indeed, since $\lim(\boldsymbol{g}^*_k \boldsymbol{f}^*_k) = \boldsymbol{1}^*_{\boldsymbol{X}}$, for every $m \in \mathbb{N}$, there exists a k_m such that for every $k \geq k_m$

$$\rho^*((fg, g^n_{i,k} f^n_{g(i),k}), (1_{\mathbb{N}}, 1^n_{X_i})) = d^*(\boldsymbol{g}^*_k \boldsymbol{f}^*_k, \boldsymbol{1}^*_{\boldsymbol{X}}) \leq \frac{1}{m+1}.$$

It means that for every $i < m$ there exist an $i' \geq i, fg(i)$ and an n_i such that for every $n \geq n_i$

$$g^n_{i,k} f^n_{g(i),k} p_{fg(i)i'} = p_{ii'}, \ k \geq k_m.$$

Thus, for every m every $i \leq m$ and every $k = n \geq \max\{k_m, n_i\}$,

$$g^n_i f^n_{g(i)} p_{fg(i)i'} = g^n_{i,k} f^n_{g(i),k} p_{fg(i)i'} = p_{ii'},$$

which means that

$$(fg, g^n_i f^n_{g(i)}) \sim_m (1_{\mathbb{N}}, 1^n_{X_i})$$

holds. In the same way, starting with $\lim(\boldsymbol{f}^*_k \boldsymbol{g}^*_k) = \boldsymbol{1}^*_{\boldsymbol{Y}}$, one obtains, for every m, the needed relation

$$(gf, f^n_j g^n_{f(j)}) \sim_m (1_{\mathbb{N}}, 1^n_{Y_j}).$$

□

Remark 5. *The uniformity condition added to the coarse equivalence to become the uniform coarse equivalence is the same one added to* $\overline{q}$*-equivalence to become the* q^**-equivalence in the case of a sequence of commutative morphisms (see [19], Remark 8(b) and [8], Section 5).*

5.3. The weak shape

The weak shape theory is another generalization of shape theory which generalizes the coarse shape theory as well (see, [23]). The commutative functorial diagram (for an appropriate category pair $(\mathcal{C}, \mathcal{D})$) is as follows:

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow S & S^* \downarrow & S_* \searrow & \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{J}{\to} & Sh^*_{(\mathcal{C},\mathcal{D})} & \underset{W}{\to} & Sh_{*(\mathcal{C},\mathcal{D})} \end{array}$$

where the functors J and WJ are faithful keeping the objects fixed. The realizing category of the weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$ is the "$*$-reduced pro-category" $pro^{\sim}_{*}$-$\mathcal{D} = (inv^{\sim}_{*}$-$\mathcal{D})/\simeq$. Generally, for every category $\mathcal{A}$ the morphism set

$$pro^{\sim}_{*}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \equiv \boldsymbol{Y}^{\boldsymbol{X}}_{*} = (inv^{\sim}_{*}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}))/\simeq$$

is not empty if and only if $M = \Lambda$ (preordered, directed, cofinite, infinite, having no maximal element). Every morphism $\boldsymbol{f}_*$ is the equivalence class $[(f_{\boldsymbol{\mu}})]$ of a hyperladder $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ (morphism of $inv_*^{\sim}$-$\mathcal{A}$).

In [21], it is showed that every set $\boldsymbol{Y}_*^{\boldsymbol{X}}$ admits a complete ultrametric structure having a lot of useful properties. However, that metric structure is not naturally comparable to $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ of [20] neither to $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ of this paper. We will show hereby how to change slightly the previous ultrametric on $\boldsymbol{Y}_*^{\boldsymbol{X}}$ to obtain a complete ultrametric space $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$ such that, at least for *inverse sequences*, $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ are its closed subspaces.

Definition 4. *Let $(f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms (hyperladders) of $inv_*^{\sim}$-$\mathcal{A}$, $M = \Lambda$, and let $m \in \mathbb{N}$. Then $(f_{\boldsymbol{\mu}})$ is said* ***to be*** m***-equivalent to***$(f'_{\boldsymbol{\mu}})$*, denoted by $(f_{\boldsymbol{\mu}}) \sim_m (f'_{\boldsymbol{\mu}})$, if the following condition is fulfilled:*

$$(\forall \mu_1 \in M)(\forall \mu'_1 \geq \mu_1, |\mu'_1| < m)(\exists \lambda_* \geq \mu'_1)(\forall \mu_2 \geq \lambda_*)\ f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}\ rel(\mu'_1, \lambda_*),$$

where $\boldsymbol{\mu} = [\mu_1, \mu_2]$.
(Herein "rel (μ'_1, λ_*)" means that for every $\mu \in [\mu_1, \mu'_1]$, $f_\mu p_{f(\mu)\lambda_*} = f'_\mu p_{f'(\mu)\lambda_*}$.)

The next properties are immediate consequences of the definition.

(i) For every $m \in \mathbb{N}$, the relation $\sim_m$ is an equivalence relation on each set $inv_*^{\sim}$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$;

(ii) if $(f_{\boldsymbol{\mu}}) \sim_{m'} (f'_{\boldsymbol{\mu}})$ and $m \leq m'$, then $(f_{\boldsymbol{\mu}}) \sim_m (f'_{\boldsymbol{\mu}})$;

(iii) $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$ if and only if for every $m \in \mathbb{N}$, $(f_{\boldsymbol{\mu}}) \sim_m (f'_{\boldsymbol{\mu}})$;

(iv) If $(f_{\boldsymbol{\mu}}) \sim_{m'} (f'_{\boldsymbol{\mu}})$ and $(f'_{\boldsymbol{\mu}}) \sim_{m''} (f''_{\boldsymbol{\mu}})$, then $(f_{\boldsymbol{\mu}}) \sim_m (f''_{\boldsymbol{\mu}})$, where $m = \min\{m', m''\}$;

(v) for every m and every $(h_{\boldsymbol{\lambda}}) \in inv_*^{\sim}$-$\mathcal{A}(\boldsymbol{W}, \boldsymbol{X})$, if $(f_{\boldsymbol{\mu}}) \sim_m (f'_{\boldsymbol{\mu}})$, then $(f_{\boldsymbol{\mu}})(h_{\boldsymbol{\lambda}}) \sim_m (f'_{\boldsymbol{\mu}})(h_{\boldsymbol{\lambda}})$;

(vi) Let $m, m' \in \mathbb{N}$, let $(f_{\boldsymbol{\mu}}) \sim_m (f'_{\boldsymbol{\mu}})$ and let $(g_{\boldsymbol{\nu}}) \in inv_*^{\sim}$-$\mathcal{A}(\boldsymbol{Y}, \boldsymbol{Z})$ fulfills the following condition:
$(\forall \nu_1 \in N = M)(\forall \nu'_1 \geq \nu_1)(\exists \mu^1 \geq \nu'_1)(\forall \nu_2 \geq \mu^1)$
$|\nu'_1| < m' \Rightarrow |g(\nu'_1)| < m$,
where g is the index function of the ladder $g_{\boldsymbol{\nu}} \in (g_{\boldsymbol{\nu}})$ assigned to $\boldsymbol{\nu} = [\nu_1, \nu_2]$. Then $(g_{\boldsymbol{\nu}})(f_{\boldsymbol{\mu}}) \sim_{m'} (g_{\boldsymbol{\nu}})(f'_{\boldsymbol{\mu}})$.

The relation $\sim_m$ on the hyperladders admits to define a certain pseudoultrametric (see also [22])

$$\delta_* : inv_*^{\sim}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \times inv_*^{\sim}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}$$

by putting

$$\delta_*((f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}})) = \begin{cases} \inf\{\frac{1}{m+1} \mid (f_{\boldsymbol{\mu}}) \sim_m (f'_{\boldsymbol{\mu}}),\ m \in \mathbb{N}\} \\ \quad 1, \text{otherwise} \end{cases}.$$

By definition, $\delta_*((f_{\boldsymbol{\mu}}),(f'_{\boldsymbol{\mu}})) = 0$ if and only if $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$. Further, if $(f'_{\boldsymbol{\mu}}) \simeq (f''_{\boldsymbol{\mu}})$, then $\delta_*((f_{\boldsymbol{\mu}}),(f'_{\boldsymbol{\mu}})) = \delta_*((f_{\boldsymbol{\mu}}),(f''_{\boldsymbol{\mu}}))$, for every $(f_{\boldsymbol{\mu}})$. Thus, there exists an ultrametric $d_* : \boldsymbol{Y}_*^{\boldsymbol{X}} \times \boldsymbol{Y}_*^{\boldsymbol{X}} \to \mathbb{R}$ defined by

$$d_*(\boldsymbol{f}_*, \boldsymbol{f}'_*) = \delta_*((f_{\boldsymbol{\mu}}),(f'_{\boldsymbol{\mu}})),$$

where $(f_{\boldsymbol{\mu}}) \in \boldsymbol{f}_*$, $(f'_{\boldsymbol{\mu}}) \in \boldsymbol{f}'_*$ is any pair of representatives. To see that the ultrametric space $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$ is complete, let us consider a Cauchy sequence $(\boldsymbol{f}_*^k)$ in $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$. Then there exists a strictly increasing sequence (k_m) in $\mathbb{N}$ such that

$$d_*(\boldsymbol{f}_*^k, \boldsymbol{f}_*^{k'}) \leq \frac{1}{m+1},\ k, k' \geq k_m.$$

It means that for any representing sequence $((f_{\boldsymbol{\mu}}^k))$ of $(\boldsymbol{f}_*^k)$

$$(\forall \mu_1)(\forall \mu'_1 \geq \mu_1, |\mu'_1| < m)(\exists \lambda_* \geq \mu'_1)(\forall \mu_2 \geq \lambda_*) \quad f_{\boldsymbol{\mu}}^k \simeq f_{\boldsymbol{\mu}}^{k'}\ rel(\mu'_1, \lambda_*),\ k, k' \geq k_m,$$

holds, where $f_{\boldsymbol{\mu}}^k \in (f_{\boldsymbol{\mu}}^k)$ and $f_{\boldsymbol{\mu}}^{k'} \in (f_{\boldsymbol{\mu}}^{k'})$ are the corresponding ladders assigned to $\boldsymbol{\mu} = [\mu_1, \mu_2] \in \boldsymbol{\Lambda}$. Now, by choosing, for every $\boldsymbol{\mu} = [\mu_1, \mu_2] \in \boldsymbol{\Lambda}$, the ladder

$$f_{\boldsymbol{\mu}}^0 = f_{\boldsymbol{\mu}}^{k_m} : \boldsymbol{X} \to \boldsymbol{Y}, \quad m = |\mu_2| + 1,$$

we obtain the family $(f_{\boldsymbol{\mu}}^0)$, $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Then, one can easily verify that $(f_{\boldsymbol{\mu}}^0)$ is a hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$, and that, for every $m \in \mathbb{N}$,

$$\rho_*((f_{\boldsymbol{\mu}}^k),(f_{\boldsymbol{\mu}}^0)) \leq \frac{1}{m+1},\ k \geq k_m.$$

Finally, by putting

$$\boldsymbol{f}_*^0 = [(f_{\boldsymbol{\mu}}^0)] : \boldsymbol{X} \to \boldsymbol{Y},$$

it immediately follows that $\lim(\boldsymbol{f}_*^k) = \boldsymbol{f}_*^0$ in $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$.

Similarly to Theorem 3 (see also Lemma 6 of [23]), if $\lim(\boldsymbol{f}_*^k) = \boldsymbol{f}_*^0$, then there exist representing hyperladders $(f_{\boldsymbol{\mu}}^k)$, $k \in \mathbb{N}$, and $(f_{\boldsymbol{\mu}}^0)$ having a unique common increasing index function $f_0 \geq 1_\Lambda$.

Recall now the canonical injection of *pro*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ into *pro**-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{f} = [(f, f_\mu)] \mapsto i(\boldsymbol{f}) = \boldsymbol{f}^* = [(f, f_\mu^n = f_\mu)]$ (Theorem 2). Further, by Lemma 7 of [23] (see also its proof), there exists a canonical (functorial) function of *pro*$^*_\sim$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$ to *pro*$^\sim_*$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, $\boldsymbol{f}^* = [(f, f_\mu^n)] \mapsto j(\boldsymbol{f}^*) = \boldsymbol{f}_* = [(f_{\boldsymbol{\mu}})]$, which is injective for inverse sequences. (Given a $\boldsymbol{\mu} = [\mu_1, \mu_2] \in \boldsymbol{\Lambda}$, the ladder $f_{\boldsymbol{\mu}}$ is defined to be the maximal commutative restriction of (f, f_μ^n) to $\boldsymbol{\mu}$ for $n = |\mu_2| + 1$; in general, j is *not* an injection!)

Lemma 7. *The function* $j : (\boldsymbol{Y}^{\boldsymbol{X}*}, d^*) \to (\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$ *is continuous.*

Proof. Let $\lim(\boldsymbol{f}_k^*) = \boldsymbol{f}_0^*$ in $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$. We have to prove that $\lim(j(\boldsymbol{f}_k^*)) = j(\boldsymbol{f}_0^*)$ in $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$. Therefore, it suffices to verify that, for every $m \in \mathbb{N}$,

$$d^*(\boldsymbol{f}^*, \boldsymbol{g}^*) \leq \frac{1}{m+1}$$

implies

$$d_*(j(\boldsymbol{f}^*), j(\boldsymbol{g}^*)) \leq \frac{1}{m+1}.$$

Let (f, f_j^n) and (g, g_j^n) be representatives of $\boldsymbol{f}^*$ and $\boldsymbol{g}^*$, respectively, and let $(f_{\boldsymbol{j}})$ and $(g_{\boldsymbol{j}})$ be the corresponding induced hyperladders. (The same letter j for the function and an index should not cause ambiguity!) Since

$$\rho^*((f, f_j^n), (g, g_j^n)) \leq \frac{1}{m+1},$$

i.e., for each $j_0 = |j_0| + 1 < m$,

$$(f, f_j^n) \sim_{j_0} (g, g_j^n),$$

the construction of $(f_{\boldsymbol{j}})$ and $(g_{\boldsymbol{j}})$ assures that for every j_1 and every $j_1' \geq j_1$ such that $|j_1'| = j_1' - 1 < m$, there exists an $i_* \geq j_1', f(j_1'), g(j_1')$ so that for every $j_2 \geq i_*$

$$f_{\boldsymbol{j}} \simeq g_{\boldsymbol{j}} \ rel(j_1', i_*), \ \boldsymbol{j} = [j_1, j_2].$$

It means that $(f_{\boldsymbol{j}}) \sim_m (g_{\boldsymbol{j}})$, i.e.,

$$\rho_*((f_{\boldsymbol{j}}), (g_{\boldsymbol{j}})) \leq \frac{1}{m+1},$$

and the conclusion follows. □

Problem 3. *Is* $j[\boldsymbol{Y}^{\boldsymbol{X}*}]$ *closed in* $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$*?*

The answer is affirmative in the sequential case.

Theorem 11. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be inverse sequences in a category* $\mathcal{A}$*. Then the canonical injections* $ji : \boldsymbol{Y}^{\boldsymbol{X}} \to \boldsymbol{Y}_*^{\boldsymbol{X}}$ *and* $j : \boldsymbol{Y}^{\boldsymbol{X}*} \to \boldsymbol{Y}_*^{\boldsymbol{X}}$ *are isometric closed embeddings of spaces* $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ *and* $(\boldsymbol{Y}^{\boldsymbol{X}*}, d^*)$ *into* $(\boldsymbol{Y}_*^{\boldsymbol{X}}, d_*)$*, respectively.*

Proof. According to Lemma 7 of [23] and our Lemma 7 and Theorems 1 and 2, it suffices to verify that in the case of inverse sequences

$$d_*(j(\boldsymbol{f}^*), j(\boldsymbol{g}^*)) = d^*(\boldsymbol{f}^*, \boldsymbol{g}^*)$$

holds. Let (f, f_j^n) and (g, g_j^n) be representatives of $\boldsymbol{f}^*$ and $\boldsymbol{g}^*$, respectively, and let $(f_{\boldsymbol{j}})$ and $(g_{\boldsymbol{j}})$ be the corresponding induced hyperladders. (The same letter j for the function and an index should not cause ambiguity!) If $d_*(j(\boldsymbol{f}^*), j(\boldsymbol{g}^*)) = 0$, then the definition of j and Lemma 7 of [23] immediately imply that $d^*(\boldsymbol{f}^*, \boldsymbol{g}^*) = 0$. Let $d_*(j(\boldsymbol{f}^*), j(\boldsymbol{g}^*)) = 1$. Since $\rho_*((f_{\boldsymbol{j}}), (g_{\boldsymbol{j}})) = 1$, i.e., $(f_{\boldsymbol{j}}) \not\sim_1 (g_{\boldsymbol{j}})$, we infer that (for $j_1' = j_1 = 1$)

$$(\forall i_* \geq 1, f(1), g(1))(\exists j_2 \geq i_*) \quad f_{\boldsymbol{j}} \not\simeq g_{\boldsymbol{j}} \ rel(1, i_*),$$

where $\boldsymbol{j} = [1, j_2]$. It implies that there are cofinally many $n = j_2 \in \mathbb{N}$ such that for every $i \geq f(1), g(1)$,

$$f_1^n p_{f(1)i} \neq g_1^n p_{g(1)i}.$$

Thus,

$$(f, f_j^n) \not\sim_1 (g, g_j^n), \quad \text{i.e.,} \quad d^*(\boldsymbol{f}^*, \boldsymbol{g}^*) = \rho^*((f, f_j^n), (g, g_j^n)) = 1.$$

Let, finally,

$$d_*(j(\boldsymbol{f}^*), j(\boldsymbol{g}^*)) = \rho_*((f_{\boldsymbol{j}}), (g_{\boldsymbol{j}})) = \frac{1}{m+1},$$

for some $m \in \mathbb{N}$. This means that

$$(f_{\boldsymbol{j}}) \sim_m (g_{\boldsymbol{j}}) \ \wedge \ (f_{\boldsymbol{j}}) \not\sim_{m+1} (g_{\boldsymbol{j}})$$

hold. Since every $j = |j| + 1 \in \mathbb{N}$, and since for every $n \in \mathbb{N}$ there is $j = |j| + 1 = n$, the first relation and the definition of function j imply that

$$(f, f_j^n) \sim_m (g, g_j^n), \quad \text{i.e.,} \quad d^*(\boldsymbol{f}^*, \boldsymbol{g}^*) = \rho^*((f, f_j^n), (g, g_j^n)) \leq \frac{1}{m+1}.$$

On the other hand, the second relation implies (similarly to the previous case) that there are cofinally many $n = j_2 \in \mathbb{N}$ such that for every $i \geq f(m+1), g(m+1)$,

$$f_{m+1}^n p_{f(m+1)i} \neq g_{m+1}^n p_{g(m+1)i}.$$

Hence,

$$(f, f_j^n) \not\sim_{m+1} (g, g_j^n), \quad \text{i.e.,} \quad d^*(\boldsymbol{f}^*, \boldsymbol{g}^*) = \rho^*((f, f_j^n), (g, g_j^n)) > \frac{1}{m+2}.$$

Therefore,

$$d^*(\boldsymbol{f}^*, \boldsymbol{g}^*) = \frac{1}{m+1}$$

which completes the proof. □

Similarly to the facts concerning the old structure (Theorems 4 and Corollary 2 of [21]), the analogue facts hold for the new complete ultrametric structure on $\boldsymbol{Y}_*^{\boldsymbol{X}}$. Especially, the corresponding hom-bifunctor is continuous and invariant at least for inverse sequences. Therefore, for HcM - the homotopy category of compact metrizable spaces and $HcPol$ - the homotopy category of compact polyhedra (or $HcANR$ - the homotopy category of compact ANR's for metric spaces) the following corollary holds.

Corollary 8. *For every ordered pair (X, Y) of compact metrizable spaces, there exist complete ultrametric structures on the corresponding shape, coarse shape and weak shape morphism sets, $Sh(X, Y)$, $Sh^*(X, Y)$ and $Sh_*(X, Y)$, respectively, such that the canonical (functorial) injections*

$$(Sh(X, Y), d) \to (Sh^*(X, Y), d^*) \to (Sh_*(X, Y), d_*)$$

are isometric closed embeddings.

Proof. Since $\mathcal{D} = HcPol$ (or $HcANR$) is a sequentially pro-reflective (i.e., tow-reflective) subcategory of $HcM = \mathcal{C}$ (Corollaries I. 5. 4 and I. 5. 6. of [11]), the appropriate realizing categories for the shape, coarse shape and weak shape are *tow*-$\mathcal{D}$, *tow**-$\mathcal{D}$ and *tow*$_*$-$\mathcal{D}$, respectively. By applying Corollary 3.9 and Theorem 4.2 of [20], our Corollary 2, Theorem 6, the above observation concerning $\boldsymbol{Y}_*^{\boldsymbol{X}}$ and Theorem 11, the conclusion follows. □

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